A proof of a conjecture of Barát and Thomassen

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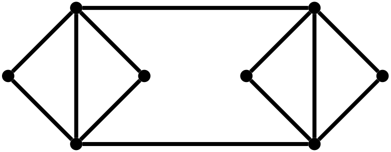
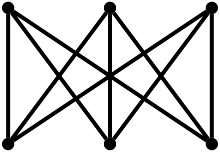
Bordeaux Graph Workshop
November 8, 2016

Decomposition of graphs

T-decomposition: **edge-partition** into **copies of T**.





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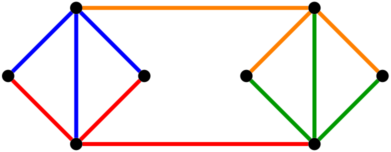
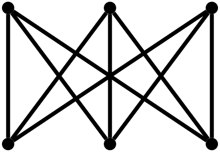


S_4-decomposition



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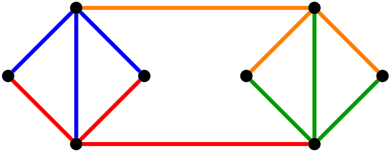

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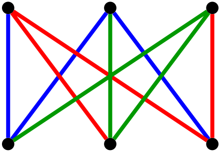

P_3-decomposition

Wilson's Theorem

Theorem (Wilson 1976)

For any tree T, K_n admits a T-decomposition, for n sufficiently large (provided divisibility condition).

Minimum degree condition

Theorem (Barber, Kuhn, Lo, Osthus 2016)

For every T, $\exists\ \epsilon_T > 0$ s.t. if G has minimum degree $(1-\epsilon_T)|V(G)|$, then G has T-decomposition (provided divisibility condition).

Barát-Thomassen conjecture

Conjecture [Barát, Thomassen – 2006]

For every fixed tree T, there exists a positive constant c_T such that every c_T-edge-connected graph with size divisible by $|E(T)|$ admits a T-decomposition.

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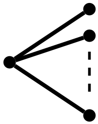
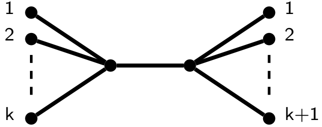


stars
[Thomassen – 2012]



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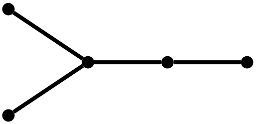

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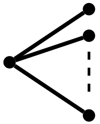

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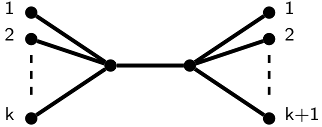

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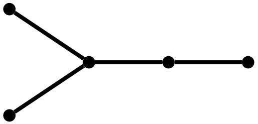

of deg. sequence $(1, 1, 1, 2, 3)$
[Barát, Garbner – 2014]

... and actually whenever $\mathrm{diam}(T) \leq 4$ [Merker – 2015+].

Barát-Thomassen conjecture for paths

When T is a path: $T = P_\ell$

- $\ell \in \{3, 4\}$ [Thomassen – 2008],
- $\ell = 2^k$ for any k [Thomassen – 2013],
- $\ell = 5$ [Botler, Mota, Oshiro, Wakabayashi – 2015+],
- ℓ is any value [Botler, Mota, Oshiro, Wakabayashi – 2015+].

Is edge-connectivity necessary?

The following would be best optimal:

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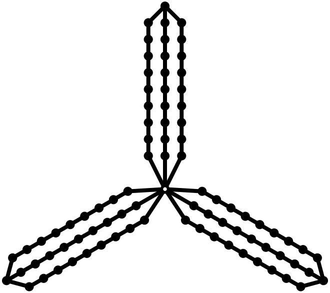
Note: 2-edge-connectivity does not suffice; *e.g.* for



... and make δ increase with preserving non P_9-decomposability.

Relations to Tutte's nowhere zero 3-flow conjecture

Theorem (Tutte's Conjecture)

Every 4-edge-connected graph admits a nowhere zero 3-flow.

- $K_{1,3}$-decompositions relate to flows: Tutte's conjecture implies every 10-e.c. graph has $K_{1,3}$-decomposition.
- Conversely, if every 8-e.c. G admits a $K_{1,3}$-decomposition, then Tutte holds with e.c $= 8$.

Barát-Thomassen Conjecture

Theorem (Bensmail, Le, Merker, Thomassé, H. – 2015+)

The Barát-Thomassen conjecture is true.

Preliminaries: Bipartite graphs

Theorem (Barát-Gerbner (2014), also Thomassen (2013))

It is sufficient to prove the conjecture for G bipartite.

Proof technique

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- 'Absorbing' technique STABILITY RESULT $+$ NOISE

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- 1. Prove that from G can extract a 'rich/stable' structure S
- 2. Use probabilistic tools to get a 'nearly good' decomposition on S.
- 3. Use the structure S to repair 'blemishes'.

Preliminaries: T-equitable coloring

Definition

$G = (A, B)$ bipartite, $T = (T_A, T_B)$ a tree. An edge-colouring $\phi : E(G) \to E(T)$ is called ***T*-equitable**, if for any pair of vertices $v \in V(G), t \in V(T)$ in the **same** part, we have $d_j(v) = d_k(v)$ for all pair of colors j, k incident to t.

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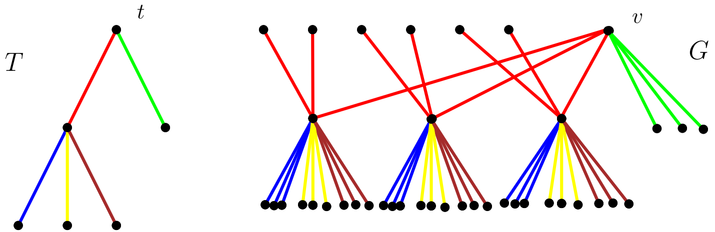

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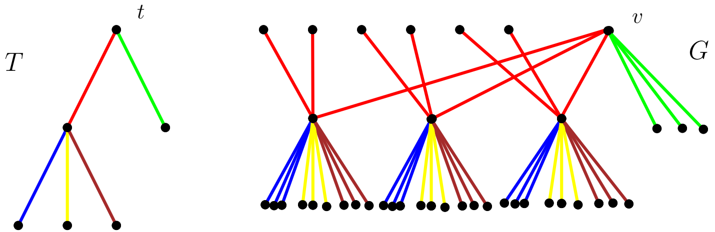

Theorem (Merker 2015+)

A highly edge connected bipartite G (+ other divisibility assumptions) has a T-equitable coloring where the min. degree in each color is large.

Match edges randomly

- For each $v \in V(G)$ and $t \in V(T)$ in the *same* part, let v "play the role" of t by matching randomly all the colored edges around t on v.

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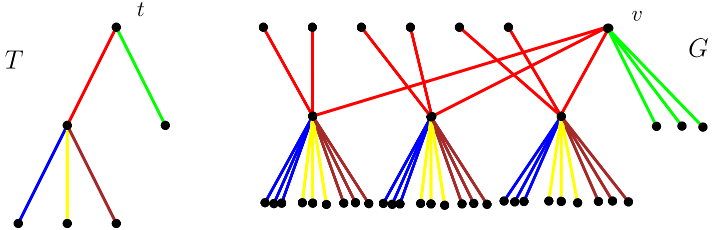

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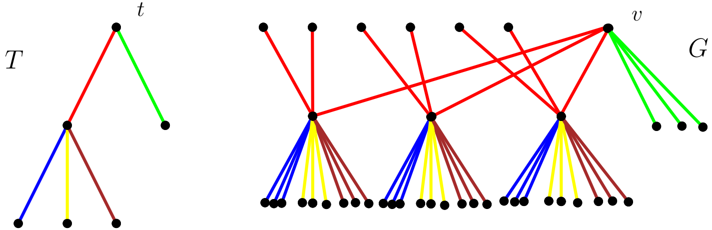

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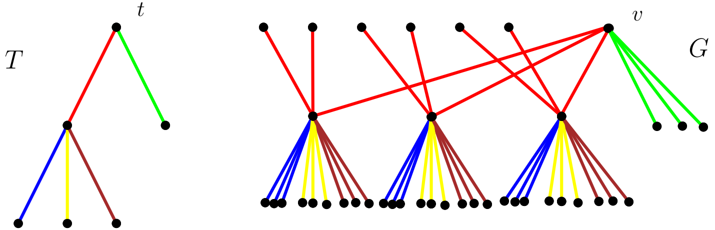

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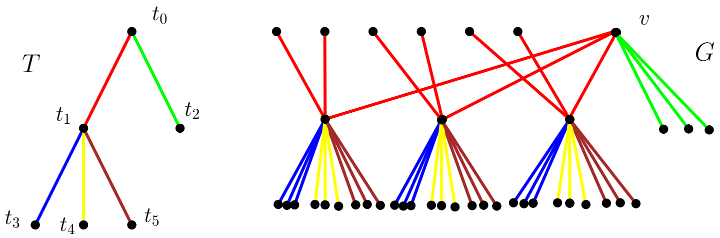
- Overwhelming majority of copies are isomorphic to T.

Number of non-isomorphic trees



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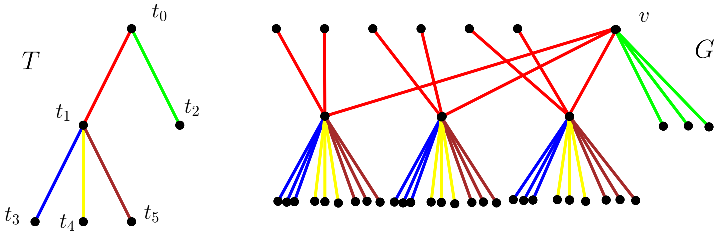

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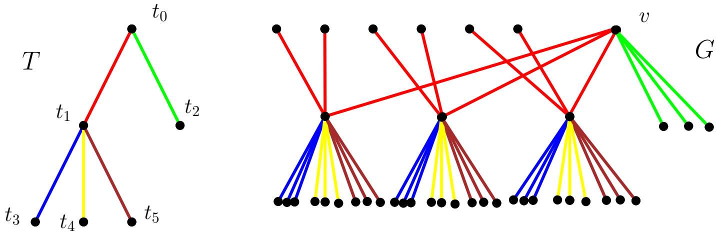

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$\mathbb{E}[X_v(t_0, t_j)] \leq 1!$

Probabilistic Machinery involved

McDiarmid's Inequality (Simplified version)

Let X be a non-negative random variable, determined by m independent random permutations $\Pi_1, \ldots, \Pi_m$ satisfying the following conditions for some $d, r > 0$

- interchanging two elements in any one permutation can affect X by at most d;
- for any s, if $X \geq s$ then there is a set of at most rs choices whose outcomes certify that $X \geq s$,

then for any $0 \leq t \leq \mathbb{E}[X]$,

$$Pr[|X - \mathbb{E}[X]| > t + 60d\sqrt{r\mathbb{E}[X]}] \leq 4e^{-\frac{t^2}{8d^2 r\mathbb{E}[X]}}.$$

Probabilistic Machinery involved

Lovász Local Lemma

Let $A_1, ..., A_n$ be events in some probability space Ω with $\mathbb{P}[A_i] \leq p$ for all $i \in \{1, \ldots, n\}$. Suppose that each A_i is mutually independent of all but at most d other events A_j. If $4pd < 1$, then $\mathbb{P}[\cap_{i=1}^{n} \overline{A_i}] > 0$.

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- Repeat for t_5, t_6 etc.

Conclusion

- **Conjecture:** There is a function f such that, for any fixed tree T with maximum degree Δ_T, every $f(\Delta_T)$-edge-connected graph with its number of edges divisible by $|E(T)|$ and minimum degree at least $f(|E(T)|)$ can be T-decomposed.

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Theorem (Bensmail, Le, Thomassé, H. 2016+)

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