TRISECTIONS OF 4-MANIFOLDS

Anthony SAINT-CRIQ
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__Context:__ M^4 is a smooth, closed, connected and oriented 4-manifold.

__Goal:__
- give a hint towards what trisections are;
- give the statement and the philosophy behind the proofs of the existence and uniqueness of trisections;
- give a diagramatic / combinatorial / 2-dimensional description of trisections;
- give some examples, mainly $\mathbb{CP}^2$.

__Definition:__ Given $k \leq g$ two integers, a (g, k)-trisection of M is a decomposition $M^4 = X_1 \cup X_2 \cup X_3$ with

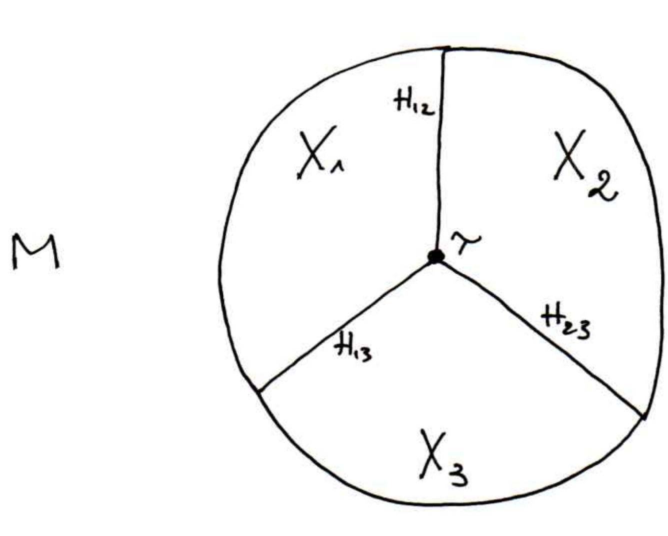
(1) X_i is diffeomorphic to $\natural^k (\mathbb{S}^1 \times \mathbb{D}^3)$ and smoothly embedded in M^4;

(2) $H_{ij} = X_i \cap X_j$ is diffeomorphic to $\natural^g (\mathbb{S}^1 \times \mathbb{D}^2)$

(3) $\Sigma = X_1 \cap X_2 \cap X_3$ is diffeomorphic to Σ_g.

The integer g is called the genus of the trisection, and Σ is the trisecting surface.

A trisection can be represented schematically as:



We can already make a few comments:

- $\partial X_i = H_{ij} \cup H_{ik}$ is a Heegaard splitting. In particular, from the Waldhausen – Haken theorem that classifies the splittings of $\#^k(\mathbb{S}^1 \times \mathbb{S}^2)$, we see that

$$\partial X_i = H_{ij} \cup H_{ik} = Y_{k,g}^+ \cup Y_{k,g}^-$$

is the standard splitting, meaning that $k \leq g$ is necessary.

- In dimension 3, Poincaré duality provided $\chi(N^3) = 0$, so that somehow this brought no information whatsoever. However, in dimension 4, by the inclusion-exclusion principle, we obtain:

$$\chi(M) = 2 + g - 3k,$$

so that once we know the genus, k is fixed.

We will see throughout the talk how analogous to Heegaard splittings trisections are. First, let us describe the trivial splitting of $\mathbb{S}^4$, so that it is a straight generalization of the genus zero splitting of $\mathbb{S}^3$; embed $\mathbb{S}^4 \subset \mathbb{C}^2 \times \mathbb{R}^3$. Then, define:

$$X_i = \{(z, x) \in \mathbb{S}^4 \,/\, z = re^{i\theta} \text{ with } \theta \in \left[\frac{2i\pi}{3}, \frac{2(i+1)\pi}{3}\right]\}$$

Then we have $X_i \cong \mathbb{D}^4$, $H_{ij} \cong \mathbb{D}^3$, $\Sigma \cong \mathbb{S}^2$ and $\mathbb{S}^4 = X_1 \cup X_2 \cup X_3$.
It turns out that by work from Laudenbach and Poénaru [LP], we have a result from Heegaard splittings that also holds for trisections:

<u>Lemma</u>: if M has a genus 0 trisection, then it is difeomorphic to $\mathbb{S}^4$.

Before we give hints about the proof of existence of trisections, there is a very useful result that relates them with handle decompositions: any (g,k)-trisected manifold has a handle decomposition with $1 : k : g-k : k : 1$ - handles. This has some consequences:

(1) If M has a $(g,0)$-trisection, then M has a handle decomposition with no 1- and 3-handles, and so is simply-connected. This happens iff $\chi(M) = 2+g$.

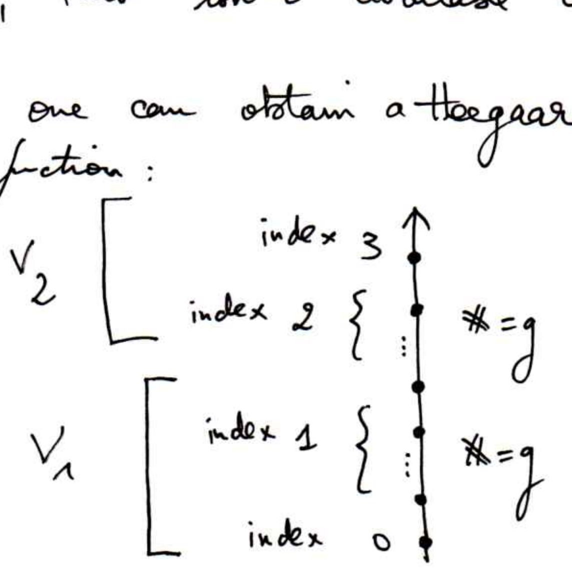
(2) If M has a (g,g)-trisection, then by [LP], we get that M is diffeomorphic to to $\#^g(S^1 \times S^3)$. This happens iff $\chi(M) = 2-2g$.

(3) $\pi_1 M$ has a presentation with k generators and $g-k$ relations, meaning that $k \geq \mathrm{rk}\,\pi_1 M$. In particular, we get a lower bound on the genus of the trisection:

$$g \geq \chi(M) - 2 + 3\,\mathrm{rk}\,\pi_1 M .$$

Originally, the proof of existence of Heegaard splitting relied on the existence of a triangulation (Moise). One could thicken the 1-skeleton of the triangulation to obtain a handlebody (with big genus, but we don't care). The other handlebody is obtained via the dual 2-skeleton. However, this isn't available anymore in dimension 4!

Instead, one can obtain a Heegaard splitting from an ordered Morse function:



$\rightsquigarrow$ genus g splitting

$N^3 = V_1 \cup V_2$

The idea for trisections is to construct a suitable Morse 2-function $f : M^4 \to \mathbb{D}^2$, that is, a generic smooth map that locally looks like a generic homotopy between Morse functions.

One can extract a trisection from such a function, called a trisecting Morse 2-function (and any trisection can be obtained this way).

To do so, we take a handle decomposition of M with $1 : i_1 : i_2 : i_3 : 1$ - handles, and we construct it step by step on each set of handles, eventually gluing the pieces together by applying homotopies when needed.

Towards a result of uniqueness now, we shall call two trisections $M = X_1 \cup X_2 \cup X_3 = Y_1 \cup Y_2 \cup Y_3$ equivalent if there is a difeomorphism $f : M \to M$ such that

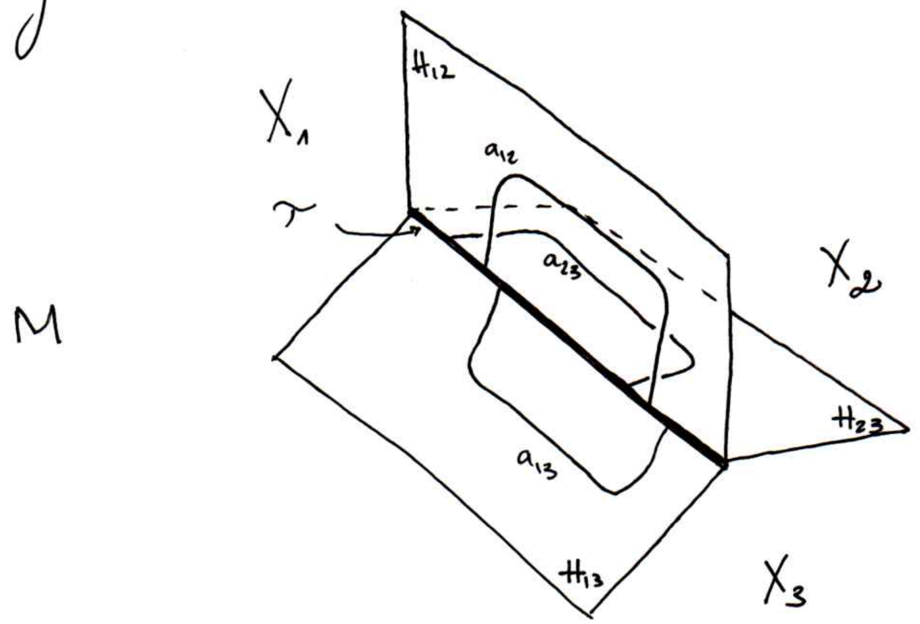
$$f(X_i) = Y_i \quad (\text{or } f(X_i) = Y_{\sigma(i)} \text{ for some } \sigma \in \mathfrak{S}_3).$$

Any trisections with diferent genera have no hopes to be equivalent. However, how about we modify the genus of each so that they match and we can ask the question again? This will give rise to an analogue to the Reidemeister - Singer theorem, by means of stabilizations of trisections.

Heuristically, we just increase the genus of each X_i by one.

<u>Definition</u>: take a trisected manifold $M = X_1 \cup X_2 \cup X_3$.
Take embedded arcs $a_{ij} \subset H_{ij}$ with endpoints in Σ, with all endpoints disjoint.
Consider regular neighborhoods $a_{ij} \subset N_{ij} \subset M$ of these arcs.

Generically, we are in the following situation, where the schematic representation has been thickened to display the arcs:



We want to add a handle to each X_i, but in a way that the X_i intersect nicely; define:

$$X_i' = (X_i \cup N_{jk}) - (\mathring{N}_{ij} \cup \mathring{N}_{ik}).$$

Then we check that this construction defines a new trisection $M = X_1' \cup X_2' \cup X_3'$.

What is the genus of this new trisection, assuming the previous one was a (g, k)-trisection? We know we have increased k by one, and the cheaty way to answer the question is to use:

$$\chi(M) = 2 + g - 3k.$$

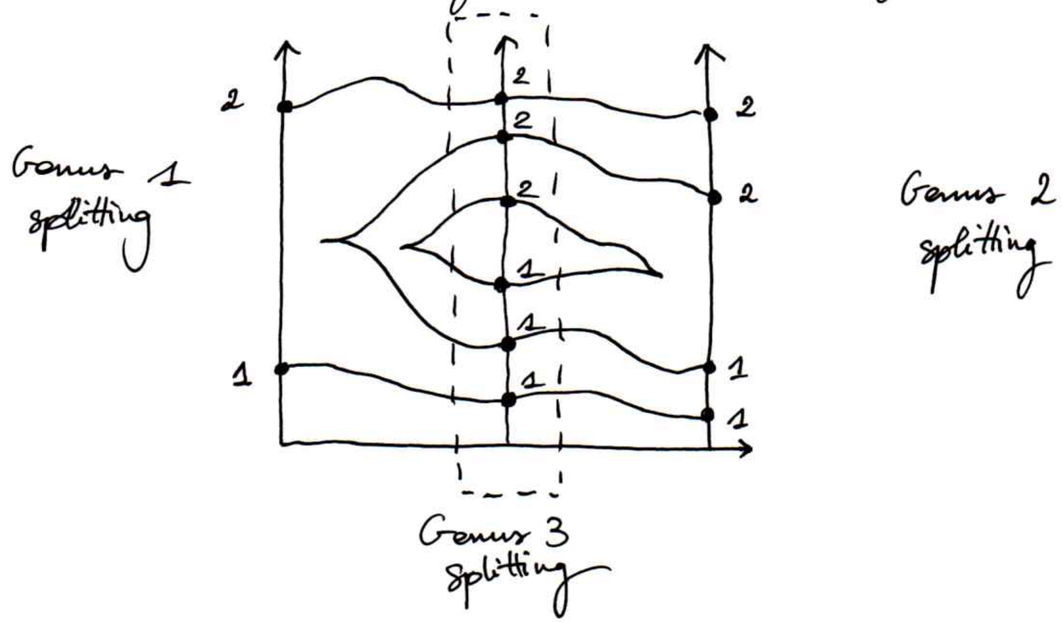
Necessarily, the newly-produced trisection is of type $(g+3, k+1)$.
One remark: this construction does not depend on the choices of the arcs if we ask them to be boundary parallel.

Now, this is the way in which there is uniqueness with trisections:

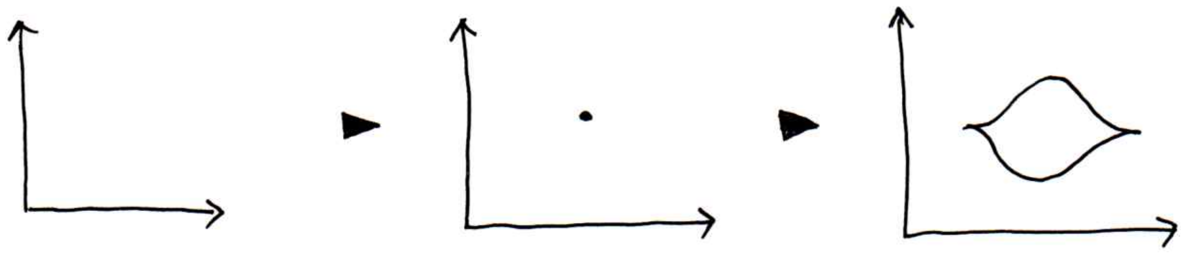
<u>Theorem</u>: (Gay-Kirby) Any two trisections of the same manifold can be made equivalent after a suitable number of stabilizations of each.

Now, again, the proof of the analogous result concerning Heegaard splittings relied on combinatorial methods; particularly, it was using the Hauptvermutung, assuring that any two triangulations can be made the same by cutting them sufficiently small.

In dimension four this result is no longer true, but we can continue with Morse- and Cerf-theoretic arguments. By Cerf theory, we know that Morse functions associated to Heegaard splittings can be homotoped one into the other by a nice homotopy:



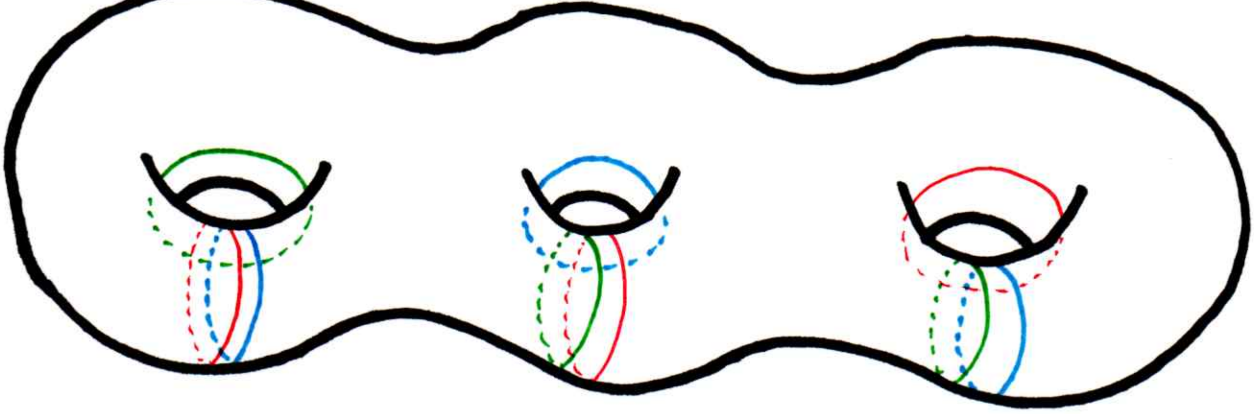
The idea is exactly the same for the Gay-Kirby theorem; take trisecting Morse 2-functions, and homotope them in a nice way:



Now that we know how to deal with trisections, we want a way to somehow visualise them; this will be achived through trisection diagrams.

Let $M = X_1 \cup X_2 \cup X_3$ be a trisected 4-manifold. Then, each H_{ij} being a $(3,1)$-handlebody, there is a system of g compressing curves on Σ such that compression along these curves gives the H_{ij}.

Call α the set of curves associated to H_{13}, β the one associated to H_{12}, and γ the one to H_{23}. Call those sets of curves the red curves, blue curves and green curves, respectively. For instance, after stabilization of the genus zero splitting of S^4, we obtain:

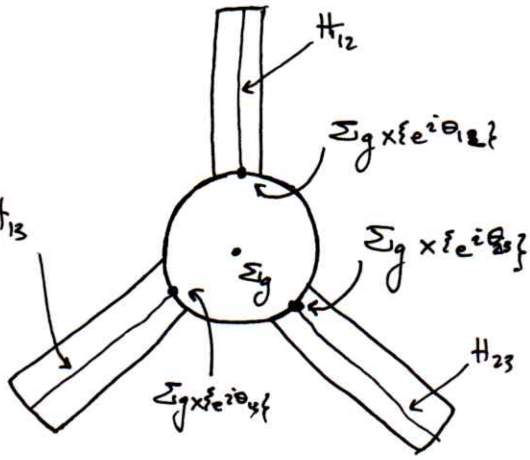


Now, conversely, start with three sets of compressing curves on Σ_g. (We sweep under the rug all problems related to standard Heegaard diagrams and incompatible sets of curves). How to produce a (hopefully unique) 4-manifold from such data?

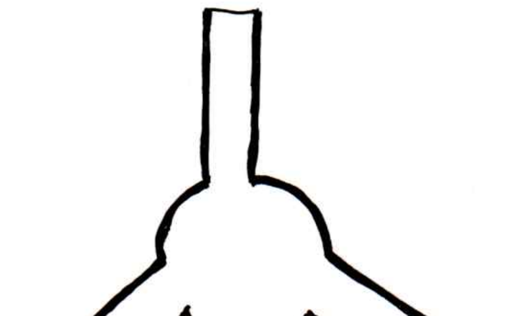
First, we can obtain H_α, H_β and H_γ by compression along the corresponding sets of curves. We may call them innocently:

$$H_\alpha = H_{13}, \quad H_\beta = H_{12}, \quad H_\gamma = H_{23}.$$

Now, we can glue these three handlebodies on $\Sigma_g \times \mathbb{D}^2$ by thickening them; this is represented as follows:

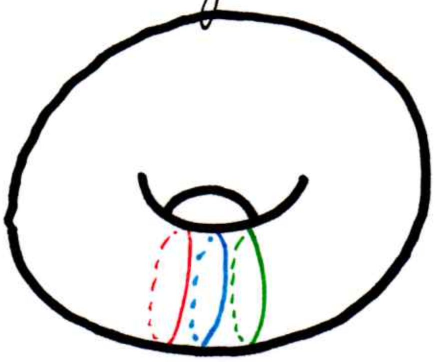


This manifold has three boundary components, all diffeomorphic to $\#^k(\mathbb{S}^1 \times \mathbb{S}^2)$. They are represented in bold:

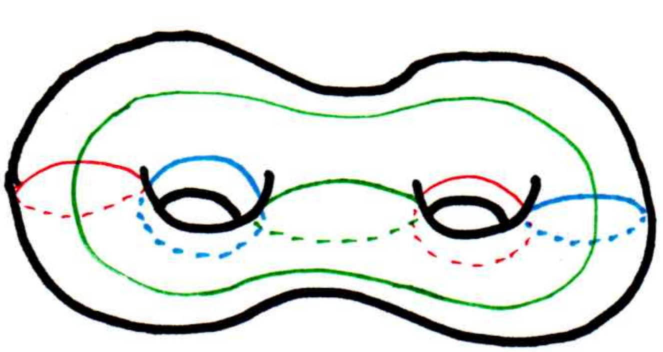


By [LP], we know that there is only one unique way to attach the $X_i = \natural^k(\mathbb{S}^1 \times \mathbb{D}^3)$ to this (i.e. the 3-handles), as well as the 4-handle.
This produces a unique 4-manifold along with a trisection of it.

Here are two more examples of trisections, given by a corresponding trisection diagram:



$\mathbb{S}^1 \times \mathbb{S}^3$, $(1,1)$



$\mathbb{S}^2 \times \mathbb{S}^2$, $(2,0)$

Now, we see that there is no reason that a trisection diagram associated to a trisection is unique, even up to a diffeomorphism of the surface. However, two diagrams representing the same trisection must be related by a diffeomorphism of the surface and a sequence of handle slides. Without getting into too much details, this summarizes roughly to:

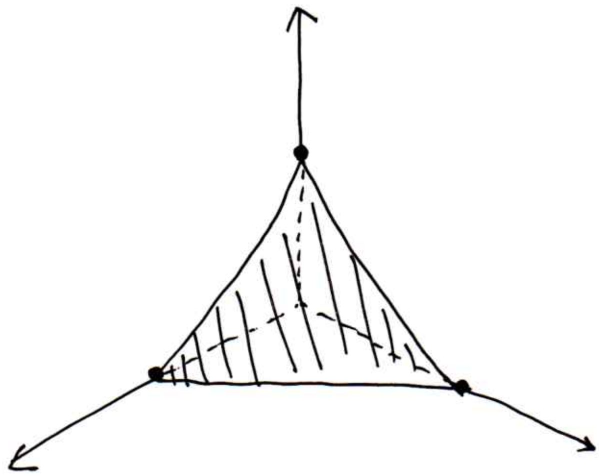
$$\{\text{trisected manifolds}\}/\text{equivalence} \leftrightarrow \{\text{diagrams}\}/\begin{array}{l}\text{diffeomorphisms}\\ \text{\& handle slides}\end{array}$$

At last, how to trisect the manifold of interest to us?
$\mathbb{CP}^2$ has a toric action, with moment map:

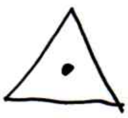
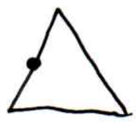
$$\mu([z_0 : z_1 : z_2]) = \left(\frac{|z_0|^2}{|z_0|^2+|z_1|^2+|z_2|^2}, \frac{|z_1|^2}{|z_0|^2+|z_1|^2+|z_2|^2}, \frac{|z_2|^2}{|z_0|^2+|z_1|^2+|z_2|^2} \right)$$

If we wanted, we could homotope this function to make it become a trisecting Morse 2-function. However, this is the painful way to do it...

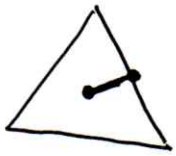
The clever way is to directly use this map. Take its image in $\mathbb{R}^3$:



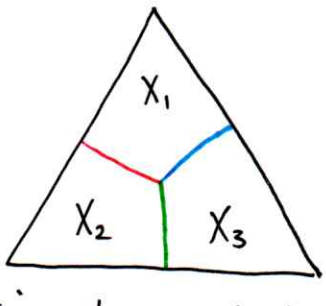
We can compute the different pre-images of points:

 $\mu^{-1} \cong \mathbb{T}^2$  $\mu^{-1} \cong \mathbb{S}^1$  $\mu^{-1} \cong *$

Somehow, the preimages of generic points in the interior collapse into a circle as we get close to the generic point on the edge. This provides that the preimage of the following line segment is a solid tori:

 $\mu^{-1} \cong \mathbb{S}^1 \times \mathbb{D}^2$

We can take the barycentric subdivision of the triangle, and preimage each piece:

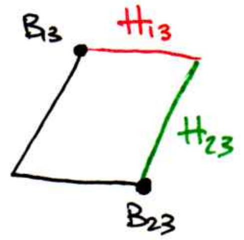


We can do computations to see that $X_i \cong \mathbb{D}^2 \times \mathbb{D}^2$ (do this in an affine chart $\{z_j = 1\}$). By the previous observations, we see that these piece together into a $(1,0)$-trisection of $\mathbb{CP}^2$.

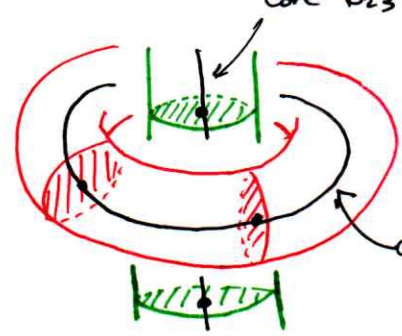
On the boundary ∂X_i, we have the standard splitting, which can be seen by:

$$X_i \cong \mathbb{D}^2 \times \mathbb{D}^2 \Rightarrow \partial X_i \cong \mathbb{S}^1 \times \mathbb{D}^2 \cup \mathbb{D}^2 \times \mathbb{S}^1 .$$

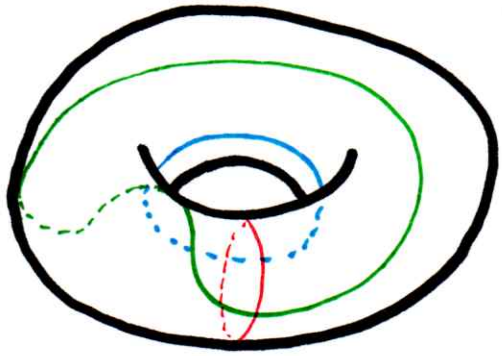
Each of these solid tori have a core that comes from the moment map:



$B_{13} \cup B_{23}$ is the Hopf link



At last, a theorem from Meier states that this trisection of $\mathbb{CP}^2$ is Stein, that is, each sector X_i is a $\{|f_1([z_0:z_1:z_2])|, \dots, |f_r(\,)| \leq 1\}$ for some holomorphic functions $f_1, \dots, f_r$.
The previous trisection has the following diagram:



Some last comments:

- There are only three trisections with genus 1, and only one with genus two: $\mathbb{S}^1 \times \mathbb{S}^3$, $\mathbb{CP}^2$, $\overline{\mathbb{CP}^2}$ and $\mathbb{S}^2 \times \mathbb{S}^2$.
- there are infinitely many with genus 3!
- Trisections allow for (some) computations: homology & intersection form.
- Open questions: Walhausen? Haken? New invariants?