Inference of a random environment from random process realizations: Formalism and application to trajectory prediction

Cécile ICHARD [a,b], Christophe BAEHR [a],
e-mail: cecile.ichard@meteo.fr, christophe.baehr@meteo.fr
[a] *Météo-France-CNRS, CNRM-GAME URA 1357, 42 Avenue Coriolis, 31057 Toulouse Cedex 1, France*
[b] *ENAC, MAIAA, 7, Avenue Edouard Belin, F-31055 Toulouse, France*

Abstract. We are interested in aircraft trajectories seen as stochastic processes. These processes evolve in an unknown atmospheric random environnment. As several aircraft parameters are unknown, such as true airspeed (TAS) and wind, we have to estimate them.

To this end, we suggest to use ensemble weather forecasts, which give different scenarios for the atmosphere, with a system of trajectory predictions. In this way, we evaluate the likelihood of each element and we construct a random weather environment organized by the element weight. To get this result, we use sequential Monte Carlo methods (SMC) in the special context of random environment.

We propose to use particle Markov chain Monte Carlo method (pMCMC) to estimate the aircraft parameters.

Keywords. Non-linear filtering, random environment, estimation, sequential Monte Carlo method, particle Markov chain Monte Carlo method, aircraft trajectories

Introduction

To satisfy the future demand in terms of air transportation, the present air-traffic management system needs to be improved. To this end two projects, NEXTGen in the United-States and SESAR in Europe, have been launched. In both cases, the selected appproach consists in constraining in time and space the aircraft position (4D-trajectory). Moreover, the SESAR project aims to ensure free flights avoiding any delaying tactics. Therefore trajectory predictors have to be accurate and reliable. In that way, the workload of air-traffic controllers can be reduced using decision support tools. Moreover the capacity of the airspace can be used to its maximal capacities.

To compute aircraft trajectories in advance, trajectory predictors need different information. Some concern the flight intent, others are directly related to the aircraft and finally some are environmental parameters, such as wind and temperature. An important source of uncertainty in aircraft trajectory prediction concerns the meteorological parameters. Indeed a part of the along track error in predicting the aircraft trajectories is due to the weather forecasting error.

Up to now, aircraft trajectory predictors use only one weather deterministic forecast. A solution which was proposed was to use statistical errors on weather forecast to get statistical errors on trajectory prediction. The problem with this method is that the statistics used are not space depending whereas the weather forecasting error is. This work aims to give a solution to this problem using ensemble weather forecasts. Indeed national meteorological center are able to provide them. These forecasts give several atmospheric evolution scenarios which reflects the lack of knowledge about the initial state. These scenarios enable to explore the uncertainties about the state of the atmosphere. Another fact at this point is that ensemble forecasts are not delivered with a probability distribution. This problem can be tackled using stochastic methods to weight the elements of the ensemble weather forecasts regarding to air-traffic observations.

In this work we suppose that we have air-traffic observations and an aircraft trajectory predictor. Each aircraft trajectory prediction has an error part and all the aircrafts trajectories in the same area are sharing the same meteorological situation. Now, considering we have a set of weather forecasts, we can evaluate a performing score regards to trajectory prediction errors over the last minutes.

In order to formalise these two ideas, the first part is dedicated to give the formal framework of this problem. Then the ensuing algorithms are explained and finally we give some numerical results on an academic example.

1. Formalism

To get the likelihood of wind proposals with respect to air-traffic radar observations, a mathematical modelization has to be done. We choose to modelize aircraft trajectories as stochastic processes evolving in a random meteorological environment. Before going deeper into the mathematical formalism, we adopt the following notations. For a probability measure μ and a measurable function f, $\mu(f)$ is the expectation of the function f for the measure μ. For a probability operator $Q(x,dy)$ giving the probability to arrive in the element dy starting from x, $\mu Q(dy) = \int \mu(dx)Q(x,dy)$ is the probabiliy of the event dy for the operator Q averaged by the measure μ. Finally $\mu Q(f) = \int \mu(dx)Q(x,dy)f(y)$ is the expectation of the function f for the operator Q through the measure μ.

1.1. Learning the Trajectory Processes in a Random Environment

Before considering aircraft trajectories, we decompose the real wind at time n, W_n^r into two parts, the forecasted part W_n^f and the forecasting error part X_n^1. We consider $X_n^1 \in E_n^{(1)}$ as a random environment where the aircraft trajectories live. This process is a Markov chain of transition kernel $M_n^{(1)}$ from $\left(E_{n-1}^{(1)}, \varepsilon_{n-1}^{(1)}\right)$ to $\left(E_n^{(1)}, \varepsilon_n^{(1)}\right)$ and initial distribution $\eta_0^{(1)}$. The aircraft positions at time n are denoted by $X_n^2 \in E_n^{(2)}$. This is also a Markov process of transition kernel $M^{(2)}_{x_n^{(1)},n,\theta}$ from $\left(E_{n-1}^{(2)}, \varepsilon_{n-1}^{(2)}\right)$ to $\left(E_n^{(2)}, \varepsilon_n^{(2)}\right)$ which depends on the parameter $\theta \in (S, \mathscr{S})$[1] and initial distribution $\eta^{(2)}_{x_0^{(1)},0,\theta}$. We consider that the fixed

[1] θ can indicate more than one parameter which are grouped under this unique notation such as true airspeed, mass, throttle

parameter θ is a realization of the random variable Θ whose density is λ. In this work θ denotes the true airspeed (TAS) of the aircraft. The transition kernel $M^{(2)}_{x_n^{(1)},n,\theta}$ uses the forecasted wind W_n^f and depends on the environment realization $x^{(1)}$. Finally we have air-traffic radar observations at time n denoted by $Y_n \in F_n$ which are observations of X^2.

Considering the family $\left((X_n^1, X_n^2), Y_n\right)_{n\geq 0} \in \Omega = \prod_{n\geq 0} (E_n \times F_n) = \prod_{n\geq 0} E_n^{(1)} \times E_n^{(2)} \times F_n$, for each θ we associate a signal/observation model. We endowed Ω with the σ-algebra $\mathscr{G} = (\mathscr{G}_n)_{n\geq 0}$ and the distribution of probability $\mathbb{P}$. Let $\overline{\mathbb{P}}$ the distribution on $(\Omega, \mathscr{H})$ defined by its restrictions $\overline{\mathbb{P}}_n$ to $\Omega_n = \prod_{p=0}^{n} E_p^{(1)} \times E_p^{(2)} \times F_p$. The stochastic process $\left\{X_n^1, X_n^2, Y_n\right\}_{n\geq 0}$ is a Markov chain under $\overline{\mathbb{P}}_n$ with respect to its natural filtration $\mathscr{H}_n$ and with transition kernel $T_{n,\theta}$ from $(\Omega_{n-1}, \mathscr{H}_{n-1})$ to $(\Omega_n, \mathscr{H}_n)$ defined by

$$
\begin{aligned}
T_{n,\theta}\left((x_{n-1}^{(1)}, x_{n-1}^{(2)}, y_{n-1}), d(x_n^{(1)}, x_n^{(2)}, y_n)\right) = {} & g_{n,\theta}(x_n^{(1)}, x_n^{(2)}, y_n) M_n^{(1)}\left(x_{n-1}^{(1)}, dx_n^{(1)}\right) \\
& M^{(2)}_{x_n^{(1)},n,\theta}\left(x_{n-1}^{(2)}, dx_n^{(2)}\right) q_n(dy_n)
\end{aligned}
\tag{1}
$$

where $g_{n,\theta}$ is a collection of bounded measurable functions from $\left(E_n^{(1)} \times E_n^{(2)} \times F_n\right)$ to $(0, \infty)$ and represents the likelihood functions. The initial distribution of this chain is defined by Eq. (2).

$$
\eta_{0,\theta}\left(d(x_0^{(1)}, x_0^{(2)}, y_0)\right) = \eta_0^{(1)}(dx_0^1)\eta^{(2)}_{x_0^{(1)},0,\theta}(dx_0^2) g_{0,\theta}(x_0^{(1)}, x_0^{(2)}, y_0) q_0(dy_0) \tag{2}
$$

where $\eta_0^{(1)} \in \mathscr{P}(E_n^{(1)})$ and $\eta^{(2)}_{x_0^{(1)},0,\theta} \in \mathscr{P}(E_n^{(2)})$.

Equation (1) means that the transition kernel of the chain (X_n^1, X_n^2, Y_n) from time step $n-1$ to time step n can be decomposed in three steps. The first one corresponds to the evolution of the environment with the kernel $M^{(1)}$, the second step consists in the evolution of the trajectory process X^2 depending on the current environment realization $x_n^{(1)}$ and the parameter θ. The third step is an updating step which is done using the likelihood function $g_{n,\theta}$.

We fix the sequence of observations such that $Y = y$, and we denote $g_{n,\theta}(x_n^{(1)}, x_n^{(2)}, y_n)$ by $G_{n,\theta}(x_n^{(1)}, x_n^{(2)})$.

Before considering the case where the environment is not known, we fix it such that $X_{[0,n]}^{(1)} = (x_0^{(1)}, \ldots, x_n^{(1)}) \in \prod_{p=0}^{n} E_p^{(1)}$ and we get :

$$
\begin{aligned}
& \mathbb{P}^{(2)}_{x_{[0,n]}^{(1)},n,\theta}\left(X_{[0,n]}^2 \in d(x_0^{(2)}, \ldots, x_n^{(2)}) \middle| Y_{[0,n-1]} = (y_0, \ldots, y_{n-1})\right) = \frac{1}{\mathscr{Z}^{\theta}_{x_{[0,n]}^{(1)},n}} \\
& \left\{\prod_{p=0}^{n-1} G_{p,\theta,x_p^{(1)}}(x_p^{(2)})\right\} \eta^{(2)}_{x_0^{(1)},0,\theta}\left(dx_0^{(2)}\right) M^{(2)}_{x_1^{(1)},1,\theta}\left(x_0^{(2)}, dx_1^{(2)}\right) \ldots M^{(2)}_{x_n^{(1)},n,\theta}\left(x_{n-1}^{(2)}, dx_n^{(2)}\right)
\end{aligned}
\tag{3}
$$

with normalizing constant :

$$\mathscr{Z}^{\theta}_{x^{(1)}_{[0,n]},n} = \mathbb{E}_{x^{(1)}_{[0,n]},\theta}\left(\prod_{p=0}^{n-1} G_{p,\theta,x^{(1)}_p}\left(X^2_p\right)\right) > 0, \tag{4}$$

and random potential functions :

$$G_{p,\theta,x^{(1)}_p} : x^{(2)}_p \in E^{(2)}_p \mapsto G_{p,\theta,x^{(1)}_p}(x^{(2)}_p) = G_{p,\theta}(x^{(1)}_p, x^{(2)}_p). \tag{5}$$

The quantity defined by Eq. (3) is also denoted by $\mathbb{Q}_{x^{(1)}_{[0,n]},n,\theta}$ and called the quenched Feynman-Kac path measure. This formula represents the distribution of the path sequence $(X^2_0,\dots,X^2_n)$ of the Markov chain X^2 where the potential function $G_{p,\theta,x^{(1)}_p}$ represents the likelihood functions.
We can associate to this measure the quenched distribution flow which corresponds to the terminal time marginals of these path distribution denoted by $\eta^{(2)}_{x^{(1)},n,\theta}$ and defined by Eq. (6) for any measurable function f_n of $E^{(2)}_n$.

$$\eta^{(2)}_{x^{(1)}_{[0,n]},n,\theta}(f_n) = \mathbb{P}(X^2_n | Y_{[0,n-1]} = (y_0,\dots,y_{n-1}), X^1_{[0,n]} = (x^{(1)}_0,\dots,x^{(1)}_n), \Theta = \theta) \tag{6}$$

The updated version of this distribution is then defined by Eq. (7)

$$\hat{\eta}^{(2)}_{x^1_{[0,n]},n,\theta} = \mathbb{P}(X^2_n | Y_{[0,n]} = (y_0,\dots,y_n), X^1_{[0,n]} = (x^{(1)}_0,\dots,x^{(1)}_n), \Theta = \theta) \tag{7}$$

As it was proved in [1], the quenched Feynman-Kac distribution flow satisfies the non-linear equation (8)

$$\begin{aligned}\eta^{(2)}_{x^{(1)}_{[0,n]},n,\theta} &= \phi^{(2)}_{n,\theta,x^{(1)}_{[0,n]}}\left(\eta^{(2)}_{x^{(1)}_{[0,n-1]},n-1,\theta}\right) \\ &= \psi_{n-1,x^{(1)}_{[0,n-1]},\theta}(\eta^{(2)}_{x^{(1)}_{[0,n-1]},n-1,\theta}) M^{(2)}_{x^1_n,n,\theta}\end{aligned} \tag{8}$$

where $\psi_{n-1,x^{(1)}_{[0,n-1]},\theta}$ is defined for any measurable function f_{n-1} of $E^{(2)}_{n-1}$ by

$$\psi_{n-1,x^{(1)}_{[0,n-1]},\theta}(\eta^{(2)}_{x^{(1)}_{[0,n-1]},n-1,\theta})(f) = \frac{1}{\eta^{(2)}_{x^{(1)}_{[0,n-1]},n-1,\theta}(G_{x^{(1)}_{n-1},n-1,\theta})}\eta^{(2)}_{x^{(1)}_{[0,n-1]},n-1,\theta}(G_{x^{(1)}_{n-1},n-1,\theta} f)$$

Equation (8) means that our terminal time marginal distribution evolves into two steps, one updating step using the function $\psi_{n-1,x^{(1)}_{[0,n-1]},\theta}$ and one prediction step done with the kernel $M^{(2)}_{x^{(1)}_n,n,\theta}$. The problem which arises here is that the random environment where the stochastic process evolves is not known. Therefore quenched Feynman-Kac measure are not sufficient to describe the problem. So another Feynman-Kac measure has to be used. This time it is in distribution space, which allow us to consider the environment as a random variable. To do so, let $X'_n = (X^1_n, \eta^{(2)}_{X^1_{[0,n]},n,\theta}) \in E'_n = E^{(1)}_n \times \mathscr{P}(E^{(2)}_n)$,

X_n' denoting the couple random environment and distribution of the aircraft positions with respect to the random variable X^1. As it was proved in [1], X_n' is a Markov chain under $\mathbb{P}_{\eta_0}$ with transition kernel $M_{n,\theta}'$ defined for any measurable function f_n' of E_n' and $(x_n^1, \eta^{(2)}_{x^1_{[0,n]},n,\theta}) \in E_n'$ by (9).

$$M'_{n,\theta}\left((x^1_{n-1}, \eta^{(2)}_{x^1_{[0,n-1]},n-1,\theta}), d(x^{(1)}_n, \eta^{(2)}_{x^{(1)}_{[0,n]},n,\theta})\right)(f_n') =$$

$$\int_{E_n^{(1)}} M_n^{(1)}(x^{(1)}_{n-1}, dx^{(1)}_n) f_n'(x^{(1)}_n, \tilde{\phi}^{(2)}_{n,\theta,x^{(1)}_{[0,n-1]}}(x^{(1)}_n, \eta^{(2)}_{x^{(1)}_{[0,n-1]},n-1,\theta})) \tag{9}$$

and initial distribution $\eta'_{0,\theta} \in \mathscr{P}(E_0')$ defined by $\eta'_{0,\theta}(d(x,\nu)) = \eta_0^{(1)}(dx)\delta_{\eta^{(2)}_{x_0^{(1)},0,\theta}}(d\nu)$ where the application $\tilde{\phi}^{(2)}_{n,\theta,x^{(1)}_{[0,n-1]}}$ is defined by Eq.(10).

$$\tilde{\phi}^{(2)}_{n,\theta,x^{(1)}_{[0,n-1]}}(x^{(1)}_n, \eta^{(2)}_{x^{(1)}_{[0,n-1]},n-1,\theta})(dx^{(2)}_n) = \phi^{(2)}_{n,\theta,x^{(1)}_{[0,n]}}(\eta^{(2)}_{x^{(1)}_{[0,n-1]},n-1,\theta})(dx^{(2)}_n) \tag{10}$$

The transition kernel $M'_{n,\theta}$ can be interpreted as the integration of the non-linear equation (8), which modelizes the evolution of the aircraft positions, over the current environment. The most important point to keep in mind to distinguish$\phi^{(2)}$ from $\tilde{\phi}^{(2)}$ is that in the quenched framework we know the environment and its evolution in time whereas in this distribution space X^1 is a random variable.

Now we define the terminal time marginal distributions $\eta'_{n,\theta}$ and $\hat{\eta}'_{n,\theta}$ for any measurable function f_n' of E_n' by Eqs. (11) and (15) respectively.

$$\eta'_{n,\theta}(f_n') = \mathbb{P}(X_n^1, \eta^{(2)}_{X^1_{[0,n]},n,\theta} | Y_{[0,n-1]} = (y_0, \ldots, y_{n-1}), \Theta = \theta) \tag{11}$$

As it was proved in [1], η' is also verifying the non-linear equation (12).

$$\begin{aligned} \eta_n' &= \phi'_{n,\theta}(\eta'_{n-1,\theta}) \\ &= \psi'_{n-1,\theta}(\eta'_{n-1,\theta})M'_{n,\theta} \end{aligned} \tag{12}$$

where $\psi'_{n-1,\theta}$ is defined for any measurable function f_n' of E_n' by Eq. (13).

$$\psi'_{n-1,\theta}(\eta'_{n-1,\theta})(f_n') = \eta'_{n-1,\theta}(G'_{n-1,\theta}f_n')/\eta'_{n-1,\theta}(G'_{n-1,\theta}) \tag{13}$$

with the potential functions $G'_{p,\theta}$ defined by Eq. (14),

$$G'_{p,\theta} : (x,\mu) \in E_n' \mapsto G'_{p,\theta}(x,\mu) = \int_{E_n^{(2)}} \mu(dy) G_{p,\theta}(x,y) = \mu(G_{n,\theta}(x,.)), \tag{14}$$

These functions represent the probability that the observation of y_n is made given that $Y_{[0,n-1]} = (y_0, \ldots, y_{n-1})$, $X_n^1 = x_n^{(1)}$ and $\Theta = \theta$. The updated distribution $\hat{\eta}'_{n,\theta}$, is defined by Eq. (15).

$$\hat{\eta}'_{n,\theta} = \mathbb{P}(X^1_n, \eta^{(2)}_{X^1_{[0,n]},n,\theta} | Y_{[0,n]} = (y_0, \dots, y_n), \Theta = \theta) \tag{15}$$

By the multiplicative structure of these measures, it is easy to sequentially simulate them using particle approximation techniques. Section 2.1 is dedicated to the island filtering method.

1.2. Bayesian Inference in a General State Space Model

Another part of the problem which has to be tackle concerns the estimation of the fixed parameter θ whose the aircraft positions X^2 depend. We know the a priori law λ, and we want to estimate the a posteriori law $\mu_n(d\theta)$ given by Eq. (16)

$$\begin{aligned} \mu_n(\Theta \in d\theta) &= \mathbb{P}(\Theta \in d\theta | Y_{[0,n]} = (y_0, \dots, y_n)) \\ &= \frac{\mathbb{P}(Y_{[0,n]} \in d(y_0, \dots, y_n) | \Theta = \theta)\mathbb{P}(\Theta \in d\theta)}{\mathbb{P}(Y_{[0,n]} \in d(y_0, \dots, y_n))} \end{aligned} \tag{16}$$

where,

$$\mathbb{P}(Y_{[0,n]} \in d(y_0, \dots, y_n) | \Theta = \theta) = \prod_{p=0}^{n} \underbrace{\mathbb{P}(Y_p \in dy_p | Y_{[0,p-1]} = (y_0, \dots, y_p), \Theta = \theta)}_{h_p(\theta)}.$$

From the section 1.1, we have that the potential functions $G'_{p,\theta}$ represent the likelihood function and that the distribution $\eta'_{\theta,n}$ is the terminal time marginal. Therefore, Eq. (16) can be written as follow :

$$\mu_n(\theta) = \frac{1}{\hat{\mathscr{Z}}_n} \prod_{p=0}^{n} \underbrace{\eta'_{\theta,p}(G'_{p,\theta})}_{h_p(\theta)} \lambda(\theta)$$

where the quantity $\hat{\mathscr{Z}}_n = \int_S \hat{\mathscr{Z}}_n'^{\theta} \lambda(\theta) d\theta$, with $\mathscr{Z}_n'^{\theta} = \mathbb{E}\left[\prod_{p=0}^{n} G'_{p,\theta}(X'_p)\right]$.

The $h_p(\theta)$ functions are not known. The solution proposed by [2] is to use their particle approximations which are unbiased. This is the object of section 2.2.

2. Particle Approximations

The posterior densities $\eta'_{\theta,p}$ need to be approximated. In the case of unknown parameter the sequence of marginal likelihoods for a given θ : $\eta'_{\theta,p}(G'_{p,\theta})$, need also to be estimated. We propose here to use particle approximation. Therefore we present two algorithms, one in order to get the particle approximation of $\eta'_{\theta,p}$ which is called island particle algorithm (IPF or IKF) and another one to get the estimation of the parameter θ called particle Markov chain Monte Carlo method (pMCMC).

2.1. Island Particle Algorithm and Interacting Kalman filters

The quantity $\eta'_{\theta,p}$, corresponds to the time marginal density of the couple $(X^1_n, \eta^{(2)}_{X^1_{[0,n]},n,\theta})$. To get its estimation we use an N weighted random sample, called particles. In our case, particles are composed of two elements. One is a realization of the random environment; the other one is the density of the position process attached to this environment. As basic particle algorithms, it consists into three steps : an initialization according to the initial distribution $\eta'_{\theta,0}$, a prediction step using $M'_{n,\theta}$, and a resampling step defined by $\psi'_{n,\theta}$.

In the first experiment we lead, we consider that aircrafts fly in straight lines which means that $M^{(2)}_{x^{(1)}_n,n,\theta}$ is linear. Moreover we suppose that observations were obtained with an additional Gaussian noise. That means that once the environment is known the position process can be estimated using Kalman filter which is optimal. Considering now we have several realization of the random environment, we can use Interacting Kalman filters (IKF) technique.That is using different environment realizations, we will have different Kalman filters, which enter into competition thanks to a selection stage. The first step of IKF consists in initializing all the particles such that one Kalman Filter is associated to one environment. Then comes one step of correction for any Kalman filters, then they are selcted according to their difference to the observations. Then the environment and the Kalman filters evolve with the kernel $M^{(1)}$ and $M^{(2)}_{\zeta^i_p,p,\theta}$ respectively. See Algorithm 1 for more details.

In the nonlinear case, to get the associated estimated distribution $\eta^{(2)}_{\zeta^i_n,n,\theta}$ to each realization of the environment a set of particles is generated. Therefore all the environment realizations are endowed with a set of particles called an island. All these islands are entering into competition in a selection step. The Island particle (IPF) algorithm is detailed in Algorithm 2. As one can see, it is composed of five steps : one initialization, two steps of mutation and two steps of selection. After initializing the particles, there is a first step of selection concerning the island level. This selection is made with respect to the probability proportional to the empirical mean of the individuals in each island. Then a particle selection is made with respect to their own likelihood within the island. Then the environment evolves and then the particles within the island.

2.2. Particle Markov Chain Monte Carlo Methods

When θ is unknown, we have to estimate the joint distribution of θ and $X'_{[0,n]}$ conditionnaly on the sequence of observations $y_{[0,n]}$. The main idea of pMCMC is to use particle approximation within the framework of Markov chain Monte-Carlo method (MCMC) to obtain the estimate.

Indeed MCMC methods such as Metropolis-Hasting (M-H) method are sampling methods based on the construction of a Markov chain which has for invariant measure the measure to sample, say μ. In our case the measure to sample is defined by Eq. (16), the problem is that we do not know the h_p functions. [2] suggests to use a particle approximation of them at each step of the Metropolis-Hasting algorithm.

Starting with any configuration $\theta(0)$, basic M-H algorithm consists in an iterating two steps scheme. The first one proposes a random perturbation of the current state $\theta(i)$ to get a new configuration θ^*, this perturbation is made with a symmetric probability transition kernel q. The second consists in generating a random number $U \sim \mathscr{U}_{[0,1]}$ and

Algorithm 1 Interacting Kalman Filter - IKF

Require: η'_0, M' et ψ'

Ensure: Particle approximation of $p(x'_n|y_{1:n},\theta)$ and $p(x'_n|y_{1:n-1},\theta)$

$\boxed{\text{Begin}}$

1. <u>INITIALIZATION</u> $p = 0$

for $i = 1,\dots,N_2$ **do**

Sample $\varepsilon^i = (\zeta_0^i, \nu^i_{\zeta_0^i,0,\theta}) \sim \eta'_0$,

$\zeta_0^i \overset{i.i.d}{\sim} \eta_0^{(1)}$, and $\nu^i_{\zeta_0^i,0,\theta} = \mathcal{N}(m_{\zeta_0^i}, \sigma_{\zeta_0^i})$

end for

$p = 1$

2. <u>SELECTION OF ISLANDS</u>

Sample $I_p = (I_p^i)_{i=1}^{N_2}$ multinomially with proba $\propto (\mathcal{N}(m_{\zeta_p^i}, \sigma_{\zeta_p^i})(G_{p,\zeta_p^i}))_{i=1}^{N_2}$

for $i = 1,\dots,N_2$ **do**

3. <u>CORRECTION OF KALMAN FILTERS</u>

Use a classical correction step of Kalman filter : $\nu^i = \mathcal{N}(m_{\zeta_p^i}, \sigma_{\zeta_p^i}) \longrightarrow \hat{\nu}^i = \mathcal{N}(\hat{m}_{\zeta_p^i}, \hat{\sigma}_{\zeta_p^i})$

4. <u>MUTATION OF ISLANDS</u>

Sample independantly ζ_{p+1}^i according to $M^{(1)}(\zeta_p^{I_p^i}, .)$

5. <u>MUTATION OF KALMAN FILTERS</u>

Sample ν_{p+1}^i according to $M^{(2)}_{p,\zeta_{p+1}^i,\theta}(\mathcal{N}(\hat{m}_{\zeta_p^{I_p^i}}, \hat{\sigma}_{\zeta_p^{I_p^i}}), .)$

end for

$p \longleftarrow p+1$ go to step 2.

$\boxed{\text{End}}$

letting $\theta(i+1) = \theta(i)$ if the ratio $\mu(\theta^*)/\mu(\theta)$ is greater than U and letting $\theta(i+1) = \theta(i)$ otherwise.

pMCMC keeps these two steps and within the evaluation of the ratio, it adds another approximation level. Indeed to get the estimation of the likelihood functions $\eta'_{\theta,p}(G'_{p,\theta})$ it uses particle estimation. pMCMC method is detailed in the Algorithm 3.

3. Results

In our different numerical experiments, we only consider simple air traffic: three aircrafts have to go from one waypoint to another one following a straight line, with constant altitude and airspeed (2 with a TAS of 740km/h *i.e.* about 400kt, 1 with a TAS of 700km/h *i.e.* about 377kt).

Concerning the random environment, we consider that the wind error is permanent and non-uniform. The domain can be decomposed into two areas such that the unknown wind error is constant in direction and strength over each area. The problem is that these permanent uniform areas are delimited by an unknown vertical border. Therefore, the forecasting wind error in both areas in terms of strength and direction, and the location of the border have to be estimated. Moreover the TAS parameter has to be approximated.

Algorithm 2 Island Particle Filter - IPF

Require: η'_0, M' et ψ'

Ensure: Particle approximation of $p(x'_n|y_{1:n},\theta)$ and $p(x'_n|y_{1:n-1},\theta)$

$\boxed{\text{Begin}}$

1. <u>INITIALIZATION</u> $p=0$

for $i=1,\dots,N_2$ **do**

Sample $\varepsilon^i=(\zeta_0^i,\nu^i_{\zeta_0^i,0,\theta})\sim\eta'_0$,

$\zeta_0^i \overset{i.i.d}{\sim} \eta_0^{(1)}$, and $\nu^i_{\zeta_0^i,0,\theta}=\frac{1}{N_1}\sum_{j=1}^{N_1}\xi_0^{i,j}$ where $\xi_0^{i,j} \overset{i.i.d}{\sim} \eta^{(2)}_{\zeta_0^i,0,\theta}$

end for

$p=1$

2. <u>SELECTION OF ISLANDS</u>

Sample $I_p=(I_p^i)_{i=1}^{N_2}$ multinomially with proba $\propto\left(\frac{1}{N_1}\sum_{j=1}^{N_1}G_p(\zeta_p^i,\xi_p^{i,j})\right)_{i=1}^{N_2}$

for $i=1,\dots,N_2$ **do**

3. <u>SELECTION OF PARTICLES INSIDE EACH ISLAND</u>

Sample $J_k^i=(J_k^{i,j})_{j=1}^{N_1}$ multinomially with proba $\propto\left(G_p(\zeta^{I_k^i},\xi_k^{I_k^i,j})\right)_{j=1}^{N_1}$

4. <u>MUTATION OF ISLAND</u>

Sample independantly ζ_{p+1}^i according to $M^{(1)}(\zeta_p^{I_p^i},.)$

for $j=1,\dots,N_1$ **do**

5. <u>MUTATION OF PARTICLES</u>

Sample $\xi_{p+1}^{i,j}$ according to $M^{(2)}_{p,\zeta_{p+1}^i,\theta}(\xi^{I_p^i,J_p^i},.)$

end for

end for

$p\longleftarrow p+1$ go to step 2.

$\boxed{\text{End}}$

The aircraft configuration has been choosen such that only one aircraft experiments the unknown limit.

Then, the observation process is gotten from radar observations. As they can have different mode, results obtained with mode-C radar observations (aircraft positions only) and those obtained with mode-S radar observations (aircraft positions and deduced wind) will be compared. In this work, the observations are obtained from a toy model perturbed with a $\mathcal{N}(0,0.1)$ on each aircraft position and in presence of mode-S radar observations with a $\mathcal{N}(0,0.1)$ on each aircraft position and for the deduced wind with a $\mathcal{N}(0,\sqrt{5})$. The period of sampling observations is 15 seconds.

Consequently, all the ingredients needed to perform and compare the results obtained with IPF or IKF method within pMCMC algorithm are available. In this example the experiment simulates 42 minutes of air-traffic and the parameter estimation is refreshed every six minutes. A Gaussian kernel with standard deviation of 0.01 has been used to perform the pMCMC algorithm, the initialization of the proposal for the TAS has been made with an independant Gaussian perturbation of the true value for each aircraft of standard deviation 10. The number of iterations of the M-H algorithm has been

Algorithm 3 Particle Metropolis Hasting algorithm

Require: η_0', $\theta(0)$, q

Ensure: estimation of $p(\Theta, x_{0:T}'|y_{1:T})$ by particle method

$\boxed{\text{Begin}}$

1. Let $\theta(0)$, filter by particle method (IKF or IPF) $p_{\theta(0)}(x_{1:T}'|y_{1:T})$, sample $X_{1:T}'(0) \sim p_{\theta(0)}(.|y_{1:T})$ and compute $\widehat{p}_{\theta(0)}(y_{1:T})$

2. Iterate $i \leq 1$

$\theta^* \sim q(.|\theta(i-1))$

Filter by particle method $\widehat{p}_{\theta^*}(x_{1:T}'|y_{1:T})$, sample $X_{1:T}'^{*} \sim \widehat{p}_{\theta^*}(.|y_{1:T})$ and compute $\widehat{p}_{\theta^*}(y_{1:T})$

With the probability

$$1 \wedge \frac{\widehat{p}_{\theta^*}(y_{1:T})p(\theta^*)q(\theta(i-1)|\theta^*)}{\widehat{p}_{\theta(i-1)}(y_{1:T})p(\theta(i-1))q(\theta^*|\theta(i-1))}$$

we have $\theta(i) = \theta^*, X_{1:T}'(i) = X_{1:T}'^{*}, \widehat{p}_{\theta(i)}(y_{1:T}) = \widehat{p}_{\theta^*}(y_{1:T})$ or $\theta(i) = \theta(i-1), X_{1:T}'(i) = X_{1:T}(i-1)$, $\widehat{p}_{\theta(i)}(y_{1:T}) = \widehat{p}_{\theta(i-1)}(y_{1:T})$

$\boxed{\text{End}}$

Table 1. RMSE integrated over aircraft trajectories between estimated aircraft positions and true positions for IKF and IPF method

	IKF	IPF
mode-C	3.33	4.09
mode-S	5.53	6.07

fixed to 10. Concerning IPF or IKF, in each area, 5 different proposals for wind force normally distributed around the true value, 5 for directions and 5 for border location uniformly distributed around the true value are considered. Random environment proposals are obtained with all possible combinations of those three ingredients, *i.e* 3125 particles.

The first numerical result, suggest that IKF are better in terms of root mean square error (RMSE) integrated over the whole trajectory, see Table 1. As we have already mentionned, Kalman filter is optimal in linear/Gaussian system. But as it may be noticed on Figure 1, the weight evolution of wind proposal for the area 1 do not suggest that there is a weight concentration on one of the direction for the IKF method. Whereas for the IPF method, there is a darker area formed around one wind proposal which corresponds to the real one. Indeed Kalman Filters anihilate the effect of the wind on trajectory errors by the correction step. Concerning the wind force, see Figure 2, one can see that the relative error of the estimated wind force to the real one is of order 0.02 and not sensible to the method used. Indeed this error is almost the same whether IPF method has been used or IKF method. Therefore, as the algorithm aimed to learn the likelihood of wind proposals and that IPF algorithm gives better results on wind direction we then consider only results obtained with IPF algorithm.

As regards the limit, Figure 3 represents the likelihood evolution of the vertical limit proposals over time. These proposals are sorted in ascending order on the x-coordinate. First all the limit proposals are equivalent as no aircrafts experiment the limit yet. Remind

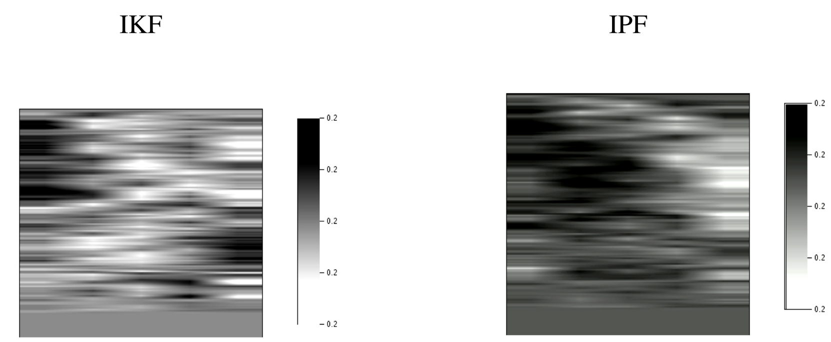

Figure 1. Likelihood evolution (in gray scale) over time (y-axis from bottm to top) of direction proposals (x-axis) obtained with IKF or IPF for one uniform area

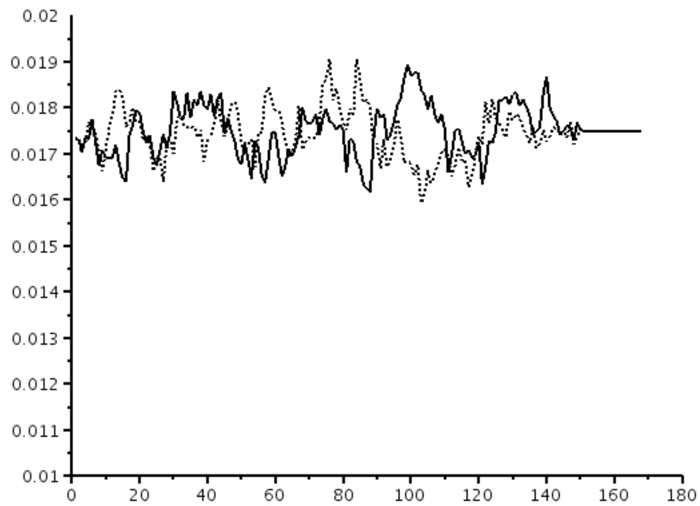

Figure 2. Evolution in time (x-axis) of the relative error of estimate wind force for one area with IKF (solid line) of IPF (dotted line) method to the real wind force

that only one aircraft experiments the limit (from right to left). Then, only the most on the right proposed limits have a low likelihood, as the aircraft which experiment it does not go throught the limit yet. Then more the aircraft goes on the left more the high likelihood limit proposal are concentrated on the left. The difference between mode-C and mode-S observations can be noticed on Figure 3. Indeed, the difference between high and low likelihood limit proposal is sharper when mode-S radar is used.

Regarding the parameter estimation, pMCMC method gives a relative error for the estimated true airspeed of standard deviation 2 whereas the first proposition was of standard deviation 10. Nevertheless, as one can observe on Figure 4, the TAS estimation is

Mode-C Mode-S

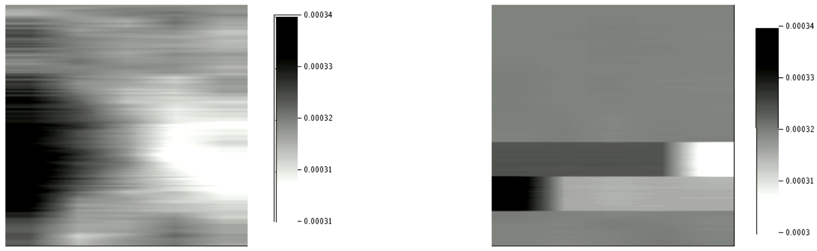

Figure 3. Time evolution (y-axis from bottom to top) of weight (grayscale) of the different limit proposals (x-axis)

not enhanced by the additional wind information. In this experiment, it should be preferable to use mode-S radar observations as they give better results.

mode-C mode-S

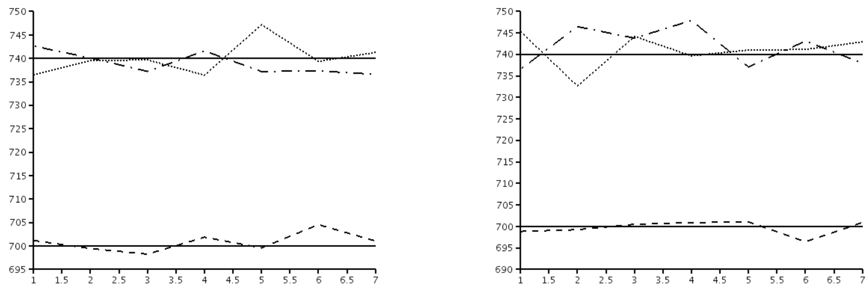

Figure 4. Estimated aircraft true airspeed by pMCMC method over the seven recycling forecast

Conclusion

The methodology developped in this work allows us to give a weight to each element set of the ensemble weather forecast regarding the traffic-observations. That is we can infer the random environment learning the likelihood of wind proposals while learning some flight parameters such as true airspeed.

References

[1] P. Del Moral, Feynman-Kac formulae: genealogical and interacting particle systems with applications. *Springer*, 2004.

[2] Andrieu, C., Doucet, A. and Holenstein, R. (2010), Particle Markov chain Monte Carlo methods. Journal of the Royal Statistical Society: Series B (Statistical Methodology), 72: 269-342. doi: 10.1111/j.1467-9868.2009.00736.x