CONNECTIONS, CHARACTERISTIC CLASSES AND CHERN–WEIL THEORY – MATEMALE 2023

ANTHONY SAINT-CRIQ

Abstract. Connections on a vector bundle should be introduced/reviewed. Parallel transport should be discussed, and in what sense does the curvature measure the failure of the parallel transport to only depend on the homotopy class of the path along which we transport.

There are invariants of bundles called characteristic classes. For instance, a hermitian bundle $E \to X$ of rank 2 on a 4-manifold is classified by its first Chern class $c_1(E) \in H^2(X; \mathbf{Z})$ and second Chern class $c_2(E) \in H^4(X; \mathbf{Z})$ (this no longer holds in dimension 5 and higher).

Chern-Weil theory expresses characteristic classes through expressions involving the curvature of a connection. We will need the expressions for the first and second Chern classes.

We let $\mathbf{k} = \mathbf{R}$ or $\mathbf{C}$. Recall that a rank r vector bundle over a smooth manifold X is a smooth surjection $p : E \to X$ with E a smooth manifold such that:

(1) for all $x \in X$, $E_x = p^{-1}(x)$ is a $\mathbf{k}$-vector space;
(2) any $x \in X$ has some neighborhood $U \ni x$ for which there exists a diffeomorphism $\psi_U : p^{-1}(U) \to U \times \mathbf{k}^r$ such that $p = \mathrm{pr}_1 \circ \psi$ on U;
(3) for any $x \in X$, the map $\psi_U : E_x \to \{x\} \times \mathbf{k}^r$ is a linear isomorphism.

We resume the situation in the following diagram:

$$\begin{tikzcd}
p^{-1}(U) \arrow[rr, "\psi_U"] \arrow[dr, "p"'] & & U \times \mathbf{k}^r \arrow[dl, "\mathrm{pr}_1"] \\
& U &
\end{tikzcd}$$

A smooth map $s : X \to E$ is called a **section** if $p \circ s = \mathrm{id}_X$. We denote as $\Gamma(E)$ the space of (global) sections of $E \to X$. A local section is a section defined on $U \subset X$, and we denote it as $s \in \Gamma(E, U)$.

Given two bundles $E \to X$ and $F \to Y$, a map $f : X \to Y$ is called a **bundle map** if there exists $g : E \to F$ such that the following diagram commutes:

$$\begin{array}{ccc} E & \xrightarrow{g} & F \\ \downarrow & & \downarrow \\ X & \xrightarrow[f]{} & Y \end{array}$$

An isomorphism of bundles is simply a bundle map such that f and g are homeomorphisms.

§1. Affine Connections on a Vector Bundle

Recall that $\Omega^k(X)$ denotes the space $\Gamma(\bigwedge^k T^*X, X)$ of differential k-forms. The special case $k = 0$ is $\Omega^0(X) = \mathscr{C}^\infty(X, \mathbf{R})$.

Definition 1. *Let $E \to X$ be a vector bundle over X. Set $\Omega^k(X, E) = \Gamma(E \otimes \bigwedge^k T^*X)$ to be the space of differential k-forms on X with values in E.*

The special case $k = 0$ is $\Omega^0(X, E) = \Gamma(E)$.

Definition 2. *Let $E \to X$ be a vector bundle over X. An **affine connection** on E is a continuous linear map $\nabla : \Omega^0(X, E) \to \Omega^1(X, E)$ such that for all $f \in \mathscr{C}^\infty(X, \mathbf{k})$ and $s \in \Omega^0(X, E)$, we have the Leibniz rule:*

$$\nabla(fs) = \mathrm{d}f \otimes s + f\nabla s.$$

Alternatively, $\nabla s \in \Omega^1(X, E) = \Gamma(T^*X \otimes E)$ can be thought of a vector bundle map $TX \to E$ (*i.e.* a fiber-preserving map which is linear on each vector space). This means that ∇ can be seen as a map $\nabla : \Gamma(E) \times \Gamma(TM) \to \Gamma(E)$ by setting $\nabla_\xi s = \nabla s(\xi)$ for $s \in \Gamma(E)$ and $\xi \in \Gamma(TX)$ a vector field.

In a local trivialization $U \subset X$, an affine connection is uniquely determined by the images $\nabla s_1, \dots, \nabla s_r$ of a basis of sections $(s_1, \dots, s_r)$. In particuler, for each i, there exist coefficients $\omega_{i,1}, \dots, \omega_{i,r}$ such that

$$\nabla s_i = \sum_{j=1}^{r} \omega_{i,j} s_j.$$

The coefficients $(\omega_{i,j})_{1 \leqslant i,j \leqslant r}$ make an $r \times r$ matrix of smooth 1-forms on U, and this is the local description of the connection.

An example is that of the exterior derivative. Let $E = X \times \mathbf{k}^r$ be the trivial rank r bundle, where $\Omega^0(X, E) = \mathscr{C}^\infty(X, \mathbf{k}^r)$. For $s : X \to \mathbf{k}^r$, we have $\mathrm{d}s : TX \to T\mathbf{k}^r = \mathbf{k}^r \times \mathbf{k}^r$ defined, for $(x, \xi) \in TX$, by: $\mathrm{d}s(x, \xi) = (s(x), \mathrm{d}_x s(\xi))$. This induces an element $\nabla s \in \Gamma(T^*X \otimes E) = \Omega^1(X, E)$. This connection is called the **standard flat connection** of rank r on X. The local description in terms of the $\omega_{i,j}$ follows from choosing a basis and inspecting the differential of those functions.

Proposition 3. *The space $\mathscr{A}(E)$ of affine connections is an affine space over the vector space $\Omega^1(X, \mathrm{End}(E))$.*

Proof. Partitions of unity (paracompacity) imply existence of affine connections (glue the standard flat connections on local trivializations). Next, if $\nabla_1, \nabla_2 \in \mathscr{A}(E)$ are two connections, then for any $f \in \mathscr{C}^\infty(X, \mathbf{k})$ and $s \in \Gamma(E, X)$, we have:

$$(\nabla_1 - \nabla_2)(fs) = \mathrm{d}f \otimes s + f\nabla_1 s - \mathrm{d}f \otimes s - f\nabla_2 s = f(\nabla_1 - \nabla_2)s.$$

That is: $\nabla_1 - \nabla_2$ is $\mathscr{C}^\infty(X, \mathbf{k})$-linear. As such: $\nabla_1 - \nabla_2 \in \operatorname{Hom}(\Gamma(E), \Gamma(T^* X \otimes E)) = \Gamma(E)^* \otimes \Gamma(T^* X \otimes E) = \Gamma(T^* X \otimes \operatorname{End}(E, E)) = \Omega^1(X, \operatorname{End}(E))$. ∎

A connection can be extended to form maps $\mathrm{d}^\nabla : \Omega^k(X, E) \to \Omega^{k+1}(X, E)$ for *any* k, by imposing the following Leibniz rule:

$$\mathrm{d}^\nabla(\eta \otimes s) = \mathrm{d}\eta \otimes s + (-1)^{\deg(\eta)} \eta \wedge \nabla s.$$

For $k = 1$, the map $\mathrm{d}^\nabla : \Omega^1(X, E) \to \Omega^2(X, E)$ satisfies the following identity, for $f \in \mathscr{C}^\infty(X, \mathbf{k})$ and $\eta \otimes s \in \Omega^1(X, E)$:

$$\mathrm{d}^\nabla(f\eta \otimes s) = \mathrm{d}f \wedge \eta \otimes s + f\mathrm{d}^\nabla(\eta \otimes s).$$

We have a sequence

$$\Omega^0(X, E) \xrightarrow{\mathrm{d}^\nabla} \Omega^1(X, E) \xrightarrow{\mathrm{d}^\nabla} \Omega^2(X, E) \xrightarrow{\mathrm{d}^\nabla} \cdots$$

which is, however, usually *not* exact (*i.e.* $\mathrm{d}^\nabla \circ \mathrm{d}^\nabla \neq 0$ in general).

Definition 4. *Let $K_\nabla : \Gamma(E) \to \Omega^2(X, E)$ be the map defined by $\mathrm{d}^\nabla \circ \mathrm{d}^\nabla$.*

Proposition 5. *The map K_∇ is $\mathscr{C}^\infty(X, \mathbf{k})$a-linear. Hence, it defines a section $K_\nabla \in \Omega^2(X, \operatorname{End}(E))$, which is called the* ***curvature*** *of the connection.*

Proof. Again, a direct computation gives:

$$K_\nabla(fs) = \mathrm{d}^\nabla(\mathrm{d}f \otimes s + f\nabla s) = \cancel{\mathrm{d}^2 f \otimes s} - \cancel{\mathrm{d}f \wedge \nabla s} + \cancel{\mathrm{d}f \wedge \nabla s} + fK_\nabla s.$$

To conclude, the same arguments as before work. ∎

If $\nabla s_i = \sum \omega_{i,j} s_j$ is the local expression of the connection in the basis $(s_1, \ldots, s_r)$, then one checks that $K_\nabla(s_i) = \sum \Omega_{i,j} s_i$, with the $\Omega_{i,j} = \mathrm{d}\omega_{i,j} - \sum \omega_{i,k}\omega_{k,j}$ making an $r \times r$ matrix $(\Omega_{i,j})_{1 \leqslant i,j \leqslant r}$ of 2-forms.

We now focus on the relation between parallel transport and curvature. Fix $\gamma : [0, 1] \to X$ a smooth path with $\gamma(0) = x$ and $\gamma(1) = y$, and let $s : [0, 1] \to E$ be a *section along* γ; that is, $p \circ s(t) = \gamma(t)$ for all $t \in [0, 1]$. The pull-back bundle $\gamma^* E$ is a vector bundle over $[0, 1]$ with fibers $(\gamma^* E)_t = E_{\gamma(t)}$. We also have a pull-back connection $\gamma^*\nabla$. Because $s \in \Gamma(\gamma^* E)$, it makes sense to look at $\gamma^*\nabla s$. The section s is called **parallel** if $\gamma^*\nabla s = 0$. This equation is an order one differential equation (by using Peetre's theorem on differential operators). By Cauchy–Lipschitz, this means that for any $v \in E_x$, the following problem has a unique solution:

$$\begin{cases} \gamma^*\nabla s = 0 \\ s(0) = v \end{cases}$$

This allows to define a linear map

$$\gamma /\!/_x^y : E_x \to E_y$$

by setting $\gamma /\!/_x^y(v) = s(1)$. The particular case of a loop $x = y$ is interesting; each loop γ induces an endomorphism $\gamma /\!/_x^x \in \operatorname{End}(E_x)$. The **holonomy** at the point $x \in X$ is the subgroup $\operatorname{Hol}_\nabla(E, x)$ of $\operatorname{End}(E_x)$ generated by those maps.

Theorem 6 ([AS53, Theorem 2])**.** *Let $x \in X$. The subgroup $\operatorname{Hol}^0_\nabla(E, x)$ of $\operatorname{Hol}_\nabla(E, x)$ generated by parallel transport along null-homotopic loops based at x is equal to the subgroup generated by the $K_\nabla(v, w)$ for $v, w \in T_x X$ (i.e. by the $(\Omega_{i,j}(v, w))_{i,j}$).*

That is: the curvature measures how dependant on the homotopy class of a loop the parallel transport is.

§2. Characteristic Classes: Chern Classes

The details of the constructions and the proofs can be found in [MS74].

Theorem 7. *There exist unique invariants $(c_k(E))_{k\in\mathbf{N}}$ of complex vector bundles $E \to X$ that satisfy the following properties:*

1. *Cohomology classes: $c_k(E) \in H^{2k}(X)$, and $c_0(E) = 1$.*
2. *Naturality: if $f : E \to F$ is a bundle map, then $c_k(E) = f^\star c_k(F)$.*
3. *Stability: if $\underline{\mathbf{C}}$ is the trivial bundle, then $c_k(E \oplus \underline{\mathbf{C}}) = c_k(E)$.*
4. *Whitney sum: $c_k(E \oplus F) = \sum_{i+j=k} c_i(E) \smile c_j(F)$.*
5. *Normalization: if $E \to \mathbf{CP}^n$ is the tautological bundle (i.e. $E_L = \{y \in \mathbf{C}^{n+1} \mid x \in L\}$), then $c_1(E)$ is a generator of $H^2(\mathbf{CP}^n)$.*

The total Chern class is the formal sum $c(E) = c_0(E) + c_1(E) + \cdots + c_n(E) \in H^*(X)$. If one interprets the Whitney sum as a product on the cohomology ring, then the formula reads as $c(E \oplus F) = c(E)c(F)$.

Example 8. *Let $E \to \mathbf{CP}^n$ be the tautological bundle. If $a = -c_1(E)$ denotes a generator of $H^2(\mathbf{CP}^n)$, then $c(T\mathbf{CP}^n) = (1 + a)^{n+1}$.*

A complex bundle comes with a natural orientation of the underlying real bundle. As such, the Euler class $e(E)$ is well-defined, and we have:

Proposition 9. *The top Chern class and the Euler class agree: $c_n(E) = e(E)$, where $n = \mathrm{rk}(E)$.*

This suggests that one can define the Chern classes recursively:

1. Set $c_n(E) = e(E)$ and $c_k(E) = 0$ for $k > n$.
2. Given $0 < k < n$, consider the bundle $F \to Y$, where $Y = E \smallsetminus 0_E$ and $F_y = E_x/\langle y\rangle$ with $y \in E_x \smallsetminus \{0\}$. Then the Gysin sequence
$$\cdots \longrightarrow H^k(X) \xrightarrow{\varphi^*} H^k(Y) \longrightarrow \cdots$$
for $\varphi : Y \to X$ gives an isomorphism for $k < 2n - 1$. Define $c_k(E) = (\varphi^*)^{-1}c_k(F)$, by noting that $\mathrm{rk}(F) = \mathrm{rk}(E) - 1$.

Theorem 10. *Complex vector bundles over a manifold X of dimension $\leqslant 4$ are uniquely determined by their rank and first two Chern classes. Moreover, for any choice of $n > 1$ and cohomology classes $\alpha \in H^2(X)$ and $\beta \in H^4(X)$, there exists a rank n complex bundle $E \to X$ with $c_1(E) = \alpha$ and $c_2(E) = \beta$.*

This comes from the universal bundle perspective; for any rank n complex bundle $E \to X$, there exists a bundle map $f : X \to BU(n)$ such that $E \cong f^*EU(n)$. Moreover, two bundles $f^*EU(n)$ and $g^*EU(n)$ are isomorphic if and only if the maps f and g are homotopic. As such, it suffices to compute $[X, BU(n)_4] = [X, K(\mathbf{Z}, 2) \times K(\mathbf{Z}, 4)] = H^2(X) \oplus H^4(X)$.

§3. Chern–Weil Theory

Consider any polynomial function $P : \mathscr{M}_n(\mathbf{C}) \to \mathbf{C}$ such that $P(AB) = P(BA)$, which we call an **invariant polynomial**. Examples include the trace, the determinant, or any coefficient of the characteristic polynomial, from the well-known $\chi_{AB} = \chi_{BA}$. In fact, these are, in some sense, the only examples. Indeed, let

$$\det(I_n + tA) = 1 + t\sigma_1(A) + \cdots + t^n \sigma_n(A).$$

Then, any invariant polynomial P can be expressed as $Q(\sigma_1, \ldots, \sigma_n)$ for some polynomial $Q \in \mathbf{C}[X_1, \ldots, X_n]$. The proof goes as follows:

(1) Trigonalize A to have arbitrarily small upper diagonal part. By continuity, this means that P only depends on the eigenvalues of A.
(2) P is therefore a symmetric function of those eigenvalues, and the classical theory of symmetric functions applies.

Recall that the curvature is $K_\nabla \in \Omega^2(X, \mathrm{End}(E))$. Given an invariant polynomial, one can look at the 2-form $P(K_\nabla) \in \Omega^2(X)$. By working in local coordinates $K_\nabla = (\Omega_{i,j}))$, one can prove that $\mathrm{d}P(K_\nabla) = 0$. In particular, this determines a De Rham cocycle, and thus an element $P(K_\nabla) \in H^*(X; \mathbf{C})$.

Proposition 11. *The element $P(K_\nabla)$ is independant of the connection ∇.*

Proof. Consider the induced bundle $\pi^\star E \to X \times I$, with $\pi : (x,t) \in X \times I \mapsto x$. If ∇_0 and ∇_1 are two connections on $E \to X$, then there are induced connections $\pi^\star \nabla_1$ and $\pi^\star \nabla_2$. For fixed t, let $\nabla = (1-t)\pi^\star \nabla_0 + t\pi^\star \nabla_1$, which is a connection on $\pi^\star E \to X \times I$. Then $P(K_{\nabla'_t}) \in H^*(X \times I; \mathbf{C})$.

Taking the inclusions $\imath_t : X \to X \times I$ to be $\imath_t(x) = (x,t)$, we see that $\imath_0^\star \nabla = \nabla_0$ and $\imath_1^\star \nabla = \nabla_1$, hence:

$$\imath_\varepsilon^\star P(K_\nabla) = P(K_{\nabla_\varepsilon}), \quad \varepsilon \in \{0,1\}.$$

But $\imath_0$ and $\imath_1$ are homotopic through $\imath_t$, so the claim follows. ∎

There is one part that we will not show: the bundle invariants $P(K_\nabla)$ are characteristic classes, *i.e.* they verify naturality among other conditions.

Theorem 12. *The cohomology class $\sigma_k(K_\nabla)$ is equal to $(2i\pi)^k c_k(E)$. In particular, we obtain the following expressions for the first two Chern classes:*

$$c_1(E) = \frac{i}{2\pi}\mathrm{Tr}(K_\nabla) \text{ and } c_2(E) = \frac{\mathrm{Tr}(K_\nabla^2) - \mathrm{Tr}(K_\nabla)^2}{8\pi^2}.$$

The two explicit formulas come from the following observations:

$$\det(I + A) = \det(\exp(\log(I + A))) = e^{\mathrm{Tr}(A - A^2/2 + A^3/3 + \cdots)}.$$

Proof. We first prove it for complex line bundles (which is an analogue of the Gauss–Bonnet theorem), then for any decomposable bundle. The general case follows from a universal bundle argument. For a bundle $E \to X$, denote as $\sigma(E)$ the following invariant:

$$\sigma(E) = \det(I + K_\nabla / 2i\pi) = \sum_k \frac{\sigma_k(K_\nabla)}{(2i\pi)^k} \in H^*(X; \mathbf{C}),$$

where we have chosen any connection ∇.

(1) If $E \to X$ is a line bundle, then $\sigma(E) = 1 + \sigma_1(K_\nabla)/2i\pi$. Now, the only $H^2(-)$-characteristic classes of real oriented plane bundles are multiples of the Euler class[1]. As such, there exists a universal constant $\alpha \in \mathbf{C}$ for which $\sigma_1 = \alpha c_1$ (because c_1 is also a multiple of e). To find the value of α, we use the Gauss–Bonnet theorem. Indeed: we can choose a suitable connection $\nabla_{\mathbf{R}}$ on the real plane bundle so that its connection and curvature matrices are skew-symmetric, *i.e.*:

$$\nabla_{\mathbf{R}} = \begin{bmatrix} 0 & -\omega_{1,2} \\ \omega_{1,2} & 0 \end{bmatrix} \text{ and } K_{\nabla_{\mathbf{R}}} = \begin{bmatrix} 0 & -\Omega_{1,2} \\ \Omega_{1,2} & 0 \end{bmatrix}$$

with $\Omega_{1,2} = \mathrm{d}\omega_{1,2}$. This can be done by fixing a metric so that the complex structure is rotation by $\pi/2$ and using the associated Levi–Civita connection. This induces a (complex) connection ∇ on the line bundle whose connection matrix is $(i\omega_{1,2}) \in \mathscr{M}_1(\mathbf{C})$ and curvature $(i\Omega_{1,2}) \in \mathscr{M}_1(\mathbf{C})$. For the bundle $T\mathbf{CP}^1 \to \mathbf{CP}^1$, whose real plane bundle is the tangent bundle of the sphere, we can apply Gauss–Bonnet to find $\alpha = 2i\pi$, by noting that $\Omega_{1,2}$ corresponds to the Riemannian curvature κ by the relation $\Omega_{1,2} = \kappa \mathrm{d}A$.

(2) If $E = E_1 \oplus \cdots \oplus E_p$ with each $E_i \to X$ a line bundle, then choose a connection ∇_i on E_i to form the connection $\nabla = \nabla_1 \oplus \cdots \oplus \nabla_p$ on E. The curvature matrix K_∇ will be block-diagonal $\mathrm{diag}(K_{\nabla_1}, \ldots, K_{\nabla_p})$, so that $\sigma(K_\nabla) = \sigma(K_{\nabla_1}) \cdots \sigma(K_{\nabla_p})$, and $\sigma(E_i) = c(E_i)$ by the previous case. This yields $\sigma(E) = c(E)$ from the Whitney sum formula.

(3) For the general case, we denote as $EU(n) \to BU(n)$ the universal rank n bundle, where $BU(n) = \mathscr{G}_n(\mathbf{C}^\infty)$. We can always assume that we are working over the m-skeletons $EU(n)_m \to BU(n)_m = \mathscr{G}_n(\mathbf{C}^m)$ for m large. For any rank n complex bundle $E \to X$, there exists a bundle map $f : X \to BU(n)$ such that $E \to X$ is isomorphic to $f^*EU(n)$. Therefore, it suffices to show the claim for $EU(n) \to BU(n)$, by naturality.

Now, if $\Gamma^1 \to \mathbf{CP}^m$ denotes the tautological line bundle, then we have:

$$c(\Gamma^1 \oplus \cdots \oplus \Gamma^1) = \sigma(K_{\Gamma^1 \oplus \cdots \oplus \Gamma^1}),$$

by the previous point. This bundle itself factors through $\Gamma^1 \oplus \cdots \oplus \Gamma^1 \xrightarrow{f} EU(n)_m$, so that:

$$f^*\sigma(K_{EU(n)_m}) = \sigma(K_{f^*EU(n)_m}) = c(f^*EU(n)_m) = f^*c(EU(n)_m).$$

Finally, we claim that f^* is injective from $H^{\bullet \leqslant 2m}(BU(n)_m)$ to $H^{\bullet \leqslant 2m}(\mathbf{CP}^m \times \cdots \times \mathbf{CP}^m)$, so that the result follows.

■

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[1] This is claimed in [MS74, Appendix C], referring to §14.

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