Planar digraphs of digirth five are 2-colorable

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January 20, 2016

Abstract

Neumann-Lara (1985) and Škrekovski conjectured that every planar digraph with digirth at least three is 2-colorable, meaning that the vertices can be 2-colored without creating any monochromatic directed cycles. We prove a relaxed version of this conjecture: every planar digraph of digirth at least five is 2-colorable. The result also holds in the setting of list colorings.

Keywords: Planar digraph, digraph chromatic number, dichromatic number, discharging.

1 Introduction

Let D be a digraph without cycles of length ≤ 2, and let G be the underlying undirected graph of D. A function $f\colon V(D) \to \{1, \dots, k\}$ is a *k-coloring* of the digraph D if $V_i = f^{-1}(i)$ is acyclic in D for every $i = 1, \dots, k$. Here we call the vertex set V_i *acyclic* if the induced

*Research was supported by a Digiteo postdoctoral scholarship.
†Supported in part by an NSERC Discovery Grant (Canada), by the Canada Research Chair program, and by the Research Grant P1–0297 of ARRS (Slovenia).
‡On leave from: IMFM & FMF, Department of Mathematics, University of Ljubljana, Ljubljana, Slovenia.

subdigraph $D[V_i]$ contains no directed cycles (but $G[V_i]$ may contain cycles). We say that D is *k-colorable* if it admits a k-coloring. The minimum k for which D is k-colorable is called the *chromatic number* of D, and is denoted by $\chi(D)$ (see Neumann-Lara [4]).

The following conjecture was proposed independently by Neumann-Lara [5] and Škrekovski (see [1]).

Conjecture 1.1. *Every planar digraph D with no directed cycles of length at most 2 is 2-colorable.*

The *digirth* of a digraph is the length of its shortest directed cycle (∞ if D is acyclic). It is an easy consequence of 5-degeneracy of planar graphs that every planar digraph D with digirth at least 3 has chromatic number at most 3.

There seems to be a lack of methods to attack Conjecture 1.1, and no nontrivial partial results are known. The main result of this paper is the following theorem whose proof is based on elaborate use of the (nowadays standard) discharging technique.

Theorem 1.2. *Every planar digraph that has digirth at least five is 2-colorable.*

The proof of Theorem 1.2 is deferred until Section 3. Actually, we will prove an extended version in the setting of list-colorings which we define next.

Let $\mathcal{C}$ be a finite set of colors. Given a digraph D, let $L : v \mapsto L(v) \subseteq \mathcal{C}$ be a *list-assignment* for D, which assigns to each vertex $v \in V(D)$ a set of colors. The set $L(v)$ is called the *list* (or the set of *admissible colors*) for v. We say D is *L-colorable* if there is an *L-coloring* of D, i.e., each vertex v is assigned a color from $L(v)$ such that every color class induces an acyclic set in D. A *k-list-assignment* for D is a list-assignment L such that $|L(v)| = k$ for every $v \in V(D)$. We say that D is *k-choosable* if it is L-colorable for every k-list-assignment L.

Theorem 1.3. *Every planar digraph of digirth at least five is 2-choosable.*

The rest of the paper is devoted to the proof of Theorem 1.3.

2 Unavoidable configurations

In this section we provide a list of unavoidable configurations used in the proof of Theorem 1.3. Orientations of edges are not important at this point, so we will consider only undirected graphs throughout the whole section.

We define a *configuration* as a plane graph C together with a function $\delta : U \to \mathbb{N}$, where $U \subseteq V(C)$, such that $\delta(v) \geq \deg_C(v)$ for every $v \in V(U)$. A plane graph G *contains* the configuration (C, U, δ) if there is a mapping $h : V(C) \to V(G)$ with the following properties:

(i) For every edge $ab \in E(C)$, $h(a)h(b)$ is an edge of G.

(ii) For every facial walk $a_1 \dots a_k$ in C, except for the unbounded face, the image $h(a_1) \dots h(a_k)$ is a facial walk in G.

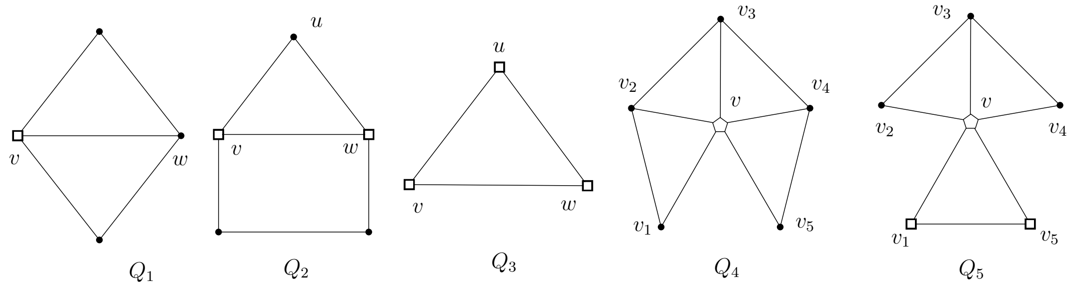

Figure 1: Configurations Q_1 to Q_5

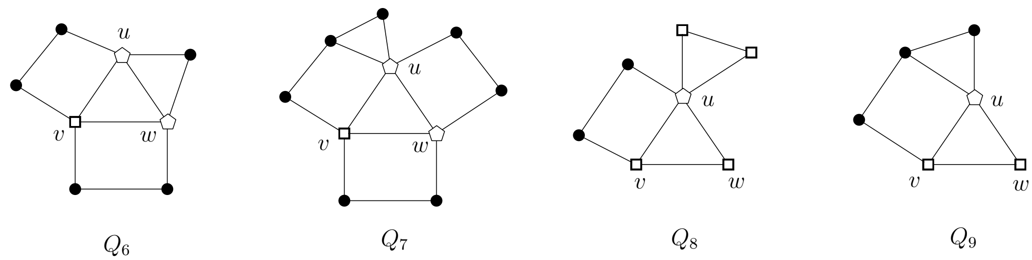

Figure 2: Configurations Q_6 to Q_9

(iii) For every $a \in U$, the degree of $h(a)$ in G is equal to $\delta(a)$.

(iv) h is *locally one-to-one*, i.e., it is one-to-one on the neighbors of each vertex of $V(C)$.

Configurations used in the paper are shown in Figures 1–4. The vertices shown as squares, pentagons, or hexagons represent the vertices in U and their values $\delta(u)$ are 4, 5, and 6, respectively. The vertices in $V(C) \setminus U$ are shown as smaller full circles. The configurations shown in these figures may contains additional notation that will be used in the proofs later in the paper.

The goal of this section is to prove the following theorem.

Theorem 2.1. *Every plane graph of minimum degree at least four contains one of the configurations $Q_1, \ldots, Q_{23}$ depicted in Figures 1–4.*

It suffices to prove Theorem 2.1 for a connected graph G. In the proof, we will use the following terminology. If v is a vertex of degree k in G, then we call it a *k-vertex*, and a vertex of degree at least k (at most k) will also be referred to as a *k^+-vertex* (*k^--vertex*). A neighbor of v whose degree is k is a *k-neighbor* (similarly k^+- and *k^--neighbor*). The *size* of a face f, denoted by $\deg(f)$, is the length of the facial walk bounding f. A face f that has size at least five is called a *major face*; if f has size at most 4 it is called a *minor face*. A *k-face* is a face of size k. By a *triangle* we refer to a face of size 3. An *r-s-t triangle* is a triangle whose vertices have degree r, s and t, respectively. An *r^+-s^+-t^+ triangle* is defined similarly. A triangle is said to be *bad* if it is a 5-4-4 triangle that is adjacent to at most two major faces.

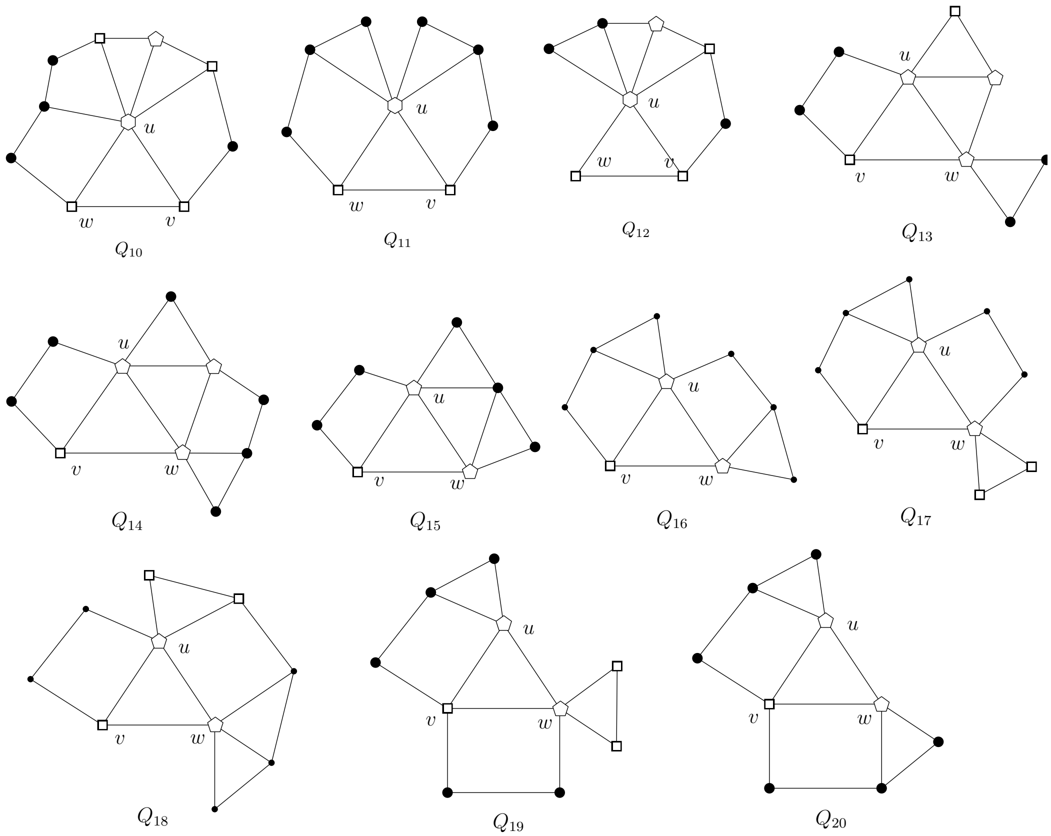

Figure 3: Configurations Q_{10} to Q_{20}

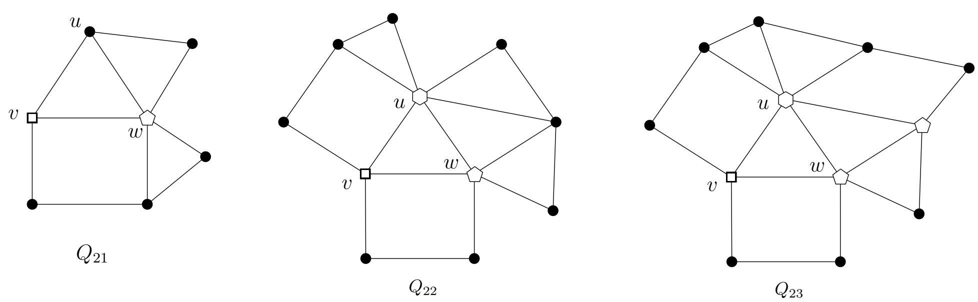

Figure 4: Configurations Q_{21} to Q_{23}

Proof of Theorem 2.1. The proof uses the discharging method. Assume, for a contradiction, that there is a plane graph G that contains none of the configurations $Q_1, \ldots, Q_{23}$ shown in Figures 1–4. Let G be a counterexample of minimum order. To each vertex or face x of G, we assign the *charge* of $c(x) = \deg(x) - 4$. A well-known consequence of Euler's formula is that the total charge is always negative, $\sum_{x \in V(G) \cup F(G)} c(x) = -8$. We are going to apply the following *discharging rules*:

R1: A k-face ($k \geq 5$) adjacent to r triangles sends charge of $(k-4)/r$ to each adjacent triangle.

R2: A 5-vertex v incident to exactly one triangle sends charge 1 to that triangle. A 5-vertex incident to exactly three triangles, sends charge $1/3$ to each triangle. A 5-vertex incident to exactly two triangles sends charge $1/2$ to each triangle unless (i) at least one of the triangles is a bad triangle in which case v sends charge of $3/5$ to each bad triangle and charge of $2/5$ to each non-bad triangle, or (ii) none of the triangles is bad, one of them is incident to a 4-vertex and the other is not, in which case v sends charge $2/3$ to the triangle with the 4-vertex and $1/3$ to the other triangle.

R3: A 6-vertex v incident to a 6-4-4 triangle T sends charge (i) $4/5$ to T if T is adjacent to exactly one major face, (ii) $3/5$ to T if T is adjacent to exactly two major faces, and (iii) $2/5$ to T if T is adjacent to three major faces.

R4: A 6^+-vertex v incident to a 6^+-5^+-5^+ triangle T sends charge $1/3$ to T unless T is a 6^+-5-5 triangle with a 6^+-5 edge incident to a 4-face and the 5-5 edge incident to a triangle, in which case v sends charge $7/15$ to T.

R5: A 6-vertex v incident to a 6-5-4 triangle T sends charge $1 - x - y$ to T, where x is the total charge sent to T by the rule R1 and y is the charge sent to T by the rule R2. Note that $y \geq 1/3$ by Claim 2 (below), so that v sends charge of at most $2/3$ to T.

R6: A 6-vertex v incident to a 6-4-7^+ triangle T sends charge $1/3$ to T.

R7: A 6-vertex v incident with a 6-6-4 triangle T sends charge $(5 - k)/10$ to T if T is adjacent to exactly k major faces ($k = 0, 1, 2, 3$).

R8: A 7^+-vertex v incident to a 7^+-4-4 triangle T sends charge $4/5$ to T.

R9: A 7^+-vertex v incident to a 7^+-5^+-4 triangle T sends charge $2/3$ to T.

R*: After rules R1–R9 have been applied, each triangle T with positive current charge equally redistributes its excess charge among those adjacent 5-5-4 triangles that have negative charge.

First, let us state two simple observations that will be used repeatedly.

Claim 1. *A major face sends charge of at least* $1/5$ *to every adjacent triangle.*

Claim 2. *A* 5*-vertex sends charge of at least* $1/3$ *to every incident triangle.*

We will make repeated use of Claims 1 and 2, which follow from R1 and R2, respectively, since the exclusion of Q_4 implies that a 5-vertex is incident with at most three triangles.

For $x \in V(G) \cup F(G)$, let $c^*(x)$ be the *final charge* obtained after applying rules R1–R9 and R* to G. We will show that every vertex and face has non-negative final charge. This will yield a contradiction since the initial total charge of -8 must be preserved.

4^+-faces: Since the charge of a 4^+-face only changes by rule R1, it is clear that every such face has a nonnegative final charge.

3-faces: Let $T = uvw$ be a 3-face. Then $c(T) = -1$. We will show that $c^*(T) \geq 0$. There are several cases to consider.

Case 1: *T* is a 4-4-4 triangle. This case is not possible since Q_3 is excluded.

Case 2: *T* is a 5-4-4 triangle. Let $\deg(u) = 5$ and $\deg(v) = \deg(w) = 4$. We may assume that u is incident to at least two triangles, for otherwise $c^*(T) \geq 0$. Since Q_1 and Q_5 are excluded, u is incident to precisely one other triangle T'. Since Q_9 is excluded, there is at least one major face incident with uv or uw. Since Q_1 and Q_2 are excluded, the face incident with the edge vw is major. If T is bad then T is adjacent to exactly two major faces and by R1 and R2, $c^*(T) \geq -1 + 1/5 + 1/5 + 3/5 = 0$. Otherwise, if T is not bad, then all its adjacent faces are major and $c^*(T) \geq -1 + 1/5 + 1/5 + 1/5 + 2/5 = 0$.

Case 3: *T* is a 6^+-4-4 triangle. Let $\deg(u) \geq 6$ and $\deg(v) = \deg(w) = 4$. Since Q_1 and Q_2 are excluded, T is adjacent to at least one major face. Since a major face always sends charge at least $1/5$ to an adjacent triangle, it follows by the rule R3 (if $\deg(u) = 6$) or R8 (if $\deg(u) \geq 7$) that $c^*(T) \geq 0$.

Case 4: *T* is a 5-5-4 triangle. Let $\deg(v) = 4$ and $\deg(u) = \deg(w) = 5$. We consider several subcases.

Subcase (a): T **is adjacent to at least two major faces.** In this case, by Claims 1 and 2, T receives total charge of at least $1/5 + 1/5 + 1/3 + 1/3 > 1$, which implies that $c^*(T) \geq 0$.

Subcase (b): T **is adjacent to no major faces.** First, suppose that all faces adjacent to T are 4-faces. Since Q_7 is excluded, each of u and w is incident to at most one other triangle besides T. If u (or w) is incident to no other triangle, then T receives a charge of 1 from u (or w) and $c^*(T) \geq 0$. Therefore, we may assume that each of u and w is incident to exactly two triangles. But now, the exclusion of Q_9 implies that neither u nor w is incident to a (bad) 5-4-4 triangle, so that each of them sends charge $1/2$ to T by the rule R2. Hence, $c^*(T) \geq 0$.

The remaining possibility (by exclusion of Q_1) is that the face incident to uw is a triangle and the faces incident to uv and vw are 4-faces. However, this gives the configuration Q_6.

Subcase (c): T **is adjacent to exactly one major face.** Since Q_1 is excluded, and using symmetry, we may assume that the three faces adjacent to T along edges uv, vw, wu have, respectively, one of the following three descriptions:

(D1) a 4-face, a 4-face, a major face;

(D2) a 4-face, a major face, a 4-face;

(D3) a 4-face, a major face, a triangle T'.

Recall that a 5-vertex is incident with at most three triangles, since Q_4 is excluded. If neither u nor w is incident with three triangles, then by rules R1 and R2, T receives charge of at least $1/5 + 2/5 + 2/5 = 1$, which implies that $c^*(T) \geq 0$. So we may assume that at least one these vertices is incident with three triangles. If the other is incident with at most two triangles neither of which is bad, then T receives charge of at least $1/5+1/3+1/2>1$, and $c^*(T)>0$. In cases (D1) and (D2), we will show that this holds.

In case (D1), we may assume by symmetry that u is incident with three triangles. By the exclusion of Q_{19} and Q_{20}, w is incident with at most two triangles neither of which is a 5-4-4 triangle, so that neither is bad. Thus, $c^*(T)\geq 0$ in this case.

In case (D2), if u is incident with three triangles then the exclusion of Q_{16} and Q_{17} implies that w is incident with at most two triangles neither of which is bad; and if w is incident with three triangles then the exclusion of Q_{16} and Q_{18} implies that u is incident with at most two triangles neither of which is bad. In each case, $c^*(T)\geq 0$.

It remains to consider case (D3). Note that neither T nor T' is a 5-4-4 triangle so neither is bad. So if one of u and w is incident with only these two triangles then $c^*(T)\geq 0$ as before. Thus, we may assume that u and w are each incident with exactly one triangle in addition to T and T', and T receives charge of at least $1/5+1/3+1/3=13/15$ by R1 and R2. Since Q_{21} and Q_{15} are excluded, it follows that G contains the configuration P_1 contained in Figure 5, where face F is not a triangle and face M is major. Clearly, $\deg(u_3)\geq 5$ since Q_1 is excluded. Note that T is the only 5-5-4 triangle adjacent to T', either clearly (if $\deg(u_3)\geq 6$) or by the exclusion of Q_{13} (if $\deg(u_3)=5$). Suppose first that F is a major face. Then T' receives in total a charge of at least $1/5+1/3+1/3+1/3=6/5$ after rules R1-R9 have been applied, and so T' sends charge of at least $6/5-1=1/5$ to T by rule R^*; thus, T receives a total charge of at least $13/15+1/5>1$, and $c^*(T)>0$. So we may assume that F is a 4-face. In this case, $\deg(u_3)\neq 5$ since Q_{14} is excluded, and so $\deg(u_3)\geq 6$. By R4, u_3 sends charge of $7/15$ to T'. Hence T' receives total charge of at least $1/3+1/3+7/15=17/15$ after rules R1-R9 have been applied, and so T' sends charge of at least $17/15-1=2/15$ to T by rule R^*; thus T receives total charge of at least $13/15+2/15=1$, and $c^*(T)\geq 0$.

Case 5: *T* is a 6-5-4 triangle. By the rule R5, T receives a total charge of 1 after the discharging rules have been applied. Hence, $c^*(T)\geq 0$.

Case 6: *T* is a 6-6-4 triangle.

Then T receives charge of at least $k/5$ in total from its k adjacent major faces by Claim 1, and $1-k/5$ in total from its two 6-vertices by R7. Hence, $c^*(T)\geq 0$.

Case 7: *T* is a 7^+-5^+-4 triangle. By rule R9, the 7^+-vertex incident to T sends charge of $2/3$ to T. By rules R2, R6 and R9, the 5^+-vertex sends charge of at least $1/3$ to T. Thus, T receives a total charge of at least 1, and $c^*(T)\geq 0$.

Case 8: *T* is a 5^+-5^+-5^+ triangle.

By rules R2 and R4, each 5^+-vertex incident with T sends charge of at least $1/3$ to T. Hence $c^*(T)\geq 0$.

4-vertices: If v is a 4-vertex, then $c(v)=0$. Since v neither receives nor gives any charge, we have $c^*(v)=0$.

5-vertices: Let v be a 5-vertex, so that $c(v)=1$. Assume the neighbors of v are $v_1,...,v_5$, ordered counter-clockwise. It follows from R2 that $c^*(v)\geq 0$ unless v is incident with exactly two triangles T_1 and T_2, each of which is bad, meaning that each is a 5-4-4 triangle adjacent to at most two major faces. Since Q_1 is excluded we may assume that $T_1=vv_1v_2$, $T_2=vv_3v_4$,

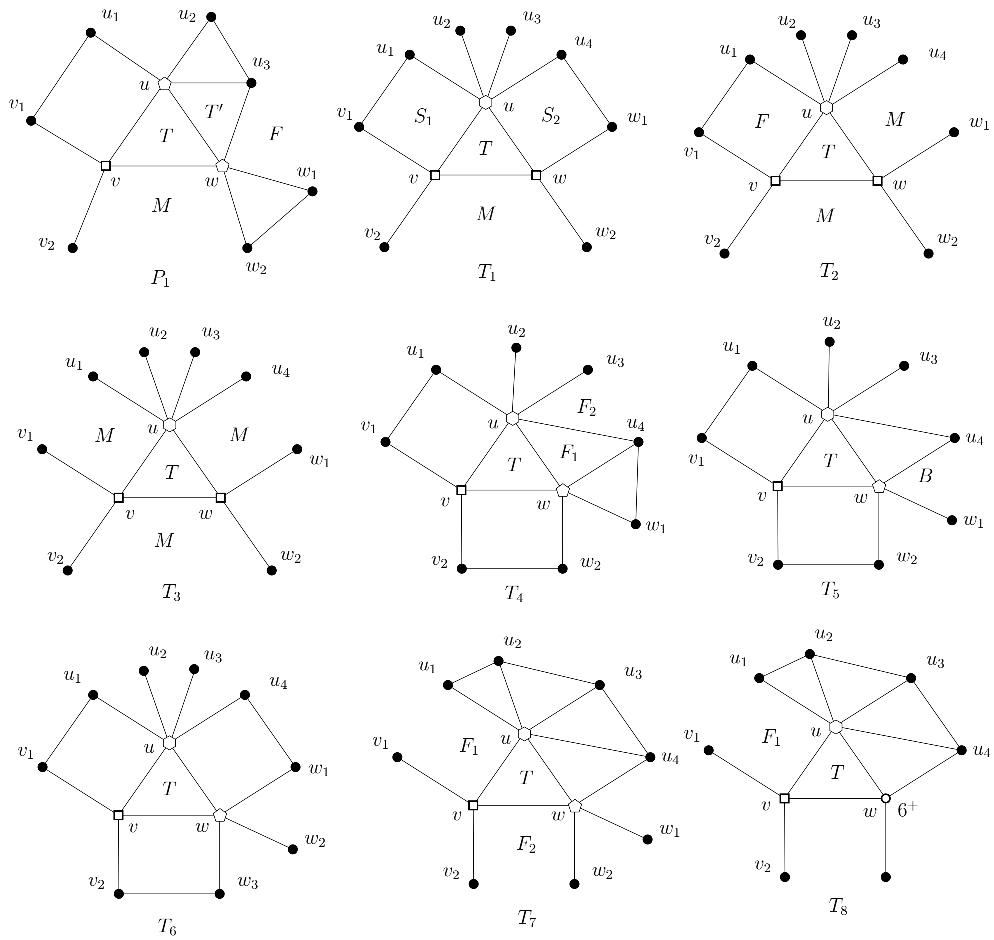

Figure 5: Discharging analysis

and each of T_1 and T_2 is adjacent to at least one 4-face. Since Q_2 is excluded, either the face containing the vertices v_2, v, v_3 is a 4-face or the faces containing the vertices v_4, v, v_5 and v_5, v, v_1 are both 4-faces. These are ruled out by the exclusion of Q_9 and Q_8, respectively.

6-vertices: We first use introduce some terminology, which we will also use for 7^+-vertices.

Let v be a d-vertex ($d \geq 6$). We will denote the neighbors of v by $v_1, ..., v_d$ in counterclockwise order, f_i will denote the face with edges vv_i and vv_{i+1} in its boundary (subscripts modulo d), and if f_i is a triangle then f_i' will denote the other face incident with the edge v_iv_{i+1}. Since Q_1 is excluded, if v is incident with d triangles then every neighbor of v is a 5^+-vertex and so v gives charge of $1/3$ to each incident triangle by R4; thus $c^*(v) \geq d-4-d/3 \geq 0$. So we may assume that v is incident with at least one non-triangular face.

If $f_i, ..., f_{i+l-1}$ are triangles and f_{i-1} and f_{i+l} are not, then we say that $F = (f_i, ..., f_{i+l-1})$ is an (l)-fan at v, and call $l+1$ the *edge-count* of F. If vv_i is an edge incident with no triangle, then we call (vv_i) a (0)-fan with edge count of 1. With this convention, the edge-counts of all fans at v sum to d. We say that an (l)-fan F is *good* if v gives charge of at most

$(l+1)(d-4)/d$ in total to the l triangles of F, and *bad* otherwise. If every fan at v is good, then v gives out charge of at most $d-4$ in total and $c^*(v) \geq 0$. Thus, we may assume that there exists at least one bad fan at v.

Now assume that v is a 6-vertex, so that $d = 6$, $c(v) = d - 4 = 2$, and $(d-4)/d = 1/3$. Note that a (0)-fan cannot be bad. We consider two cases.

Case 1: There is a bad (1)-fan at v. Let $F = (f_1)$ be a bad (1)-fan. This implies that v gives charge of more than $2/3$ to f_1. This can only happen when v gives charge of $4/5$ to f_1 by rule R3. This implies that f_1 is a 6-4-4 triangle incident with exactly one major face, which is necessarily f_1' since Q_1 and Q_2 are excluded, and f_6 and f_2 are 4-faces since Q_1 is excluded. If v is incident with at most two triangles, then v gives total charge of at most $4/5 + 4/5 < 2$, and $c^*(v) > 0$. So we may assume that v is incident with at least three triangles. Since Q_{11} is excluded, f_3 and f_5 are not both triangles. Therefore, we may assume, without loss of generality, that f_3 and f_4 are triangles, which are adjacent along the edge vv_4. Since Q_1 is excluded, v_4 is a 5^+-vertex. Thus, R3 does not apply to f_3 and f_4. The charge sent to each of f_3 and f_4 by rule R5 is at most $2/3$, and at most $1/2$ by any other rule. Since $4/5 + 2/3 + 1/2 < 2 = c(v)$, $c^*(v)$ is always positive unless v gives charge of more than $1/2$ to both f_3 and f_4 by R5. This implies that $\deg(v_3) = \deg(v_5) = 4$ and $\deg(v_4) = 5$. Since Q_{10} is excluded, f_5 is a major face. Thus, by Claim 1, f_5 sends charge of at least $1/5$ to f_4, and by Claim 2, v_4 sends charge of at least $1/3$ to f_4. Thus, v sends total charge of at most $1 - 1/5 - 1/3 = 7/15$ to f_4 by R5. Thus, the final charge sent by v is at most $4/5 + 2/3 + 7/15 < 2$.

Case 2: There is a bad (l)-fan at v, for some $l \geq 2$. Let $F = (f_1, ..., f_l)$ be a bad (l)-fan. We may assume that v gives at least as much charge to f_1 as to f_l. Since Q_1 is excluded, the vertices $v_2, ..., v_l$ are all 5^+-vertices, and so if $l \geq 3$ then v gives charge of $1/3$ to each of the faces $f_2, ..., f_{l-1}$ by rule R4. Since F is bad, v must give charge of more than 1 to f_1 and f_l together. Thus, we may assume that v gives charge of more than $1/2$ to f_1. Since f_1 is not a 6-4-4 triangle, v gives charge of more than $1/2$ to f_1 by the rule R5. This implies that $\deg(v_1) = 4$ and $\deg(v_2) = 5$.

If either f_6 or f_1' is a major face, then this face gives charge of at least $1/5$ to f_1 by Claim 1. As v_2 gives charge of at least $1/3$ by Claim 2, we have that v gives f_1 charge of at most $1 - 1/5 - 1/3 < 1/2$ by R5, a contradiction. Now, the exclusion of Q_1 implies that f_6 and f_1' are both 4-faces.

Note that since v is a 6-vertex, it cannot be incident to a bad triangle. If v_2 is incident with only two triangles, f_1 and f_2, then v_2 gives charge of at least $1/2$ to f_1 by R2. Thus, v gives to f_1 charge of at most $1 - 1/2 = 1/2$ by rule R5, a contradiction. Thus, v_2 is incident with three triangles. Since f_1' is a 4-face and Q_{21} is excluded, the face between f_1' and f_2' is not a triangle. It follows that f_2' is a triangle.

Since v gives charge of at most $2/3$ to each of f_1 and f_l, v gives charge of at most $(l+2)/3$ to the triangles of F in total. Suppose there is a (0)-fan F_0 at v. Then F and F_0 use $l+2$ edges between them, and they receive charge of at most $(l+2)/3$ from v since F_0 receives no charge. Since by Case 1 there is no bad (1)-fan at v, there cannot be another bad fan at v, since there are only $6 - l - 2 \leq 2$ unused edges. Thus, v gives charge of at most $(6-l-2)/3$

to the remaining good fan(s), and $c^*(v) \geq 2 - 6/3 = 0$.

Thus, we may assume that there is no (0)-fan at v, i.e., there are no two consecutive non-triangular faces around v. Since f_6 is a 4-face, it follows that f_5 is a triangle. By the exclusion of Q_{22}, f_3 is not a triangle. Therefore, f_4 is a triangle.

Since f_1 and f_2 are triangles and f_3 is not, we have that $l = 2$. Note that v gives charge of at most 2/3 to f_1, and recall that it gives charge of more than 1 to f_1 and f_2 together. Thus, v gives more than 1/3 charge to f_2. Since f_2 and f_2' are both triangles and Q_1 is excluded, v_3 is not a 4-vertex, and so $f_2 = vv_2v_3$ is a 6-5-5^+ triangle. In order for v to give charge of more than 1/3 to f_2 by R4, it must be that f_2 is a 6-5-5 triangle and the edge vv_3 is incident with a 4-face. In other words, $\deg(v_3) = 5$ and f_3 is a 4-face. But this gives the forbidden configuration Q_{23}, a contradiction. This completes the case when $d = 6$.

7^+-vertices: Let v be a d-vertex where $d \geq 7$. Then only the rules R4, R8 and R9 can apply to v. Let $F = (f_1, ..., f_l)$ be an (l)-fan at v. Recall that by the exclusion of Q_1, $v_2, ..., v_l$ are all 5^+-vertices. We now consider all the possible values of l.

If $l = 0$, then v gives no charge to F.

If $l = 1$, then v gives F charge of at most $4/5 = (2/5)(l + 1)$ by R8.

If $l = 2$, then v gives F charge of at most $2/3 + 2/3 = 4/3 = (4/9)(l + 1)$ by R9.

If $l = 3$, then v gives F charge of at most $2/3 + 1/3 + 2/3 = 5/3 = (5/12)(l + 1)$ by R4 and R9.

If $l \geq 4$, then v gives F charge of at most $(l + 2)/3 \leq (2/5)(l + 1)$ by R4 and R9.

Since $2/5 < 5/12 < 4/9$ and $4/9 < (d - 4)/d$ if $d \geq 8$, there are no bad fans in this case and hence $c^*(v) \geq 0$. Thus, we may assume that $d = 7$, and $c(v) = d-4 = 3$, $(d-4)/d = 3/7$. Since $5/12 < 3/7 < 4/9$, only a (2)-fan can be bad. If there are two bad (2)-fans at v then there is also a (0)-fan at v, and v gives charge of at most $4/3 + 4/3 + 0 < 3$. If there is only one bad (2)-fan, then v gives charge of at most $4/3 + 5/3 = 3$, when the remaining edges at v form a (3)-fan. In all cases, $c^*(v) \geq 0$.

□

3 Reducibility

Let us first introduce some notation that will be used in the rest of the paper. If either uv or vu is an arc in a digraph D, we say that uv is an *edge* of D. We will consider a planar digraph D, its underlying graph G, and a 2-list-assignment L, where $L(v) \subseteq \mathcal{C}$ and $|L(v)| = 2$ for every $v \in V(D)$. Let $\phi : V(D) \to \mathcal{C}$ be a function such that $\phi(v) \in L(v)$ for each vertex v. A *color-i cycle* is a directed cycle in D whose every vertex is colored with color i, for $i \in \mathcal{C}$. Recall that ϕ is an *L-coloring* if there is no color-i cycle for any $i \in \mathcal{C}$. When we speak of vertex *degrees*, we always mean degrees in G. For the digraph D, the *out-degree* and the *in-degree* of a vertex v are denoted by $d^+(v)$ and $d^-(v)$, respectively.

If D is a digraph drawn in the plane and C is a configuration (which is an undirected graph), we say that D *contains* the configuration C if the underlying undirected graph G of D contains C. A configuration C is called *reducible* if it cannot occur in a minimum counterexample to Theorem 1.3. Showing that every planar digraph D of minimum degree at least 4 and with digirth at least five contains a reducible configuration will imply that every such digraph is 2-choosable.

Throughout this section, we assume that D is a planar digraph with digirth at least five that is a counterexample to the theorem with a 2-list-assignment L such that every proper subdigraph of D is L-colorable. In most statements, we will consider a special vertex $v \in V(D)$, and we will assume that $L(v) = \{1, 2\}$. The following lemma shows that the minimum degree of D is at least four and that each vertex has in-degree and out-degree at least two.

Lemma 3.1. *Let $v \in V(D)$. Then in every L-coloring of $D - v$, each color in $L(v)$ appears at least once among the out-neighbors and at least once among the in-neighbors of v. Consequently, every $v \in V(D)$ has $d^+(v) \geq 2$ and $d^-(v) \geq 2$; therefore, D contains no 3^--vertices and every 4-vertex has $d^+(v) = d^-(v) = 2$.*

Proof. Suppose that a color $c \in L(v)$ does not appear among the outneighbors of v in an L-coloring of $D - v$. Then coloring v with c gives an L-coloring of D since a color-c cycle would have to use an outneighbor of v. The same contradiction is obtained if a color in $L(v)$ does not occur among the in-neighbors, and this completes the proof. □

Having an L-coloring ϕ of a subdigraph $D - u$ ($u \in V(D)$), we may consider coloring u with a color $i \in L(u)$. Since D is not L-colorable, this creates a color-i cycle; let $C_i = C_i(u)$ be such a cycle. Such cycles will always be taken with respect to a partial coloring ϕ that will be clear from the context. If $L(u) = \{a, b\}$, then $C_a(u)$ and $C_b(u)$ are disjoint apart from their common vertex u. Since D is drawn in the plane, these cycles cannot cross each other at u, and we say that they *touch*.

Lemma 3.2. *Let v be a vertex incident to a triangle $T = vwu$, let ϕ be an L-coloring of $D - v$, and let $i \in L(v)$. Then $C_i(v)$ cannot contain both edges vu and vw.*

Proof. Since $C_i(v)$ is directed, we may assume that $uv, vw \in E(D)$. Since D has digirth greater than three, this implies that $uw \in E(D)$. But then we have a color-i cycle in $D - v$ consisting of the path $C_i(v) - v$ and the arc uw, a contradiction. □

Lemma 3.3. *Let v be a vertex incident to a 4-cycle $S = vwux$, let ϕ be an L-coloring of $D - v$ and let $i \in L(v)$. Then $C_i(v)$ cannot contain all three edges ux, xv and vw.*

Proof. Suppose that $C_i(v)$ contains the edges ux, xv and vw. Since $C_i(v)$ is directed, we may assume that $ux, xv, vw \in E(D)$. Since D has digirth greater than four, this implies that $uw \in E(D)$, and we have a color-i cycle through the arc uw in $D - v$, a contradiction. □

The next lemma shows some restrictions on the colors around a 4-vertex that is contained in a triangle. Recall our assumption that $L(v) = \{1, 2\}$.

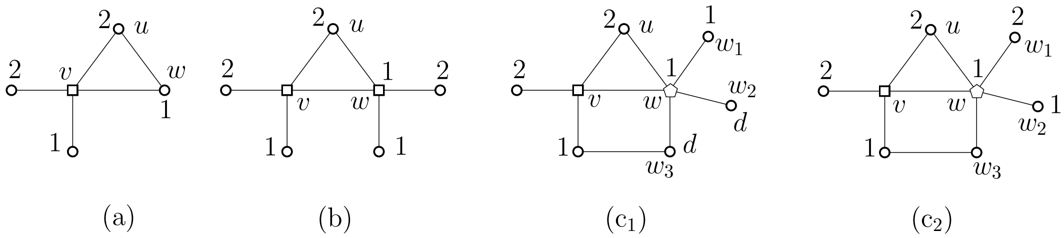

Figure 6: Colors around a 4-vertex contained in a triangle

Lemma 3.4. *Let $T = vuw$ be a triangle in D and $\deg(v) = 4$. Let ϕ be an L-coloring of $D - v$ such that $\phi(w) = 1$.*

(a) *The colors of the neighbors of v are as shown in Figure 6(a).*

(b) *If $\deg(w) = 4$, then $L(w) = L(v) = \{1, 2\}$ and the colors of the neighbors of v and w are as shown in Figure 6(b).*

(c) *If $\deg(w) = 5$, the other face containing the edge vw is a 4-face, and the clockwise neighbors of w are v, u, w_1, w_2, w_3, then either (i) $w_1 \in V(C_1(v))$ and the colors of the neighbors of v and w are as shown in Figure 6(c$_1$), where d is the color in $L(w) \setminus \{1\}$, or (ii) $L(w) = L(v) = \{1, 2\}$, and the colors of the neighbors of v and w are as shown in Figure 6(c$_2$).*

Proof. (a) By assumption, $w \in V(C_1(v))$. By Lemma 3.2, $u \notin V(C_1(v))$. Since $C_1(v)$ and $C_2(v)$ touch at v, the colors of the neighbors of v must be as claimed.

(b) By uncoloring w and coloring v with color 1, we obtain an L-coloring ϕ' of $G - w$. The claim follows by applying part (a) to $D - w$ and ϕ'.

(c) By (a), colors around v are as claimed. By Lemma 3.3, the cycle $C_1(v)$ does not contain w_3. We are done if it contains w_1, which gives Figure 6(c$_1$). Thus, we may assume that $C_1(v)$ contains w_2. Let us consider the coloring ϕ' of $D - w$ as used in the proof of part (b). Let $d \in L(w) \setminus \{1\}$. Clearly, $C_1(w) = C_1(v)$. Since $C_d(w)$ and $C_1(w)$ touch at w, the cycle $C_d(w)$ contains the edges uw and ww_1. Since $\phi(u) = 2$, we have $d = 2$ and the coloring is as shown in Figure 6(c$_2$). □

Let $Q_1, \dots, Q_{23}$ be the configurations shown in Figures 1–4. Our goal is to prove that each of these configurations is reducible. We will use the notation about vertices of each of these configurations as depicted in Figures 1–4 and in additional figures in this section.

Lemma 3.5. *Configurations Q_1, Q_2, and Q_3 are reducible.*

Proof. Assume D contains one of these configurations and let ϕ be an L-coloring of $D - v$. Without loss of generality, $L(v) = \{1, 2\}$ and $\phi(w) = 1$. If D contains Q_1, then Lemma 3.4(a) applied to the top triangle in Q_1 shows that the cycle $C_1(v)$ uses two edges of the bottom triangle, a contradiction to Lemma 3.2. Similarly, if D contains Q_2, Lemma 3.4(b) yields a contradiction to Lemma 3.3.

If D contains Q_3, applying Lemma 3.4(b) twice (to edges vw and vu) determines the colors of all neighbors of u, v, w, and we see that interchanging the colors of u and w gives an L-coloring of $D - v$ that contradicts Lemma 3.4(a), since the new cycles $C_1(v)$ and $C_2(v)$ cross at v instead of touching. □

Lemma 3.6. *Configurations Q_4 and Q_5 are reducible.*

Proof. We may assume that $vv_3 \in E(D)$.

Let ψ be an L-coloring of $D - vv_3$. Since D is not L-colorable, there is a color-i cycle C in D using the arc vv_3, where $i = 1$, say. Let ϕ be the L-coloring of $D - v$ obtained from ψ by uncoloring v; then $C_1(v)$ uses the arc vv_3. By Lemma 3.2, $C_1(v)$ cannot use the arcs v_2v or v_4v. Therefore, we may assume that $C_1(v)$ uses the arc v_5v. Since $C_1(v)$ and the cycle $C_2(v)$ touch at v, $C_2(v)$ uses the edges vv_1 and vv_2. In Q_4, this yields a contradiction by Lemma 3.2. So it remains to consider Q_5.

The edge v_1v_5 is incident with two faces, vv_1v_5 and, say, $\cdots uv_1v_5w\cdots$. Let ϕ' be the L-coloring of $G - v_1$ obtained from ϕ by uncoloring v_1 and coloring v with color 2. Note that v_1 has neighbors v_5, v colored 1, 2, respectively. Lemma 3.4(b) applied to ϕ' shows that $L(v_1) = L(v_5) = \{1, 2\}$ and $\phi(u) = \phi'(u) = \phi'(w) = 1$. Now let ϕ'' be the L-coloring of $G - v_5$ obtained from ϕ by uncoloring v_5 and coloring v with color 1. The same argument shows that $\phi(u) = \phi''(u) = \phi''(w) = 2$, which is a contradiction.

□

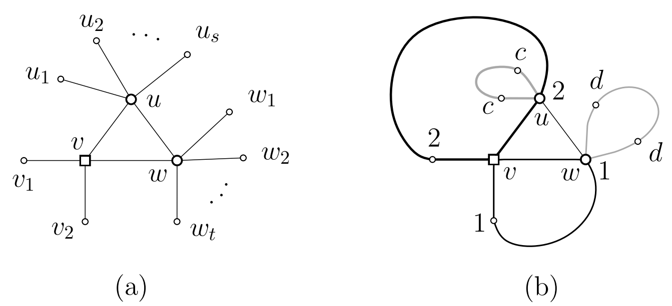

Figure 7: Triangle $T = vuw$ and its neighborhood

In the proofs of all of the subsequent lemmas, showing reducibility of particular configurations, we have a common scenario. Let us describe the common notation and assumptions that we will use.

We will always have a triangle $T = vuw$, where $\deg(v) = 4$. We will assume that $L(v) = \{1, 2\}$ and will consider an L-coloring ϕ of $D - v$. This coloring will also be denoted by ϕ_v if we would want to remind the reader that the vertex v is not colored. The neighbors of the vertices of T are denoted as in Figure 7(a), v_1, v_2 being neighbors of v, $u_1, \ldots, u_s$ neighbors of u and $w_1, \ldots, w_t$ neighbors of w, where u_i and w_j are enumerated in the clockwise order. It may be that $u_s = w_1$. By Lemma 3.4(a), we may assume that $\phi_v(v_2) = \phi_v(w) = 1$ and $\phi_v(v_1) = \phi_v(u) = 2$. We will denote the unused colors in $L(u)$ and $L(w)$ by c and d, respectively, i.e., $c \in L(u) \setminus \{2\}$ and $d \in L(w) \setminus \{1\}$. Sometimes we will be able to conclude that $c = 1$ or that $d = 2$, but in general this need not be the case.

As discussed before, ϕ_v induces cycles $C_1(v)$ and $C_2(v)$ passing through v. Form an L-coloring ϕ_u of $D-u$ from ϕ_v by coloring v with color 2 and uncoloring u. This coloring induces cycles $C_2(u)$ and $C_c(u)$ that touch at u; note that we may assume that $C_2(u) = C_2(v)$. Similarly, form an L-coloring ϕ_w of $D-w$ from ϕ_v by coloring v with color 1 and uncoloring w. The corresponding cycles $C_1(w)$ and $C_d(w)$ touch at w; we may assume that $C_1(w) = C_1(v)$. This situation is depicted in Figure 7(b), where the touching of the cycles at u and w may be different than shown (e.g., the cycle $C_c(u)$ could be in the exterior of $C_2(u)$). Note that if $c = 1$ and $d = 2$, it may happen that $C_c(u)$ and $C_d(w)$ share the edge uw (but they would be disjoint elsewhere since $c \neq d$ in this case).

Before we proceed to the next lemma, we need the following claim.

Claim 3. *The cycles $C_c(u)$ and $C_d(w)$ can be chosen so that at least one of them does not use the edge uw.*

Proof. Suppose that every choice of $C_c(u)$ and $C_d(w)$ uses uw. Then $c = 1$ and $d = 2$, and furthermore, interchanging the colors of u and w gives an L-coloring of $D - v$. This coloring contradicts Lemma 3.4 (a). □

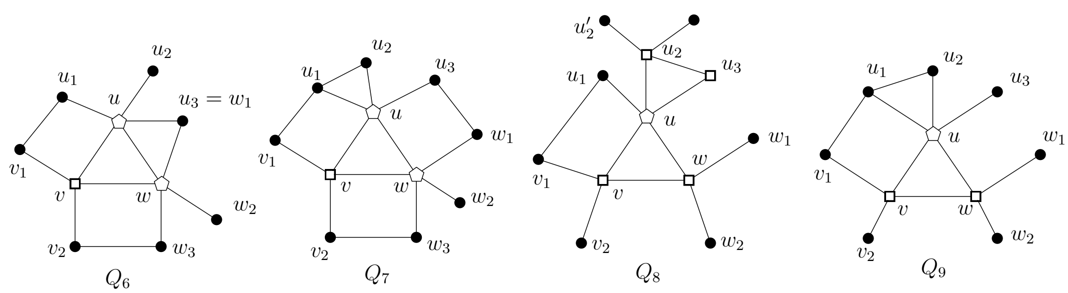

Figure 8: Configurations Q_6 to Q_9

Lemma 3.7. *Configurations Q_6, Q_7, Q_8, and Q_9 are reducible.*

Proof. We will use additional notation depicted in Figure 8. The cycle $C_2(v) = C_2(u)$ uses the edges v_1v and vu, and it therefore cannot use the edge uu_1 by Lemma 3.3. For the same reason, in Q_6 and Q_7, the cycle $C_1(v) = C_1(w)$ cannot use the edge ww_3. In Q_6, the cycle $C_2(u)$ cannot use the edge uu_2, as then the cycle $C_c(u)$, which touches $C_2(u)$ at u, must use the edges uu_3 and uw, contradicting Lemma 3.2. For the same reason, in Q_6, $C_1(w)$ cannot use the edge ww_2, and in Q_7 and Q_9, $C_2(u)$ cannot use the edge uu_3. In Q_8 and Q_9, $C_1(w)$ must use the edge ww_2, by Lemma 3.4 (b). Finally,

$$\text{if } C_2(u) \text{ uses edge } uu_2 \text{ then } C_1(w) \text{ cannot use edge } ww_2; \tag{1}$$

for, if $C_2(u)$ uses edge uu_2 and $C_1(w)$ uses edge ww_2, then $C_c(u)$ must use edges uu_3 and uw, and $C_d(w)$ must use edges uw and ww_1, contradicting Claim 3.

We now consider the four configurations separately, starting with the easiest ones.

Q_9 : By the above, $C_2(u)$ uses the edge uu_2 and $C_1(w)$ uses the edge ww_2, which contradicts (1).

Q_6 : By the above, $C_2(u)$ uses uu_3 and $C_1(w)$ uses ww_1, which is impossible since $u_3 = w_1$ and cannot have more than one color.

Q_7 : By the above, $C_2(u)$ uses uu_2 and $C_1(w)$ uses ww_1 or ww_2. By (1), $C_1(w)$ must use ww_1. Thus, $C_d(w)$ uses edges ww_2 and ww_3. Since $C_2(u)$ uses uu_2, $C_c(u)$ uses uu_3 and uw, which implies $c = 1$; and $C_c(u)$ also uses ww_1, since $\phi_u(v) = 2$ and $\phi_u(w_2) = \phi_u(w_3) = d \neq 1$. But this contradicts Lemma 3.3.

Q_8: By the above, $C_2(u)$ uses uu_2 or uu_3 and $C_1(w)$ uses ww_2. By (1), $C_2(u)$ uses uu_3. Thus, $C_c(u)$ uses edges uu_1 and uu_2. In the coloring $\phi = \phi_v$, all vertices of the cycle $C = C_c(u)$ have color c except for u, which has color 2. Form a coloring ϕ' from ϕ by coloring v with color 2 and uncoloring u_3. Since $C_2(v)$ passes through u_3, ϕ' is an L-coloring of $D - u_3$, and u_3 has neighbors u, u_2 with colors $2, c$. By Lemma 3.4 (b) applied to ϕ', $L(u_3) = L(u_2) = \{c, 2\}$, and u_2 has a neighbor u_2' of color 2 that is separated from u_3 by C. Now change ϕ' by recoloring u_2, u_3 with colors $2, c$, respectively. There can be no color-c cycle through u_3, which now has only one neighbor of color c. Also, any color-2 cycle through u_2 would have to use the edges vu, uu_2 and u_2u_2', and this is impossible since v and u_2' are separated by C and all vertices of C except for u and u_2 have color $c \neq 2$. Thus, we have obtained an L-coloring of D, a contradiction. □

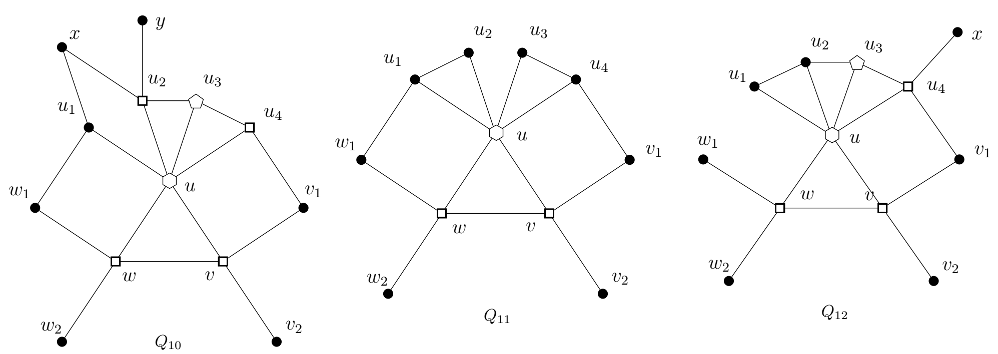

Figure 9: Configurations Q_{10}, Q_{11}, and Q_{12}

Lemma 3.8. *Configurations* Q_{10}, Q_{11}, *and* Q_{12} *are reducible.*

Proof. We will use additional notation depicted in Figure 9. By Lemma 3.4 we see that $L(w) = \{1, 2\}$. Let $C = C_2(v)$. We may assume that C contains the arcs uv and vv_1. Also, by Lemma 3.4 (b), we see that $\phi(w) = \phi(w_2) = \phi(v_2) = 1$, and $\phi(u) = \phi(w_1) = \phi(v_1) = 2$. Let $L(u) = \{c, 2\}$ and note that $C = C_2(v) = C_2(u)$.

The cycle C must use one of the arcs u_1u, u_2u, u_3u, or u_4u. By Lemma 3.3, C cannot use the arc u_4u.

Next, suppose that C uses the arc u_2u. Now, we claim that modifying ϕ_v by recoloring u with color c and w with color 2, gives an L-coloring of $D - v$. Clearly, there is no color-2 cycle through w since w has only one neighbor of color 2. Now, a color-c cycle C' through u touches C, so it uses the arcs u_3u and uu_4, contradicting Lemma 3.2. Therefore, the modified coloring is an L-coloring of $D - v$, and Lemma 3.4 (a) now implies that we can extend it to an L-coloring of D. Thus, we may suppose that C uses the arc u_1u or u_3u.

Now, if we were to modify ϕ_v by recoloring u with color c and w with color 2, by Lemma 3.4 (a) the resulting coloring ϕ^* cannot be an L-coloring of $D - v$, thus we must have a color-c cycle C' through u.

First, suppose that C uses the arc u_1u. By Lemma 3.2, C' either uses the edges u_2u and u_4u or (in Q_{11} only) uses the edges u_2u and uu_3. Suppose first that C' uses the edges u_2u and uu_4. This cannot happen in Q_{12} since in this case C' would use the edge u_4x; this would contradict Lemma 3.4(a) at the vertex u_4 by considering the coloring ϕ' of $D - u_4$ obtained from ϕ_v by coloring v with color 2, recoloring u with color c, and uncoloring u_4. Now, consider the cycle $C'' = C_2(w)$ through w in the coloring ϕ_w. Then C'' must use the arcs u_1u, uw and ww_1 since C' separates u_3 from u_1 and $c \neq 2$. This contradicts Lemma 3.3 in cases Q_{10} and Q_{11}.

The remaining case is that C' uses edges u_2u and uu_3, which can happen only in Q_{11} (by Lemma 3.3). The cycle $C_2(w)$, which uses edges w_1w and wu, cannot use u_1u by Lemma 3.3, so it must use the edge uu_4.

Next, we distinguish two cases. First, assume that $u_4u, uw, ww_1 \in E(D)$. Then $u_4v_1 \in E(D)$. Clearly, $C_2(w)$ uses a vertex $x \neq u$ on C (since C and $C_2(w)$ cross at u). But now, the directed path from v_1 to x on C, together with the arc u_4v_1 and the path from x to u_4 on $C_2(w)$ create a directed color-2 closed walk in the original coloring ϕ, a contradiction. Secondly, assume that $uu_4, wu, w_1w \in E(D)$. Clearly, $C_2(w)$ uses a vertex $y \neq u$ on C. But now, the directed path from y to u_1 on C, with the arcs u_1u, uu_4 and the directed path from u_4 to y on $C_2(w)$ creates a color-2 directed closed walk in the original coloring ϕ, a contradiction.

It remains to consider the case when C uses the arc u_3u. Consider again the coloring ϕ^* defined above, and the color-c cycle C'. Since C and C' touch, C' must use the edges u_2u and uu_1, and hence $\phi(u_1) = \phi(u_2) = c$. By Lemma 3.2, this is not possible in Q_{11} and Q_{12}, so it remains to consider Q_{10}. Since C' is a directed cycle, assume that we have the arcs u_2u and uu_1 (similar argument works for the other possibility). Now, C' cannot use the arc xu_2 by Lemma 3.3. Therefore, C' uses the arc yu_2. Considering $C_2(w)$, we also notice that $C_2(w)$ separates u_1 from w_2. Now, modify ϕ by recoloring u with color c, color v with color 2, and uncolor u_2. Since $c \neq 2$, there is no color-2 cycle through v in the resulting coloring, and since $C_2(w)$ separates u_1 from w_2 and $C_2(v)$ separates u_1 from u_4 there is no color-1 cycle through w and color-c cycle through u. Thus, the resulting coloring is an L-coloring of $D - u_2$, contradicting Lemma 3.4(a). This completes the proof. □

Lemma 3.9. *Configurations Q_{13}, Q_{14} and Q_{15} are reducible.*

Proof. We will use additional notation depicted in Figure 10. By Lemma 3.4 we see that $L(w) = \{1, 2\}$. We may assume that the cycle $C_2(v)$ uses the directed arcs uv and vv_1. By Lemma 3.3, $C_2(v)$ cannot use the arc u_1u. If $C_2(v)$ uses the arc u_2u, then $C_c(u)$ uses the

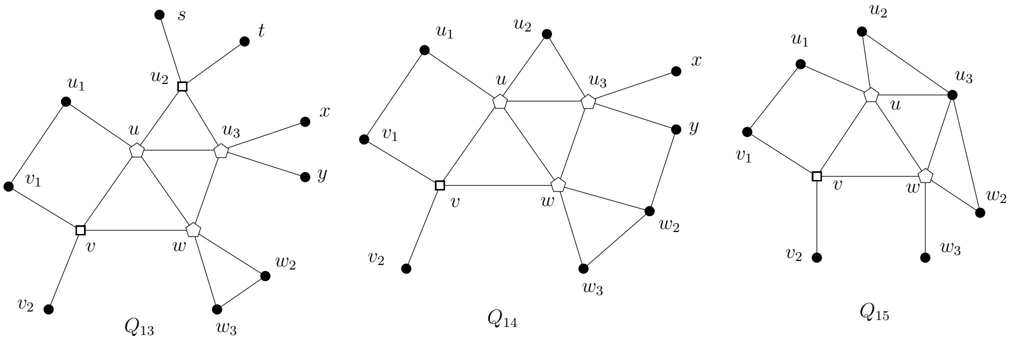

Figure 10: Configurations Q_{13}, Q_{14}, and Q_{15}

edges u_3u and uw, contradicting Lemma 3.2. Therefore, we may assume henceforth that $C_2(v)$ uses the arc u_3u and that, in particular, $\phi(u_3) = 2$. Therefore, the cycle $C_c(u)$ uses the edges u_1u and uu_2, and we have that $\phi_v(u_1) = \phi_v(u_2) = c$.

The cycle $C'' = C_d(w)$ uses two of the incident arcs to w. Since two neighbors of w are on $C_1(v)$ and u, u_3 are on $C_2(v)$, we conclude that $d = 2$. By Lemma 3.2, C'' cannot use both of the edges wu_3 and uw. Since $C_1(v)$ and C'' touch at w, $C_1(v)$ contains w_3 and C'' contains the edge ww_2.

Note that $u_3 \in V(C'')$, since u_3 is the only neighbor of u of color 2 in the coloring ϕ_w. Let us first suppose that C'' contains the arc uw. Since D has no directed triangles, we conclude that $u_3w \in E(D)$, so we may shorten C'' by eliminating vertex u, and thus we may henceforth assume that C'' contains the edge u_3w. Now, Lemma 3.2 yields a contradiction in the case of the configuration Q_{15}. So, we are left to consider Q_{13} and Q_{14}.

Suppose that $\phi_v(x) = 2$, where x is the neighbor of u_3 as shown in the figure. Then we modify the original coloring ϕ as follows: we recolor u_3 with the color $c' \in L(u_3)\backslash\{2\}$, recolor w with color 2, and color v with color 1. We claim that this is an L-coloring of D. Clearly, there is no color-1 cycle through v since v_2 is the only neighbor of v with color 1. Similarly, there is no color-2 cycle through w since u has no neighbor of color 2. Lastly, there is no color-c' cycle through u_3, since such a cycle would need to use the edges u_3u_2 and u_3y, thus it would not touch C''. This contradiction shows that $C_2(v)$ and C'' use the edge yu_3, and consequently, $\phi(y) = 2$. In particular, the cycle $C'' = C_2(w)$ uses the edges yu_3, u_3w, and ww_2. Lemma 3.3 yields contradiction in the case of the configuration Q_{14}.

It remains to consider Q_{13}. If we apply Lemma 3.4(a) to the coloring of $D - u_2$ obtained from ϕ_u by giving u color c and uncoloring u_2, we see that $\phi(s) = c$, so that $\phi(t) = \phi(u_3) = 2$.

Now, let ϕ' be the function obtained from the original coloring ϕ by giving v color 2 and recoloring u_3 with color $c' \in L(u_3)\backslash\{2\}$. There is now no color-2 cycle through v, so there must be a color-c' cycle Q through u_3, for otherwise ϕ' is an L-coloring of D. Clearly, Q and $C_2(v)$ touch at u_3 which implies that Q uses edges u_2u_3 and u_3x. Thus $c = c'$, and the neighbors t and u of u_2 are separated by Q. So recoloring u_2 with the color $c'' \in L(u_2)\backslash\{c\}$ turns ϕ' into an L-coloring of D. This contradiction completes the proof.

$\square$

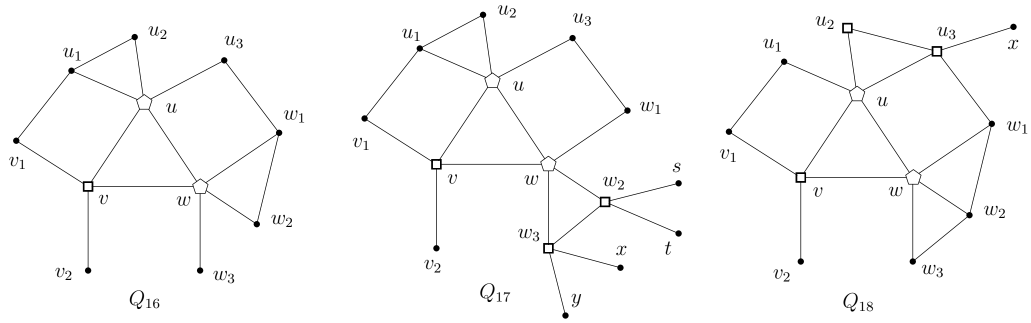

Figure 11: Configurations Q_{16}, Q_{17}, and Q_{18}

Lemma 3.10. *The configurations Q_{16}, Q_{17}, Q_{18} are reducible.*

Proof. We will use additional notation depicted in Figure 11. By Lemma 3.4 we see that $L(w) = \{1, 2\}$. Consider $C_2(v)$. We may assume uv, vv_1 are arcs in D. By Lemma 3.3, $C_2(v)$ cannot use the arc u_1u. Therefore, $C_2(v)$ uses one of the arcs u_2u and u_3u.

Let us first assume that $C_2(v)$ uses the arc u_3u. Then the cycle $C_c(u)$ uses edges u_1u and uu_2. Lemma 3.2 gives a contradiction in the case of configurations Q_{16} and Q_{17}, so it remains to consider Q_{18}.

We can turn ϕ_v into an L-coloring of $D - u_3$ by giving v color 2 and uncoloring u_3. Thus, Lemma 3.4(a) implies that $\phi_v(w_1) = \phi_v(u) = 2$, and that $\phi_v(x) = \phi_v(u_2) = c$.

Now, we consider the cycle $C'' = C_2(w)$. Since C'' and $C_1(w)$ touch at w, C'' cannot use the edge ww_3. By Lemma 3.2 applied to ϕ_w, C'' cannot use both edges ww_1 and ww_2, and so C'' must use edge uw, which means that C'' must use arcs w_1u_3, u_3u, uw and ww_2 (as using ww_1 instead of ww_2 would contradict Lemma 3.3). Since D has no directed 4-cycles or triangles, w_1w and w_1w_2 are arcs of D. Then the arc w_1w_2 followed by the segment of C'' from w_2 to w_1 is a color-2 cycle in the original coloring ϕ_v, a contradiction. This completes the proof when u_3u belongs to $C_2(v)$.

Suppose now that $C_2(v)$ uses the arc u_2u. Then the cycle $C' = C_c(u)$ uses the edges u_3u and uw, which implies that $c = 1$. By Lemma 3.3 applied to ϕ_w, C' cannot use the edge ww_1, and so C' must use the edge ww_2 or ww_3.

In either case, we can form an L-coloring ϕ'_w of $D - w$ from ϕ_v by giving v color 2, recoloring u with color $c = 1$, and uncoloring w. We now consider the cycles $C_1(w)$ and $C_d(w)$ with respect to this coloring ϕ'_v of $D - w$. Clearly, $C' = C_1(w)$. The cycle $C'' = C_d(w)$ touches C' at w. If C' used the edge ww_2, then $C_d(w)$ would have to use edges ww_3, wv and vv_1, which is not possible since w_3 and v_1 are separated by $C_1(v)$ and every vertex of $C_1(v)$ except v has color $1 \neq d$. Thus, C' uses the edge ww_3, and C'' uses the edges w_1w and ww_2. In cases Q_{16} and Q_{18} we have a contradiction to Lemma 3.2. This completes the proof for Q_{16} and Q_{18}.

It remains to consider Q_{17}. Recall that C' must use the edge ww_3 and $C_d(w)$ must use the edge ww_2. Thus we can obtain an L-coloring of $D - w_3$ from ϕ'_w by coloring w with color $c = 1$ and ucoloring w_3, and we can obtain an L-coloring of $D - w_2$ from ϕ'_w by coloring w with color d and uncoloring w_2. But Lemma 3.4 (b) says that vertices s and y must have the same color as w in both of these colorings, which is clearly impossible. This final contradiction shows that Q_{17} is reducible.

□

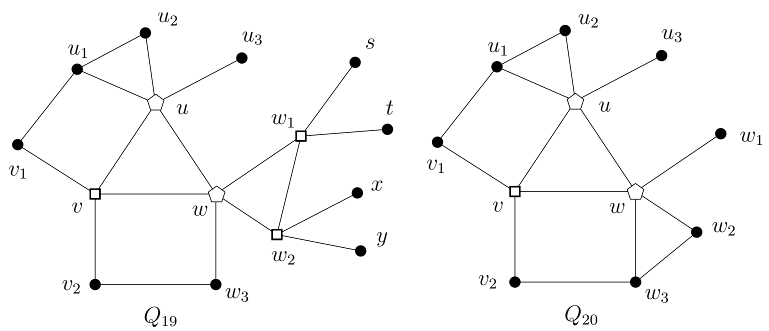

Figure 12: Configurations Q_{19} and Q_{20}

Lemma 3.11. *The configurations Q_{19} and Q_{20} are reducible.*

Proof. We will use additional notation depicted in Figure 12. By Lemma 3.4 we see that $L(w) = \{1, 2\}$. Let $C = C_2(v)$, and assume that $C_2(v)$ uses the arcs uv, vv_1. The cycle $C_2(v)$ cannot use the arc u_1u by Lemma 3.3. Therefore, $C_2(v)$ must use one of the arcs u_2u or u_3u. If $C_2(v)$ uses the arc u_3u, then $C_c(u)$ would use edges u_1u and uu_2, a contradiction to Lemma 3.2. Thus, $C_2(v)$ uses the arc u_2u. The cycles $C_c(u)$ and $C_2(v)$ touch at u; thus $C_c(u)$ must use the edges uw and uu_3. This implies that $c = 1$. Let us now consider the cycle $C_1(v)$. Clearly, it contains the edges wv and vv_2, and does not contain the edge w_3w by Lemma 3.3. Therefore, $C_1(v)$ uses one of the edges w_1w and w_2w.

If $C_1(v)$ uses the edge w_2w, then $C_d(w)$ must use the edges wu and ww_1. Since $\phi_w(u) = 2$, we have $d = 2$ and $\phi(w_1) = 2$. Observe that the cycles $C_1(u)$ and $C_2(w)$ share the edge uw, but are otherwise disjoint. However, this is not possible, since they cross each other as they enter and leave the edge uw. This contradiction shows that $C_1(v)$ uses the edge w_1w. Consequently, we have $\phi(w_1) = 1$. Then $C_d(w)$ contains the edges ww_3 and ww_2. This contradicts Lemma 3.2 for configuration Q_{20}.

It remains to consider Q_{19}. Recall that the cycle $C_1(v) = C_1(w)$ must use the edge w_1w, and $C_d(w)$ must use the edge ww_2. This gives a contradiction in exactly the same way as for Q_{17} in the previous proof (using ϕ_w here instead of ϕ'_w there). Therefore, Q_{19} is also reducible.

□

Lemma 3.12. *The configurations Q_{21}, Q_{22} and Q_{23} are reducible.*

Proof. Consider the cycle $C_1(v)$. It uses the edges wv and vv_2. By Lemma 3.3, $C_1(v)$ cannot use the edge w_2w. If $C_1(v)$ uses the edge w_1w, then $C_d(w)$ would use edges uw and wu_4, and Lemma 3.2 would yield a contradiction. This proves that $C_1(v)$ uses the edge u_4w. Now

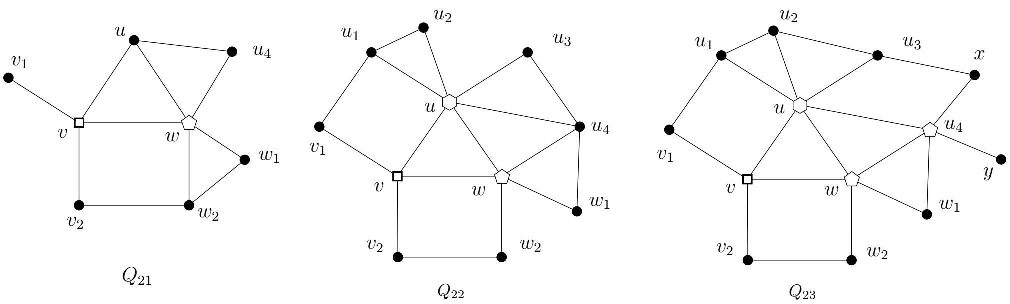

Figure 13: Configurations Q_{21}, Q_{22}, and Q_{23}

we see that the cycle $C_d(w)$ uses the edges w_1w and ww_2, and thus $\phi_v(w_1) = \phi_v(w_2) = d$. Lemma 3.2 gives a contradiction in the case of Q_{21}, so it remains to consider Q_{22} and Q_{23}.

Let $C = C_2(v) = C_2(u)$, which uses the edges uv and vv_1. By Lemma 3.3, C cannot use the edge u_1u. Since $\phi_v(u_4) = 1$, C uses one of the edges u_2u and u_3u. Let $C' = C_c(u)$, which touches $C = C_2(u)$ at u. If C uses edge u_3u, then C' must use either edges uu_1 and uu_2, or edges uw and uu_4, both of which would contradict Lemma 3.2. Thus, C uses the edge u_2u, and C' uses two of the edges uu_3, uu_4 and uw. In particular, $c = 1$ since $\phi_u(w) = \phi_u(u_4) = 1$.

By Lemma 3.2, C' cannot use the edges uu_4 and uw, and therefore it must use edge uu_3; we may assume that C' uses the arc u_3u. If C' uses arc uw then it uses arc wu_4, since $\phi_u(w_1) = \phi_u(w_2) = d \neq 1$. Since D has no directed triangle, $uu_4 \in E(D)$, and so we can shorten C' by using arc uu_4 instead of the path uwu_4. Thus, we may assume that C' uses the arc uu_4. In Q_{22}, this contradicts Lemma 3.2 applied to the triangle u_3uu_4.

It remains to consider Q_{23}. By Lemma 3.3 applied to the 4-cycle u_3uu_4x, C' cannot use the arc u_4x, and therefore it must use the arc u_4y. In particular, $\phi(y) = c = 1$.

Recall that $C_1(v)$ uses edges u_4w and wv, and $C = C_2(v)$ uses edges u_2u and uv. Form ϕ' from ϕ_u by coloring u with color 1 and recoloring u_4 with color $d' \in L(u_4)\backslash\{1\}$. Since $C_2(v)$ separates the neighbors u_1 and u_3 of u, and u is the only neighbor of w with color 1 in ϕ', it follows that there is no color-1 cycle through u in ϕ'. We also claim that there is no color-d' cycle through u_4 in ϕ': if $C_1(v)$ uses the edge u_4y, then this holds since $C_1(v)$ separates the neighbors x and w_1 of u_4; and if $C_1(v)$ uses the edge u_4x then it holds because $\phi'(x) = \phi'(y) = 1$ and w_1 is the only neighbor of u_4 that can have color d'. Thus, ϕ' is an L-coloring of D, a contradiction. This completes the proof of reducibility of Q_{23}.

□

We are ready to complete the proof of the main result.

Proof of Theorem 1.3. By Theorem 2.1, every planar graph of minimum degree at least four contains one of the configurations $Q_1, \ldots, Q_{23}$. Suppose that D is a minimum counterexample to Theorem 1.3. Then D has digirth at least five and minimum degree at least four, but cannot contain any of the configurations $Q_1, \ldots, Q_{23}$ by Lemmas 3.5-3.13. This proves that a counterexample does not exist, and the proof is complete. □

4 Concluding remarks

We raise the following questions. It would be interesting to see if the result can be pushed to digirth 4. Also, the following relaxation of Conjecture 1.1 should be of interest.

Conjecture 4.1. *There exists k such that every oriented planar graph without cycles of length $4, \dots, k$ is 2-colorable.*

The original conjecture still seems out of reach. In fact, we do not know of a simple proof of the fact that planar digraphs of large digirth are 2-colorable. In support of the conjecture, it would be nice to see whether one can find large acyclic set in a planar digraph, say of size $n/2$. In fact, the following was conjectured in [3].

Conjecture 4.2. *Every oriented n-vertex planar graph has an acyclic set of size at least $\frac{3n}{5}$.*

It is known that the bound in Conjecture 4.2 cannot be replaced by any larger value.

5 Acknowledgements

We are deeply grateful to a referee whose comments greatly improved the presentation of the paper as well as slightly shortened its length.

References

[1] D. Bokal, G. Fijavž, M. Juvan, P. M. Kayll, B. Mohar, The circular chromatic number of a digraph, J. Graph Theory 46 (2004), 227–240.

[2] G. Chartrand, H.V. Kronk, C.E. Wall, The point-arboricity of a graph, Israel J. Math. 6 (1968), 169–175.

[3] A. Harutyunyan, Brooks-type results for coloring of digraphs, PhD Thesis, Simon Fraser University, 2011.

[4] V. Neumann-Lara, The dichromatic number of a digraph, J. Combin. Theory, Ser. B 33 (1982), 265–270.

[5] V. Neumann-Lara, Vertex colourings in digraphs. Some Problems. Seminar notes, University of Waterloo, July 8, 1985 (communicated by A. Bondy and S. Thomassé).