Trisections of 4-manifolds

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Contents

Introduction and acknowledgments

Gay and Kirby introduced in their 2012 seminal paper [10] the notion of trisection of a 4-manifold, totally analogous to the 3-dimensional concept of Heegaard splittings. Their work relied on the theory of Morse 2-functions to prove both a result of existence and of uniqueness, the latter having a truly equivalent statement as the Reidemeister-Singer theorem.

The theory of Morse 2-functions was somewhat necessary, because of the lack of combinatorial results regarding triangulations of smooth manifolds in dimension 4 (mainly the *Hauptvermutung*). Moreover, the distinction between topological and smooth structures means that some results concerning trisections, such as the Alexander lemma, or the conjectured Waldhausen–Haken theorem, must be approached differently.

In the present work, we may detail the work of Gay and Kirby, as well as the combinatorial representation of a trisection that is a trisection diagram.

Along the road, we shall give some examples and results regarding trisections with specific properties, and also explain how trisections can be used to compute topological invariants. Natural (still open) questions shall also be mentioned here and there.

I would like to express my deepest gratitude to M. Fiedler, who always proved to be generous and helpful, and who allowed me to present my work at this year's session of the Matemale.

I am also thankful to Mrs. Moussard for all the time she gave me, and who was of great advice and suggested I attend the Summer Trisectors pre-workshop.

My special thanks are extended to my friend, Pablo, who was very patient in listening to me while experiencing the difficulties of the work.

Finally, this work wouldn't have been possible without the support of Lucile.

All the pictures in this document have been made in the open-source vector graphics editor `Inkscape`, and exported from `svg` to `pdf` graphics.

1 Preliminaries and notations

Throughout this work, all manifolds will be smooth, connected, compact and oriented. Besides handlebodies, we will also assume that they have no boundary.

We will denote the connected-sum of boundaryless manifolds as $\#$, and the boundary connected-sum as $\natural$.

1.1 Handlebodies and handle decompositions

An **n-dimensional k-handle**, or an **(n, k)-handle** for short, is a $\mathbb{D}^k \times \mathbb{D}^{n-k}$, along with an attaching map (see below). If the dimension n is understood, we may simply speak about **k-handles**.

Given an n-manifold M with boundary $\partial M \neq \varnothing$, an **attaching map** for a(n n-dimensional) k-handle is an embedding $f : \mathbb{S}^{k-1} \times \mathbb{D}^{n-k} \to \partial M$. The process of attaching a k-handle to M is the construction of the new manifold N by

$$N = M \cup_f (\mathbb{D}^k \times \mathbb{D}^{n-k}),$$

where $\cup_f$ denotes the gluing of the two manifolds along f :

$$N = M \amalg (\mathbb{D}^k \times \mathbb{D}^{n-k})/\sim,$$

with $x \sim y \iff f(y) = x$.
Now, an **(n, k)-handlebody of genus** g is the manifold obtained after attaching g times a k-handle to $\mathbb{D}^n$. Examples include :

$$\mathcal{H}_g = \natural^g(\mathbb{S}^1 \times \mathbb{D}^2) \text{ the genus } g \ (3, 1)\text{-handlebody},$$

and

$$\mathcal{Z}_k = \natural^k(\mathbb{S}^1 \times \mathbb{D}^3) \text{ the genus } k \ (4, 1)\text{-handlebody}.$$

The genus g surface is $\Sigma_g = \partial\mathcal{H}_g$, and we note that $\partial\mathcal{Z}_k = \#^k(\mathbb{S}^1 \times \mathbb{S}^2)$.

Given an n-manifold, a **handle decomposition** for it is a decomposition

$$\mathbb{D}^n \amalg \cdots \amalg \mathbb{D}^n = M_0 \subset M_1 \subset \cdots \subset M_n = M,$$

where M_k is obtained from M_{k-1} by attaching k-handles. If M_k is obtained from M_{k-1} by attaching r_k handles, and if M_0 has r_0 connected components, we call such a decomposition an $(r_0 : r_1 : \ldots : r_n)$-handle decomposition.

Given a handle with attaching map $f : \mathbb{S}^{k-1} \times \mathbb{D}^{n-k} \to \partial M$, we call (framed) **attaching sphere** the submanifold $f(\mathbb{S}^{k-1} \times \{0\})$. In the particular case of attaching 2-handles to a 4-manifold, we obtain a framed link in the boundary by taking all the attaching maps.

A handle decomposition always provides us with a cellular decomposition, by deformation retracting all the $\mathbb{D}^k \times \mathbb{D}^{n-k}$ and $\mathbb{S}^{k-1} \times \mathbb{D}^{n-k}$ to $\mathbb{D}^k \times \{0\}$ and $\mathbb{S}^{k-1} \times \{0\}$ respectively – note that the number of k-cells equals the number of k-handles.
In particular, a manifold with a handle decomposition consisting of no 1-, 2-, ..., k-handles is k-connected.

1.2 Heegaard splittings

Given two solid *tori* $\mathbb{S}^1 \times \mathbb{D}^2$, there are several ways to glue them along their boundary. One could choose to glue them along the identity map, which would produce $\mathbb{S}^1 \times \mathbb{S}^2$, or one could instead use a map interverting the two copies of $\mathbb{S}^1$ in the boundary, resulting in $\mathbb{S}^3$. This leads us to define a special kind of decomposition :

Definition 1.1. *Given a 3-manifold M, a **Heegaard splitting** of it is a decomposition $M = V_1 \cup V_2$ and an integer g such that V_1 and V_2 are diffeomorphic to $\mathcal{H}_g$, and such that $V_1 \cap V_2 = \partial V_1 = \partial V_2$. The integer g is called the **genus** of the splitting, and the $H = V_1 \cap V_2$, diffeomorphic to Σ_g, is called the Heegaard surface.*

In dimension three, one can always choose a triangulation of any manifold (Moise, [27]). Doing so and thickening the 1-skeleton and the dual 1-skeleton of that triangulation – by means of a regular neighborhood of it – produces a 1-handlebody of some (potentially large) genus, thus a Heegaard splitting. ■

However, if we have in mind to generalize the construction to the dimension above, we need to come up with a way to prove the result without resorting to this combinatorial argument. By connectedness and smoothness of M, one can always find a Morse function $f : M \to \mathbb{R}$ with exactly one index zero and one index three critical points, and such that $f(x) < f(y)$ for critical points x and y of respective indexes $k < \ell$. Such a function therefore has g index one and g index two critical points, and provides us with a $1 : g : g : 1$-handle decomposition for M.
This handle decomposition is what we need : the 0- and 1-handles are one handlebody V_1, and the remaining handles are the other handlebody V_2.

In the beginning of this section, we introduced a genus one Heegaard splitting for both the 3-sphere and $\mathbb{S}^1 \times \mathbb{S}^2$. It turns out $\mathbb{S}^3$ has a genus zero splitting, by taking the northern and southern hemispheres decomposition, and it is the only such splitting :

Lemma 1.2. (Alexander) *Any 3-manifold having a genus zero Heegaard splitting is diffeomorphic to $\mathbb{S}^3$.*

In the proof (which boils down to the Alexander radial extension trick, and the fact that in dimension three, **Top** = **Diff**), we even have better : we map the splitting of the manifold to the genus zero splitting of $\mathbb{S}^3$. This leads us to defining an equivalence relation on Heegaard splittings :

Definition 1.3. *Let $M = V_1 \cup V_2 = W_1 \cup W_2$ be a manifold with two Heegaard splittings. The two splittings are said to be **equivalent** if there exists a diffeomorphism of M mapping V_i to W_i.*

Obviously, two splittings of different genera can never be equivalent. However, there is still a way to state a uniqueness theorem, by means of stabilization.

Given two Heegaard splittings $M = V_1 \cup V_2$ and $N = W_1 \cup W_2$, the manifold $M \# N$ has a Heegaard splitting given by $M \# N = (V_1 \natural W_1) \cup (V_2 \natural W_2)$. In particular, the Heegaard surface for $M \# N$ is the connected sum of those of M and N.
Now, $\mathbb{S}^3$ being neutral regarding the connected sum, this is a way to increment the genus of a Heegaard splitting by taking the connected-sum with its genus one splitting described above :

Definition 1.4. *Given a Heegaard splitting M, its* ***stabilization*** *is the splitting obtained after connected-summing with the genus one splitting of* $\mathbb{S}^3$.

Here, we are speaking about *the* splitting of genus one as if it had always been unique. It turns out it is the case, by the following theorem :

Theorem 1.5. (Waldhausen) *Every splitting of* $\mathbb{S}^3$ *is equivalent to some number of stabilizations of its genus zero splitting.*

Now that we have a way to increase the genus of any splitting, we can take two, stabilize them so they have the same genus, and ask again the question whether they are equivalent or not. They will not necessarily be equivalent, but the following tells us there is a way to make it true :

Theorem 1.6. (Reidemeister-Singer) *Any two Heegaard splittings of a* 3*-manifold can be made equivalent after a suitable number of stabilizations for each.*

Again, there is a purely combinatorial proof of this result, relying on the *Hauptvermutung*, which can be seen in details in [29]. However, for generalization sake, we shall continue with Morse functions. Take two splittings, and take Morse functions associated to them. Note that this is always possible, because a genus g splitting gives a $(1 : g : g : 1)$-handle decomposition, itself giving us that Morse function.
Now, by results from section 1.4, it is always possible to homotope the first function into the second, with only births and deaths of pairs of critical points, all births appearing before all deaths (see figure 1). Birth of pairs of critical points corresponding to stabilization of a splitting, the only thing left to do is to pick a Morse function after the last birth and before the first death, which is the stabilization for both we were looking for. ■

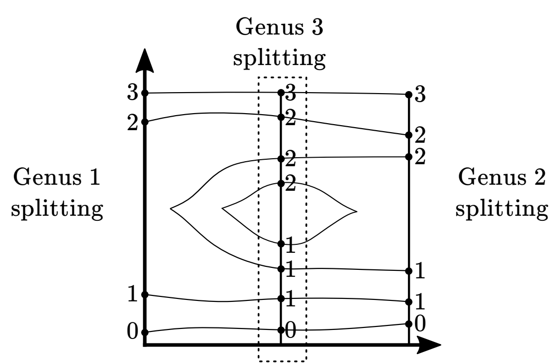


Figure 1. Homotoping one function to the other yields two equivalent stabilizations.

We will see that the same philosophy applies in proving the results for trisections. One last meaningful example is the one of the splitting of $\partial\mathcal{Z}_k$. We take $k \leqslant g$ two integers, and we take the k-fold connected-sum of the Heegaard splitting of genus one of $\mathbb{S}^1 \times \mathbb{S}^2$. This yields a genus k splitting of $\partial\mathcal{Z}_k$, which we can stabilize $g - k$ times to obtain what we will call the **standard genus** g **splitting** of it, written as :

$$\partial\mathcal{Z}_k = Y^+_{k,g} \cup Y^-_{k,g}.$$

It is called *the* standard splitting, because of the following :

Theorem 1.7. (Waldhausen–Haken) *The only Heegaard splittings of* $\#^k(\mathbb{S}^1\times\mathbb{S}^2)$ *are the stabilizations of its unique genus* k *splitting obtained by connected-summing* k *times the standard splitting of* $\mathbb{S}^1\times\mathbb{S}^2$*.*

1.3 *A Note on 4-dimensional handlebodies*

The paper [18] is about handlebodies in dimension four, and the attaching of 3- and 4-handles. The main result is that, up to isotopy, there is only one way to attach the 3- and the 4-handles to a 4-manifold (so long as the resulting 4-manifold is closed). Here are reformulations of this fact

Theorem 1.8. *Any diffeomorphism of* $\partial\mathcal{Z}_k = \#^k(\mathbb{S}^1\times\mathbb{S}^2)$ *extends to a diffeomorphism of the bounding* $\mathcal{Z}_k = \natural^k(\mathbb{S}^1\times\mathbb{D}^3)$*.*

Corollary 1.9. *Let* X *be the manifold obtained after attaching* k *3-handles to* $\natural^k(\mathbb{S}^2\times\mathbb{D}^2)$*. Assume that* $\partial X\cong\mathbb{S}^3$*. Consider a 4-handle attached through a diffeomorphism* $g:\partial\mathbb{D}^4\to\partial X$*. Then* $X\cup_g\mathbb{D}^4$ *is diffeomorphic to* $\mathbb{S}^4$*.*

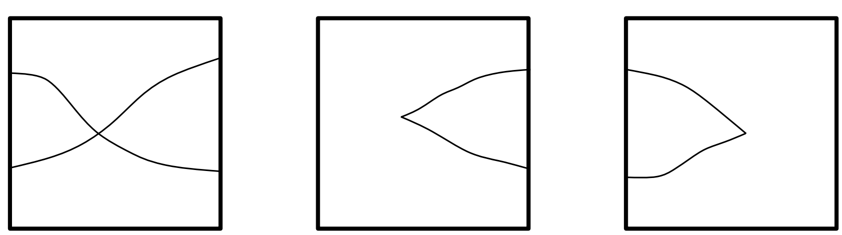
1.4 From Morse functions and Cerf theory to Morse 2-functions

In this section, we recall the definitions regarding Morse function, generic homotopies between them, Cerf theory, Morse 2-functions and generic homotopies between them. See [7] for a more in-depth look at the notions.

Definition 1.10. *A* ***generic homotopy*** *between two Morse functions* $f_0, f_1 : M\to\mathbb{R}$ *is a homotopy* $f_t : M\to\mathbb{R}$ *that is Morse at all but finitely many times* $t_*\in]0,1[$*. The homotopy is the function* $f:[0,1]\times M\to\mathbb{R}$*, and the* ***image of the fold locus*** *is the image under* h *of the singular locus of the functions* f_t*. We ask that at all times* t_* *where* f_{t_*} *is not Morse, exactly one of the following events occur :*

(i) Two critical values cross at t_**, that is, the image of the fold locus is embedded except for one double point (see figure 2a). We speak of a* ***crossing****.*

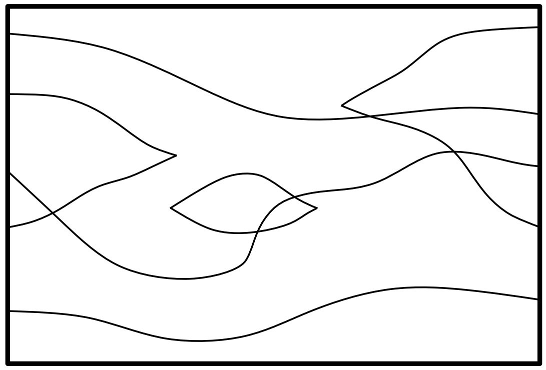
(ii) A pair of cancelling critical points of consecutive indices is born, that is, around t_**, there is a ball where* h_t *is Morse with no critical points in that ball for* $t<t_*$*, and with two critical points for* $t>t_*$ *(see figure 2b). We speak of a* ***birth****, and when that happens with the* t *parameter reversed, we speak of a* ***death*** *(see figure 2c).*



(a) (b) (c)

Figure 2. (a) A *crossing.* (b) A *birth.* (c) A *death.*

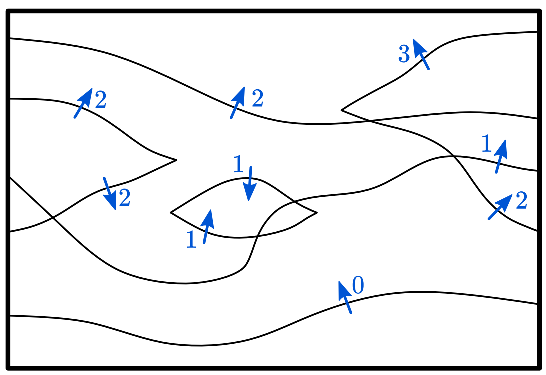
The three graphics in the previous figure are called *Cerf graphics.* Here is an example of a more general Cerf graphic :



It is a standard fact from Cerf theory that generic homotopies between Morse functions are indeed generic and stable. They allow to speak of Morse 2-functions :

Definition 1.11. *A* ***Morse 2-function*** *is a function $f : M \to \mathbb{R}^2$ that is locally at all points in M a generic homotopy between Morse functions.*

Given a Morse 2-function, it needs not make sense to speak about the index of a fold. However, choosing a transverse direction to the fold, we obtain locally a Morse function, whose index of a critical point is well-defined. For instance, for the previous example, here is a labelling of the indices of the folds :



Now, what are homotopies of Morse 2-functions ? We must talk about generic homotopies of generic homotopies of Morse functions :

Definition 1.12. *Given two generic homotopies of Morse functions $h_{0,t}, h_{1,t} : M \to \mathbb{R}$, a* ***generic homotopy*** *between them is a homotopy $h_{s,t} : M \to \mathbb{R}$ that is a generic homotopy between Morse functions at all but finitely-many times $s_* \in]0,1[$, where exactly one of the following events occurs :*

(i) $h_{s_*,t}$ is not a generic homotopy between Morse functions because exactly two of the events listed in definition 1.10 occur at the same time.

(ii) $h_{s_*,t}$ is not a generic homotopy between Morse functions because its singular locus has a non-transverse double point, see figure 3a. We call that event a ***Reidemeister II fold crossing****.*

(iii) $h_{s_*,t}$ is not a generic homotopy because its singular locus has a transverse triple point, see figure 3b. We call that event a ***Reidemeister III fold crossing****.*

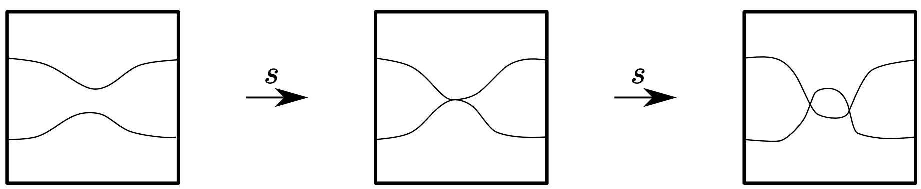
(iv) $h_{s_*,t}$ *is not a generic homotopy because its singular locus has a birth (or a death) occuring at the same location than a crossing, see figure 3c. We call that event a* ***cusp-fold crossing****.*

(v) $h_{s_*,t}$ *is not a generic homotopy because its singular locus contains an isolated point, which bounds in time a birth-death event with nothing, see figure 3d (or nothing with a birth-death with the s-time reversed). We call that event an* ***eye birth*** *(or* ***death****)* ***singularity****.*

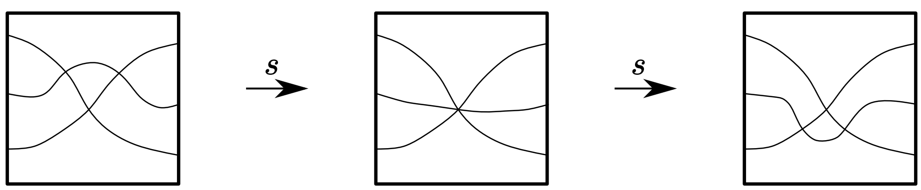
(vi) $h_{s_*,t}$ *is not a generic homotopy because a death and a birth merge at the same time, see figure 3e. We call that event a* ***merge singularity*** *(or* ***unmerge*** *when the s-time is reversed).*

(vii) $h_{s_*,t}$ *is not a generic homotopy because the fold locus has a cusp bounding in time a definite fold to a birth-crossing-death singularity (or a death-crossing-birth when the s-time is reversed), see figure 3f. We call that event a* ***swallowtail birth*** *(or* ***death****)* ***singularity****.*

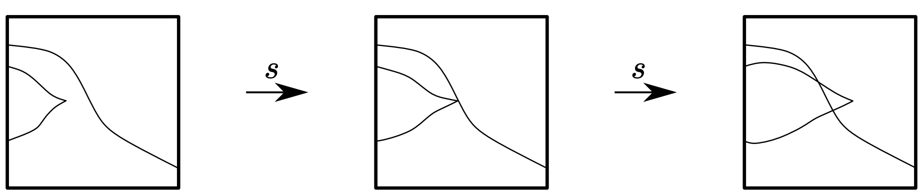
Here we list all the mentioned events, where the middle graphic is the time at which it it *not* a Cerf graphic (note that in all these events, the time can be reversed and the homotopy read "backwards") :



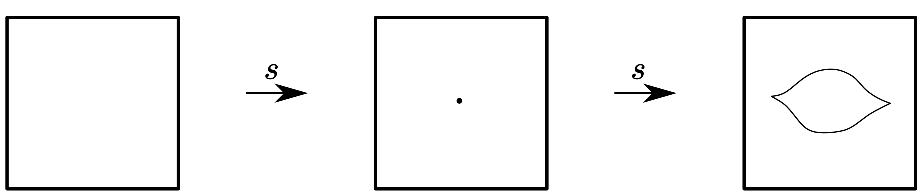
(a) A *Reidemeister II crossing.*



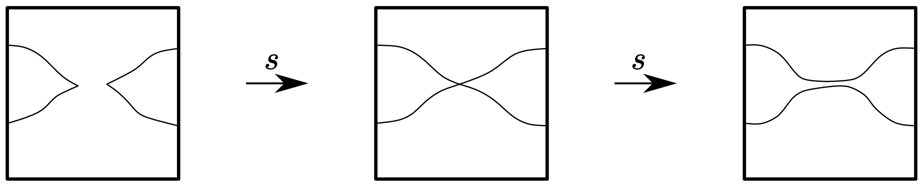
(b) A *Reidemeister III crossing.*



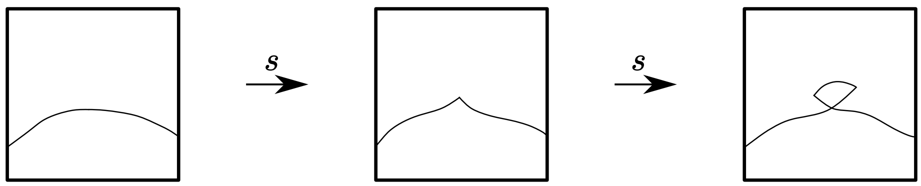
(c) A *cusp-fold crossing.* The other three mirrored versions also exist.



(d) An *eye birth singularity.*



(e) A *merge singularity.*



(f) A *swallowtail birth.* An upside-down version also exists.

Figure 3. All the possible events in a generic homotopy between generic homotopies of Morse functions.

This allows us to define :

Definition 1.13. *A **generic homotopy** between two Morse 2-functions* $f_0, f_1 : M \to \mathbb{R}^2$ *is a homotopy* $f_t : M \to \mathbb{R}^2$ *that is locally at all points in* M *a generic homotopy between generic homotopies of Morse functions.*

One could study the singular locus of a generic homotopy between Morse functions, which would give an immersed surface in $\mathbb{R}^3$, totally analogous to the Cerf graphics. However, working with these folds is another story than the mere 2D Cerf graphics we had so far, and no one had ever done this yet.

2 Trisections : building the theory

2.1 Definition and first properties

Definition 2.1. *A* ***trisection*** *of a 4-manifold M is a decomposition $M = X_1 \cup X_2 \cup X_3$ with X_i smoothly embedded, and two integers $k \leqslant g$, such that :*

(i) X_i *is diffeomorphic to* $\mathcal{Z}_k = \natural^k(\mathbb{S}^1 \times \mathbb{D}^2)$.

(ii) $H_{ij} = X_i \cap X_j$ *is diffeomorphic to* $\mathcal{H}_g = \natural^g(\mathbb{S}^1 \times \mathbb{D}^2)$.

(iii) $X_1 \cap X_2 \cap X_3$ *is diffeomorphic to* Σ_g.

The previous decomposition is called a (g, k)-***trisection****, and g is the* ***genus*** *of the trisection. The triple intersection $F = X_1 \cap X_2 \cap X_3$ is called the* ***trisecting surface****.*

For the sake of keeping things simple, we will only deal with *balanced* trisections (where we fix the genus of all three sectors X_i to be the same), although the more general theory isn't much more work.

Here is the typical way to represent a trisection, referred to as the "cartoon picture of a trisection" in [10] :

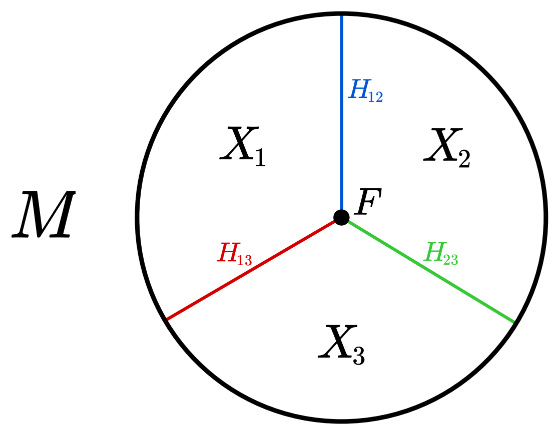


Figure 4. The schematic representation of a trisection.

In defining the genus of a trisection, why do we only care about g? In dimension three, Poincaré duality provides $\chi(M) = 0$, so we couldn't relate the genus of a Heegaard splitting to the Euler characteristic. However, in dimension four, it does carry information, which we can use by means of the inclusion-exclusion principle :

$$\chi(M) = 2 + g - 3k.$$

In particular, once we know the genus, k is fixed. Moreover, we will see later that the genus that matters is the one of the trisecting surface.

The 4-sphere has a genus zero trisection, totally analogous to the Heegaard splitting of $\mathbb{S}^3$. Embed $\mathbb{S}^4 \subset \mathbb{C} \times \mathbb{R}^3$, and define

$$X_i = \{(re^{i\theta}, \mathbf{x}) \in \mathbb{S}^4 \mid 2i\pi/3 \leqslant \theta \leqslant 2(i+1)\pi/3\}.$$

The definition is a translation of figure 4 in terms of subsets of $\mathbb{S}^4$, and this provides a genus zero trisection : $\mathbb{S}^4 = X_1 \cup X_2 \cup X_3$.

It turns out that lemma 1.2 still holds in this context :

Lemma 2.2. *If a 4-manifold has a genus zero trisection, then it is diffeomorphic to* $\mathbb{S}^4$.

Proof. This time, the Alexander radial extension trick will simply not work, because we want a diffeomorphism and not a mere homeomorphism (**Top** $\neq$ **Diff**). However, by corollary 1.9 with $k = 0$, we directly obtain the result. ∎

Pre-imaging three sectors in the image of a projection map is something quite common to define a trisection. The idea can be used to define a trisection of $\mathbb{S}^1 \times \mathbb{S}^3 \subset \mathbb{S}^1 \times \mathbb{C} \times \mathbb{R}^2$ in a similar fashion :

$$X_i := \{(s, (re^{i\theta}, \mathbf{x})) \in \mathbb{S}^1 \times \mathbb{S}^3 \mid 2i\pi/3 \leqslant \theta \leqslant 2(i+1)\pi/3\}.$$

This time, the trisection is of type $(1, 1)$, and by lemma 2.2, it is of minimal genus. This leads to the following definition of a topological invariant :

Definition 2.3. *Given a 4-manifold M, its* ***trisection genus****, denoted as $g_T(M)$, is the minimal genus of its trisections.*

We therefore have :

$$g_T(M) = 0 \iff M \cong \mathbb{S}^4 \qquad \text{and} \qquad g_T(\mathbb{S}^1 \times \mathbb{S}^3) = 1.$$

Now, from the very definition of the trisection, we see that $\partial X_i = H_{ij} \cup H_{ih}$ (with i, j and h distinct) is a Heegaard splitting. Its genus equals g, therefore it is the standard splitting, by theorem 1.7 :

$$\partial X_i = H_{ij} \cup H_{ih} = Y_{k,g}^+ \cup Y_{k,g}^-.$$

We will see later a proof of the following important fact :

Proposition 2.4. *If M has a (g, k)-trisection, then M has a $(1 : k : g - k : k : 1)$-handle decomposition.*

This has three immediate consequences :

(i) Since a handle decomposition deformation retracts to a cellular decomposition, we see that $\pi_1(M)$ has a presentation with k generators and $g - k$ relations. In particular, we see that $k \geqslant \operatorname{rk} \pi_1(M)$. Plugging this inside the Euler characteristic formula in terms of the type of the trisection, we obtain a lower bound on the trisection genus of a manifold :

$$g_T(M) \geqslant \chi(M) - 2 + 3 \operatorname{rk} \pi_1(M).$$

This estimate is sharp, in the sense that there is equality for lots of manifolds with prescribed fundamental group. See [4] for more details.

(ii) If M has a $(g, 0)$-trisection, then M is simply-connected.

(iii) Dually, if M has a (g, g)-trisection, then by [18], with some work, we obtain that M is diffeomorphic to $\#^g(\mathbb{S}^1 \times \mathbb{S}^3)$.

Now, we end this section with defining what we mean by "two trisections are the same" :

Definition 2.5. *Given two trisections $M = X_1 \cup X_2 \cup X_3 = Y_1 \cup Y_2 \cup Y_3$ of the same manifold, they are said to be* ***equivalent*** *if there is a diffeomorphism $f : M \to M$ and a permutation $\sigma \in \mathfrak{S}_3$ such that $f(X_i) = Y_{\sigma(i)}$ for all $i = 1, 2, 3$.*

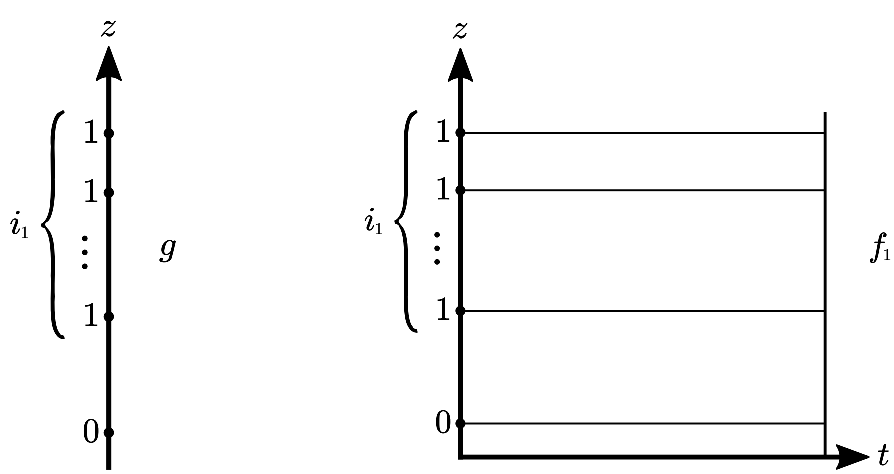
One sees that if two trisections have different genera, they have no hope to be equivalent. However, if one could find a way to produce a new trisection with a different genus, it would be possible to modify those two trisections to have their genus agree, and re-ask the question whether they are equivalent or not. The Gay–Kirby theorem will provide some way to answer this question positively.

2.2 Morse 2-functions and trisecting functions : a proof of existence

As was described earlier, thickening the 1-skeleton of a triangulation of a 3-manifold had the effect to yield a Heegaard splitting of that manifold. However, combinatorial arguments related to the *Hauptvermutung* aren't available in dimension four, so we may want to describe a Morse-theoretic proof of existence. The proof given here is taken from [10].

We start with a $(1 : i_1 : i_2 : i_3 : 1)$-handle decomposition of M^4. We will construct a certain Morse 2-function on the different parts of that handle decomposition, and glue them together by adding Cerf graphics. At last, we will homotope the resulting Morse 2-function to make it trisecting.

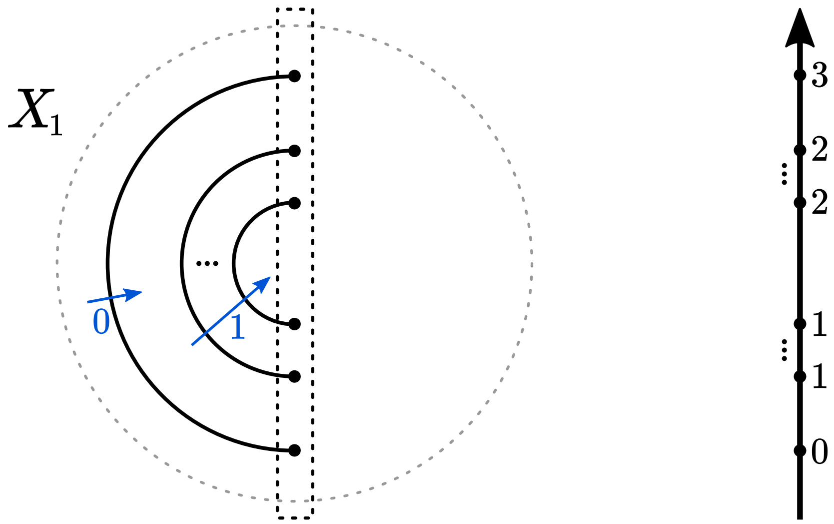
Let X_1 be the union of the 0- and the 1-handles. Then $X_1 \cong \natural^{i_1}(\mathbb{S}^1 \times \mathbb{D}^3) \cong I \times \natural^{i_1}(\mathbb{S}^1 \times \mathbb{D}^2)$. There is a standard Morse function $g : \natural^{i_1}(\mathbb{S}^1 \times \mathbb{D}^2) \to \mathbb{R}$ with one index zero critical point, and i_1 of index one that gives its handle decomposition. This induces a Morse 2-function $f_1 : X_1 \to \mathbb{R}^2$ by defining $f_1 : (t, p) \in I \times \natural^{i_1}(\mathbb{S}^1 \times \mathbb{D}^2) \mapsto (t, g(p))$ (ignoring the diffeomorphism of X_1 on the left). Its Cerf graphic is shown in figure 5b.



(a) (b)

Figure 5. (a) The standard Morse function g on $\natural^{i_1}(\mathbb{S}^1 \times \mathbb{D}^2)$. (b) The associated Morse 2-function $f_1 : X_1 \to \mathbb{R}^2$.

We can always post-compose with a diffeomorphism of the image to the left half-disc to obtain a new Morse 2-function $G_1 : X_1 \to \mathbb{D}^2$ whose image of the fold locus is as in figure 6a. Note that we indicate the indices of the folds in blue by choosing a transverse direction to those folds.



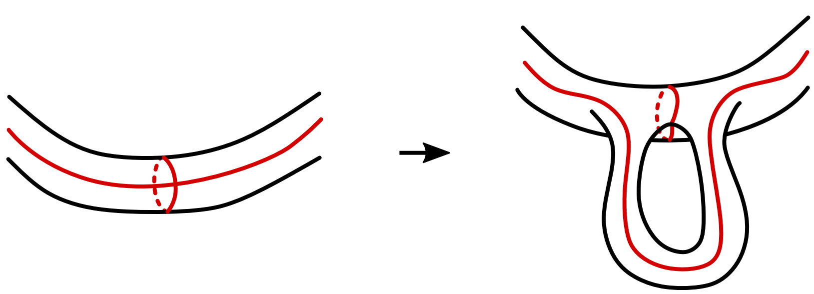
(a) (b)

Figure 6. (a) The image of the fold locus of G_1, whose image is the left half-disc. (b) The vertical Morse function on the boundary ∂X_1.

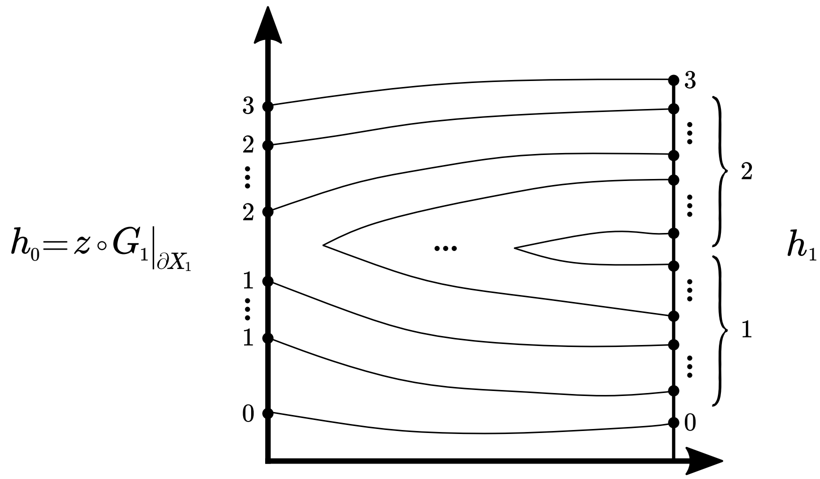
In particular, restricting the vertical Morse function $z \circ G_1$ to the boundary ∂X_1 gives a Morse function whose critical points are indicated in figure 6b above. Note that $\partial X_1 \cong \#^{i_1}(\mathbb{S}^1 \times \mathbb{S}^2)$, which means that this function induces the standard genus i_1 splitting on it.

Now, consider the framed attaching link $L \subset \partial X_1$ for the 2-handles. We can move L in ∂X_1 by isotoping it so that it generically lies in between the level sets of the last index one and the first index two critical points of the Morse function $z \circ G_1|_{\partial X_1}$. This means that L is in a regular neighborhood of the Heegaard surface Σ of the previously-mentionned Heegaard splitting $\partial X_1 = H_1 \cup_\Sigma H_2$. In particular, we can follow the gradient flow lines to project L onto an immersed curve $\overline{L}$ on Σ. Generically, we can assume that this immersed curve only has double points self-intersections.

The previous Heegaard splitting can now be stabilized to resolve those crossings as follows :



By Cerf theory, this corresponds to homotoping the Morse function $z \circ G_1|_{\partial X_1}$ into a new one by a generic homotopy h_t where its only non-Morse times correspond to birth of pairs of cancelling points :



We can now view this homotopy as a Morse 2-function to be glued to the previous one, which in turn gives an extension of G_1 to a collar neighborhood $X_1 \cup \partial X_1 \times [0,1]$, and whose image of the fold locus is indicated in figure 7.

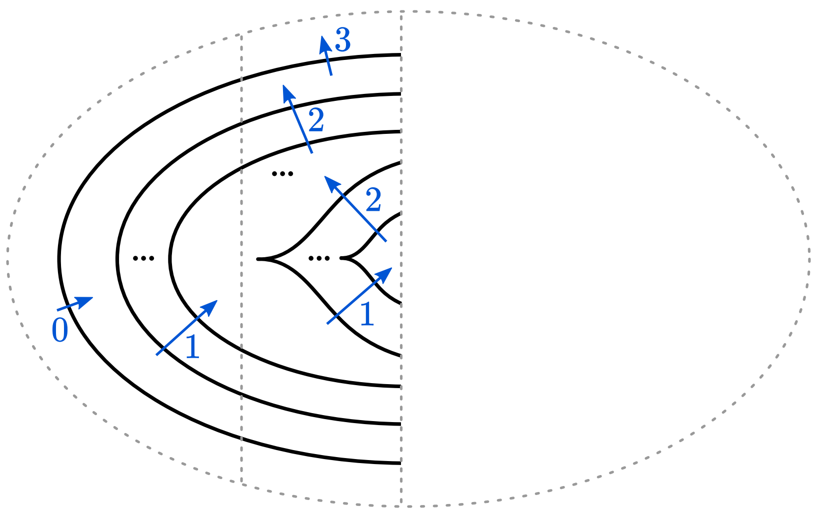


Figure 7. The fold locus of the extension of G_1 to a collar $X_1 \cup (\partial X_1 \times [0,1])$.

The attaching link for the 2-handles now lies in the stabilized surface Σ' with corresponding Heegaard splitting $\partial X_1 = H_1' \cup_{\Sigma'} H_2'$. The Morse function for that splitting is the vertical function on the boundary at the right of figure 7.

Because attaching a 3-dimensional 2-handle corresponds to having an index 2 critical point, attaching a 4-dimensional 2-handle corresponds to having an I's worth of 3-dimensional 2-handles, and in turn, an I's worth of index 2 critical points. This means that attaching a 2-handle to the previous construction corresponds to gluing the Cerf graphic made of only one definite fold of index 2.

We can apply the following procedure for as many link component L has (that is i_2, the number of 2-handles to be attached) :

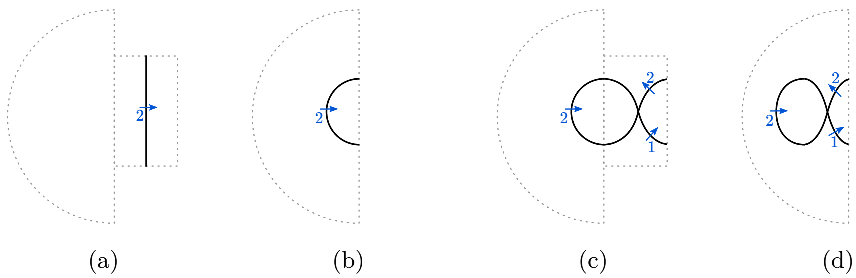
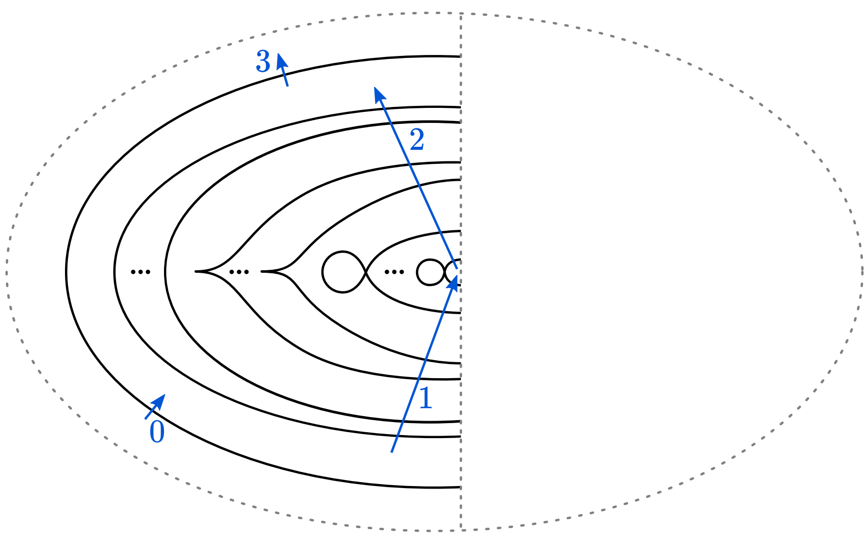


Figure 8. (a) Gluing the Cerf graphic for the attaching 2-handle. (b) Applying a diffeomorphism of the image to fit this all in the half-disc. (c) Homotoping the vertical Morse function at the boundary by performing a crossing (possible, by Cerf theory). (d) Again, applying a diffeomorphism to fit in the half-disc.

This results in a Morse 2-function $G_2 : X_2 \to \mathbb{D}^2$ on X_1' the union of the 0-, 1- and 2-handles, whose image of the fold locus is as follows :



We can now repeat the construction for X_1 to the union X_3 of the 3- and the 4-handles, and obtain another Morse 2-function on X_3 whose image of the fold locus is the vertical mirror image of figure 7. Those two Morse 2-functions can be glued together by homotoping one into the other (we know it is possible, by Cerf theory), which translates to adding a suitable Cerf graphic $\mathcal{G}$, to produce a final function $G : M \to \mathbb{D}^2$ whose image of the fold locus is as in figure 9.

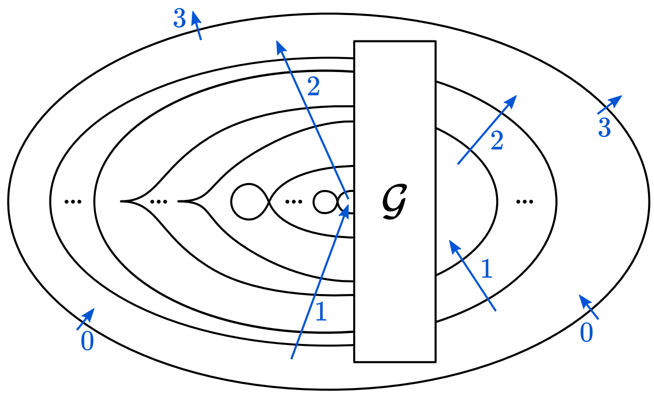


Figure 9. The image of the fold locus of the so-far constructed Morse 2-function $G : M \to \mathbb{D}^2$.

The idea is now to homotope this function into a more standard position that will allow for pre-imaging sectors into a trisection. Note that homotoping is made possible by [7]. First, we can perform *cusp-fold crossings* to the Cerf graphic $\mathcal{G}$ to move all births and deaths out to its left and right, respectively. The procedure is as follows :

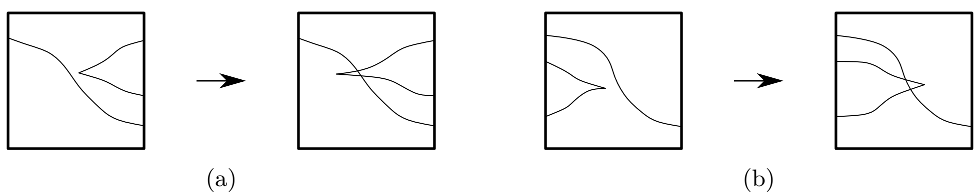
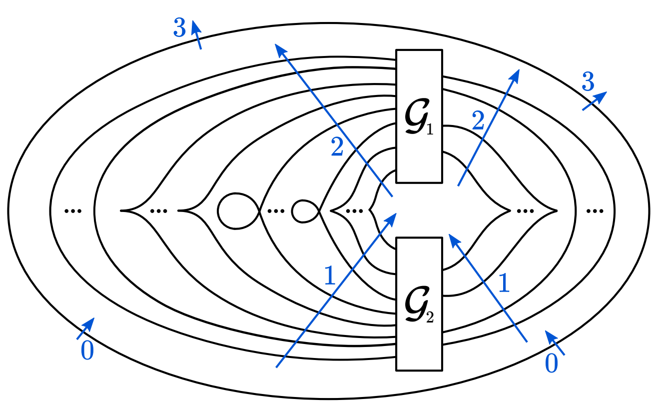


Figure 10. (a) Moving births to the left by performing *cusp-fold crossings*. (b) Moving deaths to the right.

The remaining Cerf graphic being only made of crossings of critical points, it can be split into two Cerf graphics, by separating them by the indices of the folds. This sets us in the following position :



Now, we move the cusps from the right to the left of the “kinks” by the following procedure :

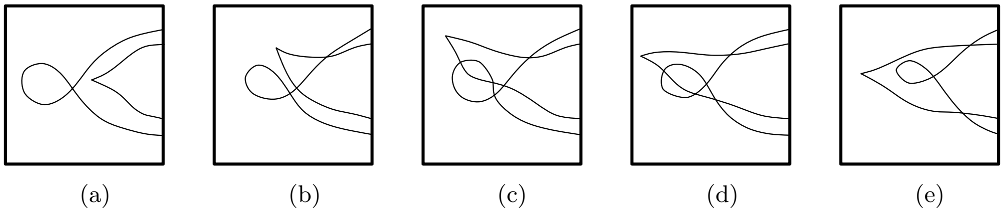
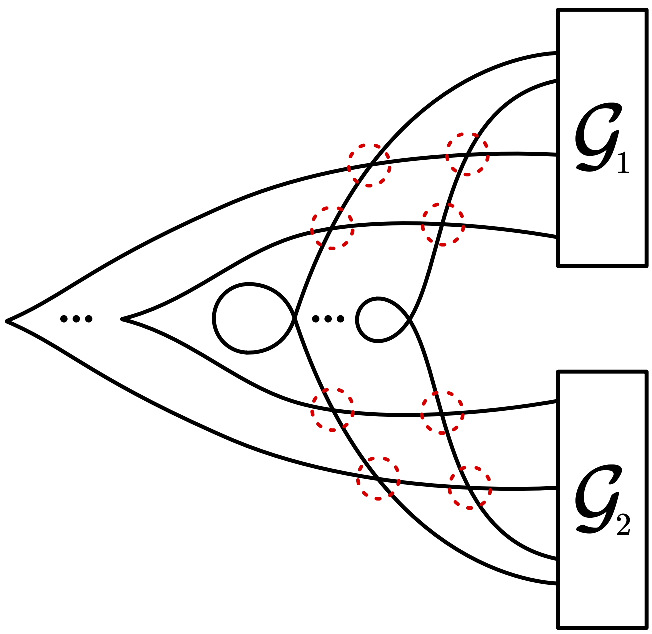


Figure 11. (a) A cusp to the right of a “kink”. (b) Perform a *cusp-fold crossing*. (c) Perform a *Reidemeister II move*. (d) Perform a *Reidemeister III move* in the middle. (e) Perform a *Reidemeister II move*.

Applying this for each kink, one cusp at a time, we end up in this situation :



Note that all the crossings created by this procedure (circled in red) can be moved inside the Cerf graphic boxes. The last move we want to perform is this one :

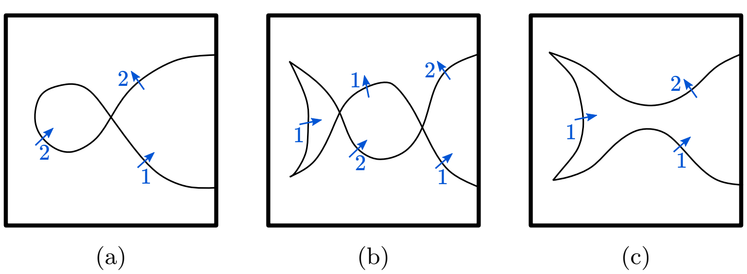
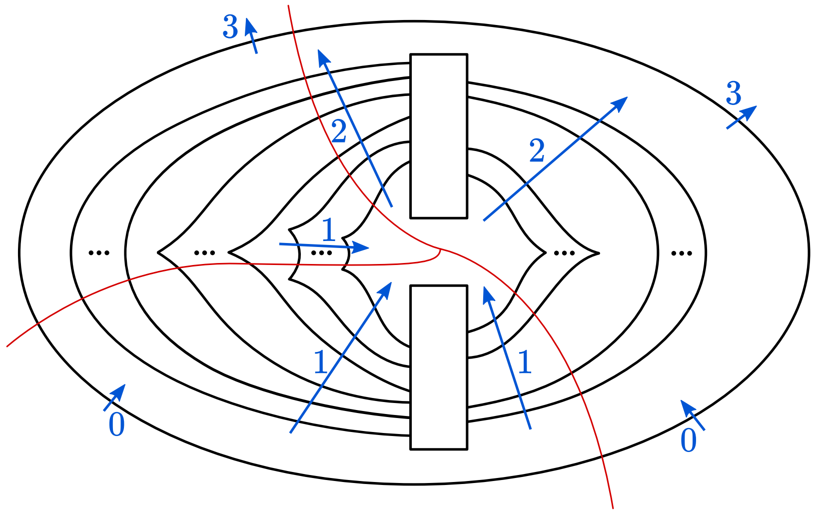


Figure 12. (a) A “kink”. (b) Perform a *swallowtail birth singularity*. (c) Perform a *Reidemeister II move*.

After doing it for each kink, this has the effect of setting us in the following position :



By applying a diffeomorphism of the image again, we can see that we are in the more general case of figure 13. One comment is that we place ourselves in a more general setting where we added an additional Cerf graphic box. This will be necessary for the next part.

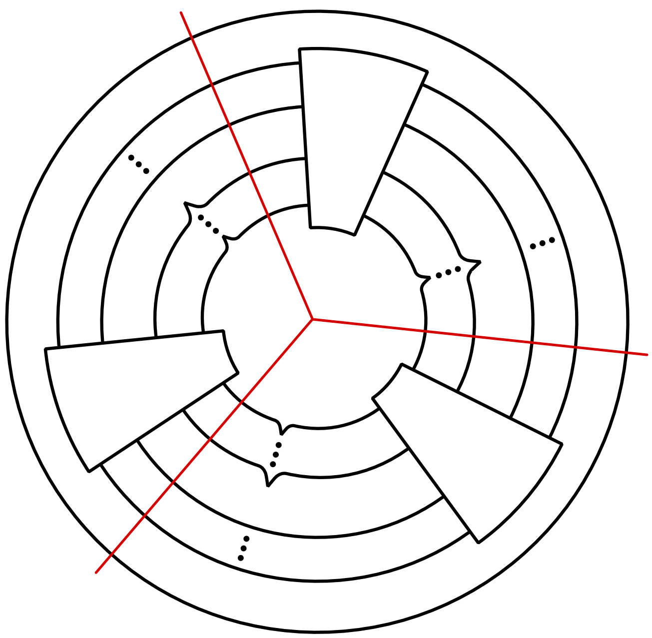
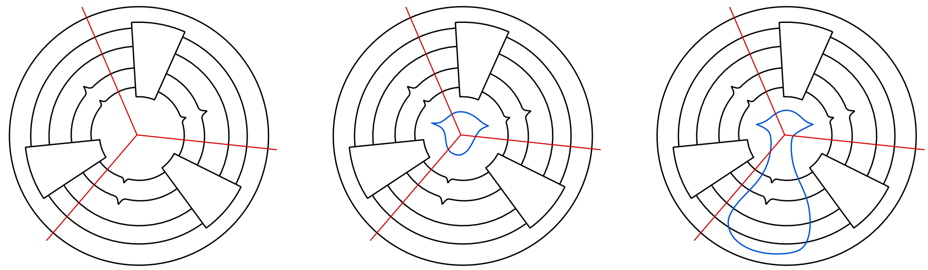


Figure 13. The general form of the fold locus. Except for the outermost fold, all folds have index one by choosing a transverse direction to be pointing "inwards". Each Cerf box is only made of crossings of critical points.

The number of cusps in each sector is not necessarily the same, and this also goes for the number of folds without cusps. However, we can perform one last modification to our Morse 2-function, to make it trisecting :



(a) (b) (c)

Figure 14. (a) The general form for the image of the fold locus. (b) Performing an *eye birth singularity.* (c) Moving the eye by performing *Reidemeister II moves* and *cusp-fold crossings.*

Doing the previous moves has the effect of increasing by one the number of folds without cusps in one prescribed sector only. Also note that after this move, up to moving all the crossings we created inside the Cerf graphic boxes, we are still in the general position of figure 13.

Finally, pre-imaging each sector indeed gives a 4-dimensional handlebody, by an argument converse to what we did in figure 7.

For the pairwise intersections, we see that it corresponds to taking the radial Morse function on the boundary of each sector. This function induces a handle decomposition on the 3-manifold, which makes it a handlebody of corresponding genus.

For the triple intersection, we see that it bounds each of the pairwise intersections. Therefore, it is a closed surface with genus.

This means that pre-imaging through this Morse 2-function indeed provides us with a trisection of M. ■

Definition 2.6. *If a trisection of manifold is obtained by a Morse 2-function as in figure 13, we call that Morse 2-function* ***trisecting*** *for that trisection.*

Any trisection admits a trisecting Morse 2-function. Indeed, each sector has a Morse 2-function as in figure 7, and all three functions can be glued together by homotoping the radial Morse functions on the boundaries. This function needs not be trisecting, but it can generically be perturbed to become trisecting.

2.3 Trisection diagrams and stabilization

We aim to give a combinatorial and lower-dimensional description of trisections, just the same way we can do for Heegaard diagrams. Recall the cartoon picture of a trisection in figure 4. For each of the handlebodies H_{ij}, pick a system of g compressing curves on Σ_g such that compression along those curves produces that handlebody.

More precisely : pick a system of g disjoint simple closed curves such that cutting Σ_g along those curves gives a $2g$-punctured sphere, embedded in M. One can glue discs to those punctures to obtain an actual sphere, and this sphere bounds a ball inside M. Then, re-glue the discs we attached to the punctures pairwise to obtain a handlebody bounding Σ_g.

Note that the resulting handlebody is embedded in M^4, but needs not be embedded in a 3-submanifold of M, and rather immersed. The following picture depicts such an example :

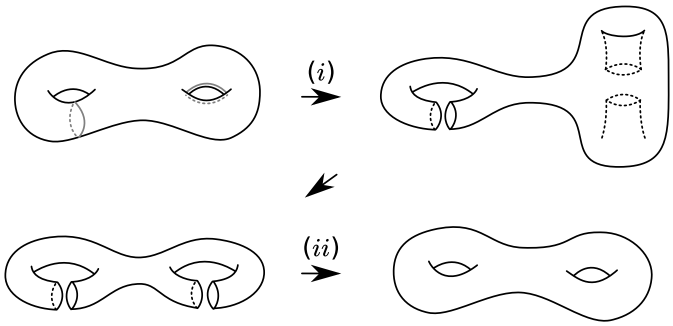


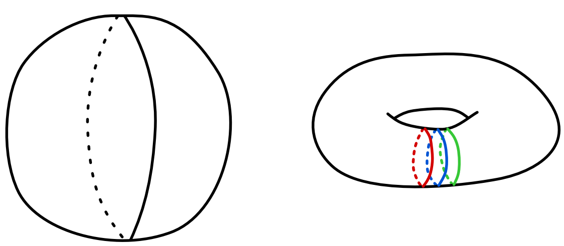
Figure 15. Compression along the two indicated curves on Σ_2. Step *(ii)* corresponds to attaching the ball inside and gluing the punctures back together. Try to picture where the ending handlebody is in the originating surface : ”inside” or “outside” ?

Denote by α a set of g compressing curves for H_{13}, as β one for H_{12} and as γ one for H_{23} (see figure 4).

Definition 2.7. *Given a trisection of a manifold, the data* $(\Sigma_g, \alpha, \beta, \gamma)$ *constructed previously is called a* ***trisection diagram*** *associated to that trisection.*

It is common to draw the curves in each set of curves in a different colour, as indicated in the construction. Although the choices made are almost surely inconsistent across the literature, we shall stick to those choices here.

For instance, here are two trisection diagrams associated to the two examples of $\mathbb{S}^4$ and $\mathbb{S}^1 \times \mathbb{S}^3$ indicated previously :



(a) (b)

Figure 16. (a) The example of the $(0,0)$-trisection of $\mathbb{S}^4$. No curves are drawn because none are needed. (b) The $(1,1)$-trisection diagram for $\mathbb{S}^1 \times \mathbb{S}^3$.

By theorem 1.7 again, it turns out that each $(\Sigma_g, \alpha, \beta)$, $(\Sigma_g, \alpha, \gamma)$ and $(\Sigma_g, \beta, \gamma)$ are standard diagrams for the Heegaard splitting $Y^+_{k,g} \cup Y^-_{k,g}$. This means that, selecting two colors out of the three, up to sliding the curves, there is a diffeomorphism of the surface Σ_g that carries those curves into standard position :

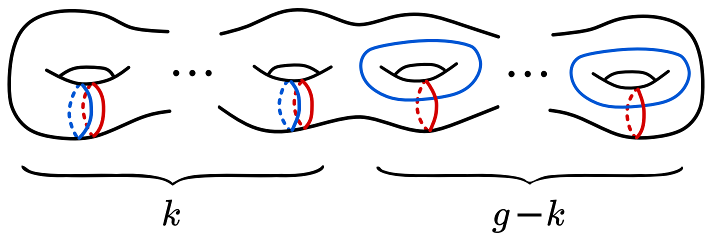


Figure 17. A trisection diagram in standard position. The green curves are not shown, and need not be either meridian or longitude.

More precisely, this is what we mean by a *Heegaard diagram in standard position* :

Definition 2.8. *A Heegaard diagram* $(\Sigma_g, \alpha, \beta)$ *is called* ***standard*** *if there is some* $k \leqslant g$ *such that :*

- $\alpha_i = \beta_i$ *for* $i \leqslant k$*.*
- $\mathrm{card}(\alpha_i \cap \beta_j) = \delta_{ij}$ *for* $i, j \geqslant k+1$*.*

This necessarily implies that the Heegaard diagram in question is the one of the Heegaard splitting $Y^+_{k,g} \cup Y^-_{k,g}$. Back to trisection diagrams, we see that even though we can always place one selected pair of colors into standard Heegaard position, this needs not act as we want on the remaining third colors (and the remaining two pairs of colors). This leads to the following definition :

Definition 2.9. *A trisection is called* ***standard*** *if it has a trisection diagram such that all three choices of pairs of colours are already in standard (Heegaard) position.*

For instance, the examples of figure 16 are both standard trisections. From [20], we cite the following result :

Theorem 2.10. (Meier–Schirmer–Zupan) *If* M *has a* (g, k)*-trisection with* $k \geqslant g-1$*, then that trisection is standard.*

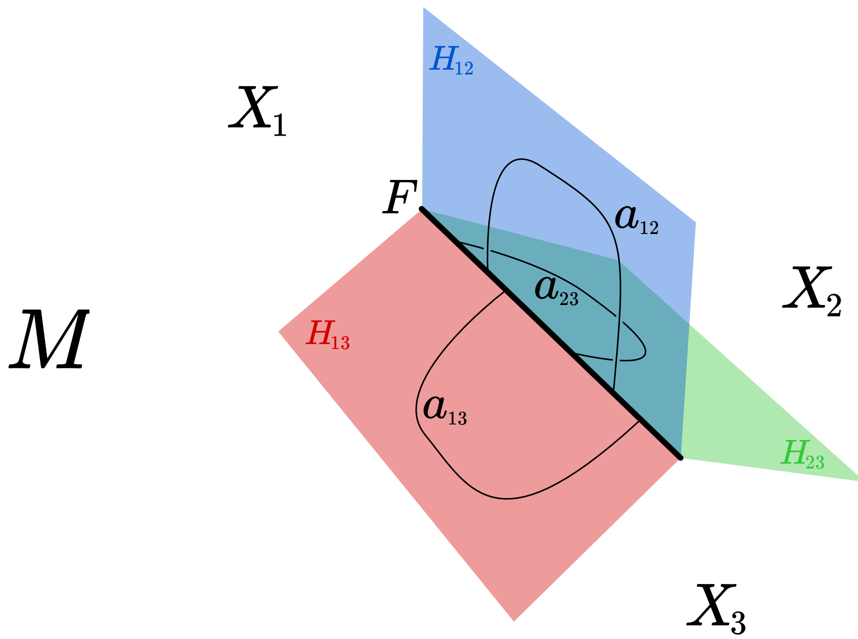
Moreover, it turns out that all genus two trisections are standard, by work from [23].

A natural question arises : given three sets of compression curves α, β and γ on Σ_g, does it come from a trisection ? A necessary condition is that each pair of colors gives a standard genus g splitting of $\#^k(\mathbb{S}^1 \times \mathbb{S}^2)$ for some fixed k common to all three choices of pairs. It turns out this condition is also sufficient :

Proposition 2.11. *Given three sets of compression curves* α, β *and* γ *on* Σ_g *such that each choice of pairs is a Heegaard diagram for* $\#^k(\mathbb{S}^1 \times \mathbb{S}^2)$*, there exists a unique trisected manifold* $M = X_1 \cup X_2 \cup X_3$ *such that* $(\Sigma_g, \alpha, \beta, \gamma)$ *is a diagram for that trisection.*

Proof. This result relies on the work from [18], see section 1.3. The idea is to thicken the surface into $\Sigma_g \times \mathbb{D}^2$, and to compress along each set of curves α, β and γ to obtain three handlebodies H_{13}, H_{12} and H_{23} respectively. Then, we can glue each $H_{ij} \times I$ to a neighborhood of $\Sigma \times \{e^{\theta_{ij}}\}$ for three different values of θ_{ij}. This gives a 4-manifold whose boundary has three copies of $\#^k(\mathbb{S}^1 \times \mathbb{S}^2)$. From [18], there is only one unique way to retrieve a trisected manifold by attaching the remaining handles. ∎

Let us now describe the stabilization operation on a trisection. Start with a trisected manifold $M = X_1 \cup X_2 \cup X_3$. Recall that $H_{ij} = X_i \cap X_j$ is a 3-dimensional handlebody, and the triple intersection $F = X_1 \cap X_2 \cap X_3$ is a genus surface. For all i, j, choose a boundary parallel arc $a_{ij} \subset H_{ij}$ with endpoints in F, such that all six endpoints of the three arcs are distinct in F. Choose regular neighborhoods $N_{ij} \subset M$ for each arc. If we extrude the cartoon picture of a trisection from figure 4, we obtain the following schematic representation of the situation :



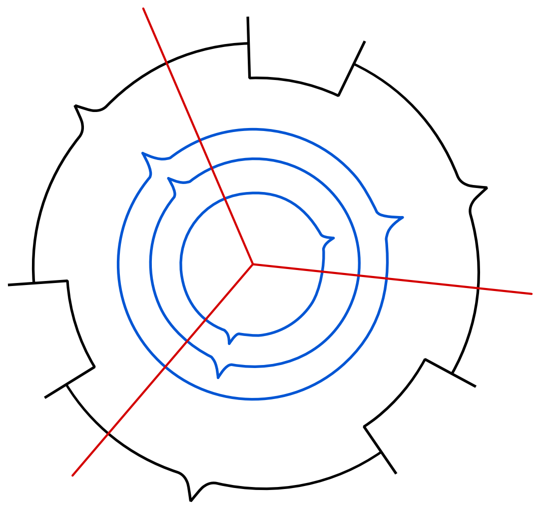
The idea is to increase the genus of each X_i, so we can attach the opposite N_{jk} to it to do so. However, this would make the double intersections not as we would like them, so we need to remove the other two neighborhoods from that X_i. This results in defining :

$$X_i' := (X_i \cup N_{jk}) - (\mathring{N}_{ij} \cup \mathring{N}_{ik}).$$

It is now an easy computation to see that this gives a new trisection. Moreover, if the originating one was of type (g, k), then this new one is of type $(g+3, k+1)$. At last, the construction is independent of the choices of the arcs (and of the regular neighborhoods), because we chose them to be boundary parallel. This yields :

Definition 2.12. *Given a trisection of a manifold, the* ***stabilization*** *of that trisection is the operation of replacing it by the new one constructed above.*

Stabilization has an interpretation in terms of the trisecting Morse 2-function. Recall the rebalancing of the number of folds without cusps from figure 14. It corresponds to stabilization in one sector ; therefore, the whole process of stabilization corresponds to applying the procedure in figure 14 with the following three eyes :



Note that the order in which we apply those eyes doesn't matter, for we can always perform the moves described in figure 14 to swap the order in which they appear. Moreover, the stabilization operation has an interpretation in terms of trisection diagrams. Indeed, here is the diagram obtained from stabilizing the genus zero trisection :

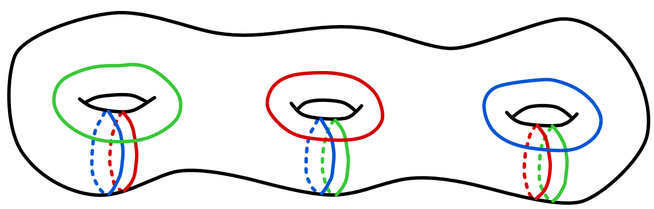


Figure 18. The stabilization diagram, a $(3,1)$-trisection of $\mathbb{S}^4$.

Stabilization of a trisection translates to taking the connected sum of its diagram with the one from figure 18 above.

2.4 Uniqueness : the Gay–Kirby theorem

First, we shall prove an extended version of proposition 2.4 :

Proposition 2.13. *Let $M = X_1 \cup X_2 \cup X_3$ be a (g,k)-trisection. Then M has a handle decomposition with $(1 : k : g-k : k : 1)$ handles, such that :*

(i) X_1 is the union of the 0- and 1-handles.

(ii) In the Heegaard splitting $\partial X_1 = H_{12} \cup H_{13}$, whose Heegaard surface is F, the attaching link L for the 2-handles lies in the interior of H_{12}.

(iii) The framed attaching link $L = K_1 \cup ... \cup K_{g-k}$ is isotopic in H_{12} to a framed attaching link $L' = K'_1 \cup ... \cup K'_{g-k}$ in F, with framings equal to the ones induced by F.

(iv) There exists a system of compression discs $\mathcal{D}_1, ..., \mathcal{D}_g$ for H_{12} such that for all $j = 1, ..., g-k$, the curve K'_j intersects $\partial\mathcal{D}_j$ exactly once, and is disjoint from the other $\partial\mathcal{D}_i$.

(v) H_{12} *has a tubular neighborhood* $N = [-\varepsilon, \varepsilon] \times H_{12}$ *such that* $N \cap X_1 = [-\varepsilon, 0] \times H_{12}$ *and* X_2 *is obtained by attaching the 2-handles to* $[0, \varepsilon] \times H_{12}$.

Proof. Take a trisecting Morse 2-function for that trisection (see definition 2.6). The image of the fold locus is as in figure 13, with k folds without cusps and $g - k$ folds with cusps. By post-composing with a diffeomorphism of the image, we can always set ourselves in the position of figure 19 (note that, without loss of generality, we pre-image differently than from the existence proof here, where X_2 doesn't contain a Cerf graphic and X_3 has two of them).

Calling that previous Morse 2-function $G : M \to \mathbb{R}^2$, we can take the horizontal Morse function $t \circ G : M \to \mathbb{R}$, whose critical points are the vertical tangencies of the Morse 2-function G. This indeed provides us with a handle decomposition with as many handles as announced, and X_1 is the union of the 0- and 1-handles. Moreover, X_2 is seen as $g - k$ 2-handles attached to the handlebody $H_{12} \times I$ seen on the blue segment. This means we only need to show that the attaching link for the 2-handles satisfies the properties we announced.

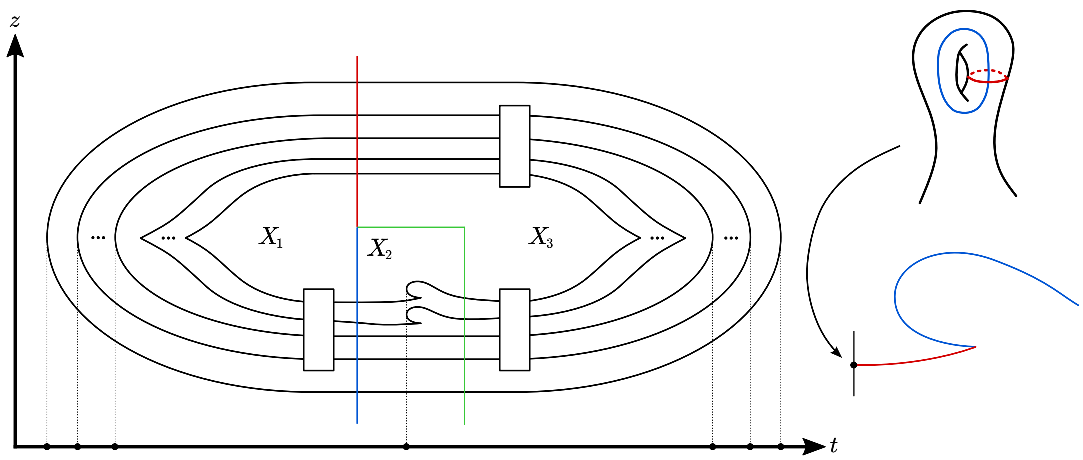


Figure 19. Reading the handle decomposition from the trisecting Morse 2-function. To the right is a zoom on a fold in the X_2 region.

This is also seen in figure 19 : each attaching knot for each 2-handle is a longitude on the surface, whereas the compression discs' boundaries are meridional curves (recall from figure 8 that attaching a 2-handle corresponds to the fold colored blue, and that the fold colored red is the 1-handle it is attached to). ■

It turns out if we have a handle decomposition of M with $(1 : k : g - k : k : 1)$ handles satisfying the properties of proposition 2.13, then M has a (g, k)-trisection. Proving this statement is a verification, and Gay and Kirby used it to give an alternative proof for the existence of trisections.

We shall now prove the following :

Theorem 2.14. (Gay–Kirby, 2012) *Any two trisections of the same manifold can be stabilized a suitable number of times each to yield equivalent trisections.*

Proof. (Sketch) Start with two trisections $M = X_1 \cup X_2 \cup X_3 = Y_1 \cup Y_2 \cup Y_3$. Proposition 2.13 can be applied to give two handle decompositions $\mathcal{H}_X$ and $\mathcal{H}_Y$. Both handle decompositions induce a Heegaard splitting of ∂X_1 or ∂Y_1 with respective framed attaching links L_X and L_Y for their 2-handles behaving as previously-stated.

Handle decompositions being purely Morse-theoretic, Cerf theory applies and tells us that we can get from $\mathcal{H}_X$ to $\mathcal{H}_Y$ by a sequence of :

(i) Add cancelling pairs of 1- and 2-, or 2- and 3-handles to both decompositions.

(ii) Slide k-handles over k-handles ($k = 1, 2, 3$).

(iii) Isotope a handle without performing handle slides.

By the trisecting function interpretation of stabilization we made in the previous section, we see that this operation translates into adding both a pair of 1- and 2- and of 2- and 3-handles. In particular, by seeing that we have as many 1- and 3-handles in each associated handle decomposition, we see that we *must* perform step *(i)* as many times for both pairs, and this can be achieved by stabilization. Also note that this operation doesn't break the properties regarding the attaching link for the 2-handles in each associated handle decomposition.

Sliding or isotoping 1- and 3-handles doesn't affect the trisection. It doesn't change the properties from proposition 2.13 neither. Therefore, it suffices to check what effects the sliding or the isotoping of 2-handles.

First, assume we only want to perform one 2-handle sliding on the decomposition $\mathcal{H}_X$. We have a Heegaard splitting $\partial X_1 = H_{12} \cup H_{13}$ from the corresponding trisection, with attaching link L for the 2-handles in H_{12}. As given by proposition 2.13, we can isotope L in $F = \partial H_{12}$ so that the components of L are geometrically dual to compression curves on F (point *(iv)* from proposition 2.13).

The handle sliding corresponds to the choice of an arc between two components K_1 and K_2 of L. We can project that framed arc onto F by following the gradient flow lines, but this gives an immersed curve with crossings. As seen in the proof for existence, we need to stabilize to resolve those crossings, and this is done by stabilization of the whole trisection. We check that this stabilization operation can be done so to assure we still have the desired properties regarding the attaching link, and so that performing the sliding of the 2-handle also maintains them. When stabilizing this trisection, we must also stabilize the other one.

Lastly, assume we only need to perform one isotopy of a 2-handle of $\mathcal{H}_X$. Then, by assumption, this isotopy extends to one of the whole M. This means that the $X_1 = Y_1$, and we have two Heegaard splittings $\partial X_1 = H_{12} \cup H_{13} \cup H'_{12} \cup H'_{13}$ induced by each trisection. The attaching link for both handle decompositions lies in H_{12} and in H'_{12}, and satisfies point *(iv)* from proposition 2.13 in both cases. Now, because both Heegaard splittings of ∂X_1 are of genus g, Waldhausen–Haken provides an isotopy of ∂X_1 taking H_{12} to H'_{12}.

However, this isotopy needs *not* carry L to itself ; we need to find one that fixes L to be done. The details – which we shall not give here – involve another Cerf-theoretic argument, which in turn involves stabilizing the Heegaard splittings. This again can be done by stabilization of the whole trisection. ■

A more natural way of proving the theorem would be by trying a method analogous to the proof of the Reidemeister-Singer theorem, which involves a homotopy of the Morse functions associated to the two Heegaard splittings.

In that case, it would be a homotopy of the two trisecting Morse 2-functions that is needed, but this involves the study of the surface of folds of homotopies of Morse 2-functions. These surfaces are highly non-trivial to deal with, and a nice Cerf-like argument is likely to be harder to prove in all generality.

Therefore, the technical proof involving the handle decompositions is impossible to avoid for now.

Remark 2.15. *Just like with Heegaard splittings, even though two trisections have the same genus, they need* not *be equivalent. Even for the 4-sphere, we do not have a complete classification of its trisections yet. This means that, possibly, there exists a genus three trisection of* $\mathbb{S}^4$ *that isn't the stabilization of the trivial trisection (i.e. the Waldhausen theorem hasn't been proven yet). The only examples are for trisections with genus* $g \leqslant 2$*, which we list in figure 21.*

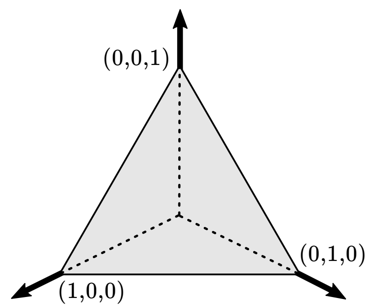
3 More thorough examples and applications

3.1 $\mathbb{CP}^2$, $\mathbb{S}^2 \times \mathbb{S}^2$ and trisections in small genera

Recall the moment map from the toric action on $\mathbb{CP}^2$:

$$\mu([z_0 : z_1 : z_2]) = \left(\frac{|z_0|^2}{|z_0|^2 + |z_1|^2 + |z_2|^2}, \frac{|z_1|^2}{|z_0|^2 + |z_1|^2 + |z_2|^2}, \frac{|z_2|^2}{|z_0|^2 + |z_1|^2 + |z_2|^2} \right).$$

The image for that moment map (the Delzant polytope) is a triangle, the convex hull of the three points $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$:



Here, we list the preimages of generic points in that triangle :

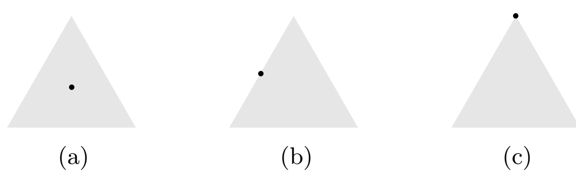
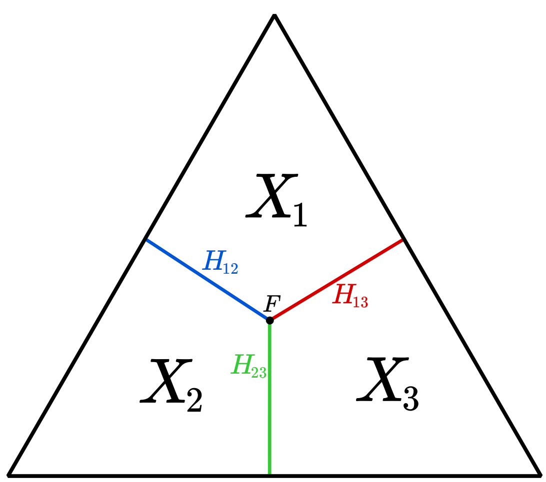


Figure 20. (a) The preimage is a generic fiber $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$. (b) One of the circle collapses to a point, the preimage is $\mathbb{S}^1 \times \{*\}$. (c) Both circles have collapsed, the preimage is $\{*\} \times \{*\}$.

For exact computations, we refer to [15]. The idea is again to preimage three sectors in a trisection of that image, just as in figure 4. We can take the barycentric subdivision of the Delzant polytope, and preimage each sector into X_i :



Each piece of the trisection has a simple description :

$$X_i = \{[z_0 : z_1 : z_2] \mid |z_j|, |z_h| \leqslant |z_i|\} \qquad \text{and} \qquad H_{ij} = \{[z_0 : z_1 : z_2] \mid |z_h| \leqslant |z_i| = |z_j|\}.$$

Therefore, we have $X_i \cong \mathbb{D}^4$, and by the observation of the collapsing circle made in figure 20, we have that each H_{ij} is a solid torus. This provides us with a $(1, 0)$-trisection of $\mathbb{CP}^2$, whose trisection diagram can be seen in figure 21.

We could use a similar principle (see [10]) with the moment map $\mu : \mathbb{S}^2 \times \mathbb{S}^2 \to I^2$ to obtain a $(2, 0)$-trisection of $\mathbb{S}^2 \times \mathbb{S}^2$. See figure 21 for its diagram.

So far, we have seen *all* the trisections in genus less than two. That is, we have the following :

Theorem 3.1. *If M has trisection genus $g_T(M) = 1$, then M is either $\mathbb{S}^1 \times \mathbb{S}^3$, $\mathbb{CP}^2$ or $\overline{\mathbb{CP}}^2$, where the trisection diagram for $\overline{\mathbb{CP}}^2$ is the mirror image of the one for $\mathbb{CP}^2$.*

For the genus two, this is highly more difficult :

Theorem 3.2. (Meier-Zupan, 2014, see [23]) *If M has trisection genus $g_T(M) = 2$, then M is either $\mathbb{S}^2 \times \mathbb{S}^2$, or a connected sum of $\mathbb{S}^1 \times \mathbb{S}^3$, $\mathbb{CP}^2$ and $\overline{\mathbb{CP}}^2$ with two summands. Moreover, each of these 4-manifolds has a unique genus two trisection, up to diffeomorphism.*

This means that we can list all the (irreducible) trisections (that is, those that are not the connected-sum of two trisections) with genus $g \leqslant 2$:

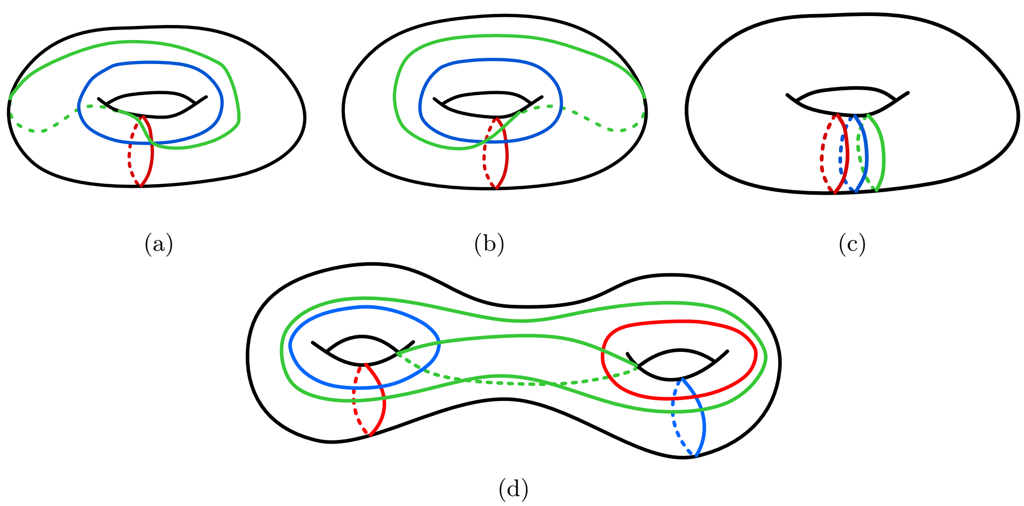


Figure 21. The exhaustive list of irreducible genus one and two trisections. (a) $\mathbb{CP}^2$. (b) $\overline{\mathbb{CP}}^2$. (c) $\mathbb{S}^1 \times \mathbb{S}^3$. (d) $\mathbb{S}^2 \times \mathbb{S}^2$. One can always take any selection of two genus one trisections and produce a genus two trisection by taking their connected-sum, but that trisection would be reducible.

Now, genus three is a whole different story. Recall that different Lens spaces admit inequivalent Heegaard diagrams (the red curve is meridional and the blue curve is the corresponding torus knot). From [19], we have the following :

Theorem 3.3. (Meier, 2017) *If M^3 has a genus g Heegaard splitting, then $\mathcal{S}(M)$ and $\mathcal{S}^*(M)$ admit $(3g, g)$-trisections.*

Here, $\mathcal{S}(M)$ and $\mathcal{S}^*(M)$ denote respectively the *spin* and *twisted-spin* of M, a construction of a 4-manifold. In fact, we have that M, $\mathcal{S}(M)$ and $\mathcal{S}^*(M)$ have same fundamental group. Taking $M = L(p,q)$ a lens space, we have $\operatorname{rk}\pi_1(M) = 1$, and $X = \mathcal{S}(M)$ has a $(3,1)$-trisection. By $\operatorname{rk}\pi_1(X) \leqslant 1$ from the handle decomposition (see 2.13), which means $k = 1$ is the minimal value we can have. From $\chi(X) = 2 + g - 3k$, we also see that $g = 3$ is the minimal value, so that the $(3,1)$-trisection is minimal.

This means that we have obtained as many $(3,1)$-trisected manifolds as there are values for p, for $\pi_1\mathcal{S}(L(p,q)) \cong \mathbb{Z}/p$, meaning that different lens spaces yield manifolds with different fundamental groups. See figure 22a for the trisection diagram for $\mathcal{S}(L(2,1))$. This proves the case $(g,k) = (3,1)$ of the following :

Corollary 3.4. (Meier, 2017) *For any $g \geqslant 3$ and $k \leqslant g-2$, there are infinitely many distinct 4-manifolds admitting minimal (g,k)-trisections.*

However, the question of knowing whether we have obtained *all* genus three trisections by these means is still open :

Conjecture 3.5. (Meier) *Every irreducible 4-manifold with trisection genus three is either the spin of a Lens space, or a Gluck twist on a specific 2-knot in the spin of a Lens space.*

3.2 Trisections in context : computing topological invariants

We want to compute the homology from a (g,k)-trisection diagram. Start with a trisection diagram $(\Sigma, \alpha, \beta, \gamma)$. There is a trisected manifold $M = X_1 \cup X_2 \cup X_3$ associated to that diagram, whose pairwise intersections are denoted as $H_{13} = H_\alpha$, $H_{12} = H_\beta$ and $H_{23} = H_\gamma$. For $\nu \in \{\alpha, \beta, \gamma\}$, define L_ν to be the subgroup of $H_1(\Sigma)$ generated by the homology classes of the curves in ν. For convenience, denote as $L_{\mu\nu} = L_\mu \cap L_\nu$.

We call M the unique trisected 4-manifold associated to that diagram. Let $Y = H_\alpha \cup H_\beta \cup H_\gamma$ be the *spine* of the trisection, and let N be a regular neighborhood of it. We see that N is the 4-manifold obtained after gluing the thickened H_ν to the $\Sigma \times \mathbb{D}^2$ (see 2.11 for the construction). In particular, N has three boundary components all diffeomorphic to $\natural^k(\mathbb{S}^1 \times \mathbb{S}^2)$, and N deformation retracts to Y (in particular, they have the same homology). The idea (see [6]) to compute the homology is to use the Mayer-Vietoris sequence on the decomposition :

$$M = (M - \overset{\circ}{N}) \cup_{\partial N} N.$$

We shall first prove the following two homological results :

Lemma 3.6. *The homology of the pair (H_ν, Σ) is given by :*

$$H_2(H_\nu, \Sigma) = L_\nu, \qquad H_3(H_\nu, \Sigma) \cong \mathbb{Z}, \qquad H_k(H_\nu, \Sigma) = 0 \text{ if } k \neq 2, 3.$$

Proof. Using the long exact sequence for the pair (H_ν, Σ), we obtain the result by making the following observations :

(i) $H_2(\Sigma) \to H_2(H_\nu)$ is the zero map, because $H_2(H_\nu) = 0$.

(ii) $H_1(\Sigma) \overset{u}{\twoheadrightarrow} H_1(H_\nu)$, so that $H_2(H_\nu, \Sigma) = L_\nu$ the kernel of u, because the curves in ν bound discs in H_ν, by a dimension argument.

■

Lemma 3.7. *We have* $H_2(\partial N) \cong L_{\alpha\beta} \oplus L_{\beta\gamma} \oplus L_{\alpha\gamma}$*, as well as* $H_\bullet(Y, \Sigma) \cong H_\bullet(H_\alpha, \Sigma) \oplus H_\bullet(H_\beta, \Sigma) \oplus H_\bullet(H_\gamma, \Sigma)$*. Moreover, for the homology of* N*, we have :*

$$H_1(N) \cong H_1(\Sigma)/(L_\alpha + L_\beta + L_\gamma), \quad H_2(N) \cong \ker[L_\alpha \oplus L_\beta \oplus L_\gamma \to H_1(\Sigma)], \quad H_3(N) \cong \mathbb{Z}^2.$$

Proof. For $H_\bullet(Y, \Sigma) \cong H_\bullet(H_\alpha, \Sigma) \oplus H_\bullet(H_\beta, \Sigma) \oplus H_\bullet(H_\gamma, \Sigma)$, this is analogous to the fact that the homology functor sends a wedge to the direct sum, by using the Mayer-Vietoris sequence in relative homology for the decomposition $Y = H_\alpha \cup_\Sigma H_\beta \cup_\Sigma H_\gamma$.

For $H_2(\partial N)$, we have $\partial N = \partial X_1 \amalg \partial X_2 \amalg \partial X_3$, so the computation boils down to calculation of $H_2(\partial X_i)$. From the Heegaard splitting $\partial X_i = H_\mu \cup_\Sigma H_\nu$, we can apply Mayer-Vietoris again to obtain $H_2(\partial X_i) = L_\mu \cap L_\nu$, because $H_2(H_\mu) = H_2(H_\nu) = 0$, so that $H_2(\partial X_i) = \ker[H_1(\Sigma) \to H_1(H_\mu) \oplus H_1(H_\nu)]$.

For the homology of N, it is the same as the homology of Y. Use the long exact sequence for the pair (Y, Σ), and use that the map $H_2(\Sigma) \to H_2(Y)$ is zero. ■

Now, we can compute the homology of the whole manifold M :

Proposition 3.8. *The homology of* M *is given by :*

$$H_1(M) \cong H_1(\Sigma)/(L_\alpha + L_\beta + L_\gamma), \quad H_2(M) \cong L_\alpha \cap (L_\beta + L_\gamma)/(L_{\alpha\beta} + L_{\alpha\gamma}), \quad H_3(M) \cong L_{\alpha\beta\gamma}.$$

Proof. It is just a matter of plugging everything into the Mayer-Vietoris sequence for the decomposition $M = (M - \mathring{N}) \cup_{\partial N} N$. Note that $M - \mathring{N}$ is a disjoint union of handlebodies, and ∂N is its boundary. ■

For an example, recall the trisection diagram for $\mathbb{S}^2 \times \mathbb{S}^2$ from figure 21. It is immediate that $L_\alpha + L_\beta + L_\gamma = H_1(\Sigma)$ and $L_{\alpha\beta\gamma} = 0$, so that $H_1(\mathbb{S}^2 \times \mathbb{S}^2) = H_3(\mathbb{S}^2 \times \mathbb{S}^2) = 0$. Next, we see that $L_\alpha \cap (L_\beta + L_\gamma) = L_\alpha$, and $L_{\alpha\beta} = L_{\alpha\gamma} = 0$, so that $H_2(\mathbb{S}^2 \times \mathbb{S}^2) \cong L_\alpha \cong \mathbb{Z}^2$.

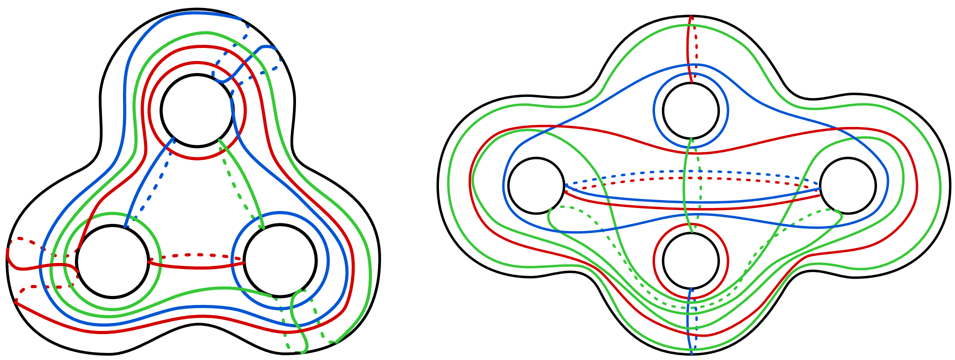
For more invariants, there is also a way to compute the intersection form of M, simply by counting the intersection numbers of pairwise choices of curves on the diagram. See [5, 6] for the details.

At last, recall that the handle decomposition (proposition 2.13) implied that $\pi_1(M)$ had a presentation with k generators and $g - k$ relations. There is a way to compute a full presentation for the fundamental group by using the trisection diagram (thanks to Delphine for that explanation). Recall that, given a trisection fiagram, the manifold associated to it was obtained by gluing 3- and 4-handles to the spine of the trisection, that is, the union of the three 3-dimensional handlebodies. Gluing those 3- and 4-handles doesn't change the fundamental group, so that it is uniquely determined by the one of that spine.

Taking one first handlebody, say, H_{13}, one has g generators. Gluing another one, say, H_{12}, we obtain relations accordingly to meridian discs. This means that we obtain a relation given by the crossings of the blue curves with the successive red curves, with the generator or its inverse determined by the framings. We do the same for H_{23}, and obtain relations accordingly.

Note that we can always consider a diagram in standard position as in figure 17, so that $g-k$ relations are killed by the red-blue relations. Therefore, we are left with k generators and relations given by the red-green crossings.

For trivial examples, one can verify that we obtain the correct fundamental groups for the diagrams in figure 21. For less trivial examples, here are diagrams taken from [1] :



(a) (b)

Figure 22. (a) The spin lens space $\mathcal{S}(L(2,1))$. (b) $\mathbb{T}^2 \times \mathbb{S}^2$.

We check that the procedure indeed produces $\pi_1 \mathcal{S}(L(2,1)) \cong \mathbb{Z}/2$ and $\pi_1(\mathbb{T}^2 \times \mathbb{S}^2) \cong \mathbb{Z}^2$.

Some people are hoping that this may be a first step in the right direction towards (dis)proving the smooth Poincaré conjecture. Here are some still-open questions in that regards :

- What *other* topological invariants can a trisection diagram allow to compute ?
- Can trisections define *new* invariants ? In particular, is it possible to describe an invariant that distinguishes between smooth structures on $\mathbb{S}^4$? (Note that g_T is one such smooth invariant, but computing it can be very difficult, especially in those exotic scenarii. See [14] for another step in that direction.)

One last big conjecture is whether the Haken lemma (see [12]) for Heegaard splittings also holds for trisections :

Conjecture 3.9. (Additivity conjecture, Lambert-Cole–Meier, see [16]) *Let $M = M_1 \# M_2$ be a connected-sum of two 4-manifolds. Then, any trisection of M is the connected-sum of two trisections of M_1 and M_2. That is : $g_T(M_1 \# M_2) = g_T(M_1) + g_T(M_2)$.*

This has an important corollary :

Corollary 3.10. (Lambert-Cole–Meier) *If conjecture 3.9 is true, then g_T is a topological invariant. In particular, there are no exotic versions of the following manifolds : $\mathbb{S}^4$, $\mathbb{CP}^2$, $\mathbb{S}^1 \times \mathbb{S}^3$, $\mathbb{S}^2 \times \mathbb{S}^2$, $\mathbb{CP}^2 \# \mathbb{CP}^2$ and $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$.*

This follows from the following theorem of Wall and Gompf, see [11, 31] :

Theorem 3.11. (Gompf) *If M_1 and M_2 are homeomorphic but not diffeomorphic, then there exists k such that $M_1 \# S(k)$ and $M_2 \# S(k)$ are diffeomorphic, where $S(k) = \#^k(\mathbb{S}^2 \times \mathbb{S}^2)$.*

3.3 Trisecting more manifolds : going further

Throughout this work, we have been dealing with *balanced* trisections. That is, each sector, as a 4-dimensional handlebody, had the same genus. However, contrary to Heegaard splittings where both handlebodies *had to* be of the same genus, there is no need to make this assumption here :

Definition 3.12. *Given a 4-manifold M, an* ***unbalanced*** $(g; k_1, k_2, k_3)$*-trisection is a decomposition $M = X_1 \cup X_2 \cup X_3$ with :*

(i) X_i is diffeomorphic to $\mathcal{Z}_k = \natural^{k_i}(\mathbb{S}^1 \times \mathbb{D}^2)$, with $k_i \leqslant g$.

(ii) $H_{ij} = X_i \cap X_j$ is diffeomorphic to $\mathcal{H}_g = \natural^g(\mathbb{S}^1 \times \mathbb{D}^2)$.

(iii) $X_1 \cap X_2 \cap X_3$ is diffeomorphic to Σ_g.

Note that each $\partial X_i = H_{ij} \cup H_{ik}$ is still a Heegaard splitting, so that we have, by Waldhausen–Haken :

$$\partial X_i = H_{ij} \cup H_{ik} = Y^+_{k_i,g} \cup Y^-_{k_i,g}.$$

The stabilization operation needs not be symmetric anymore, and we may talk about an i-stabilization when adding a handle to the sector X_i. This means that i-stabilization has the effect of transforming a $(g; k_1, k_2, k_3)$-trisection into a $(g + 1; k_1 + \delta_{i1}, k_2 + \delta_{i2}, k_3 + \delta_{i3})$-trisection. In particular, the balanced stabilization is the process of performing a 1-, a 2- and a 3-stabilization. Note that the order in which those are made doesn't matter.

Note that existence of unbalanced trisections is implied by the particular case of the balanced ones, and that the Gay–Kirby still holds, for we can always stabilize an unbalanced trisection to make it balanced again. The diagrammatic aspects of unbalanced trisections also hold in this context, but allow for more leeway since we don't ask anymore that any selection of two colors is not for the same number of summands of $\mathbb{S}^1 \times \mathbb{S}^2$ in ∂X_i.

At last, the theory of trisections can be extended to more general cases. For instance :

- *Relative trisections* are trisections of manifolds with boundary, see [2, 3].
- For trisections of *non-orientable* manifolds, see [24].
- For *bridge trisections*, that is, how trisections behave with knotted surfaces in the 4-manifold, see [21, 22].

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