Chernoff's Inequality and Best-Arm Identification

An Introduction to Sequential Decision Problems

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Introduction: Information for Deviation Lower Bounds

Chernoff Bound for Bernoulli variables

Let $\mu \in (0,1)$. Let $X_1, X_2, \dots X_n \overset{iid}{\sim} \mathcal{B}(\mu)$, and $\bar{X}_n = (X_1 + \cdots + X_n)/n$.

Theorem

For all $\mu \le x \le 1$,

$$\mathbb{P}_\mu\left(\bar{X}_n \ge x\right) \le e^{-n\,\mathrm{kl}(x,\mu)}$$

where $\mathrm{kl}(x,y) = x \log \frac{x}{y} + (1-x)\log\frac{1-x}{1-y}$ is the binary relative entropy. Similary, for all $0 \le x \le \mu$,

$$\mathbb{P}_\mu\left(\bar{X}_n \le x\right) \le e^{-n\,\mathrm{kl}(x,\mu)} \ .$$

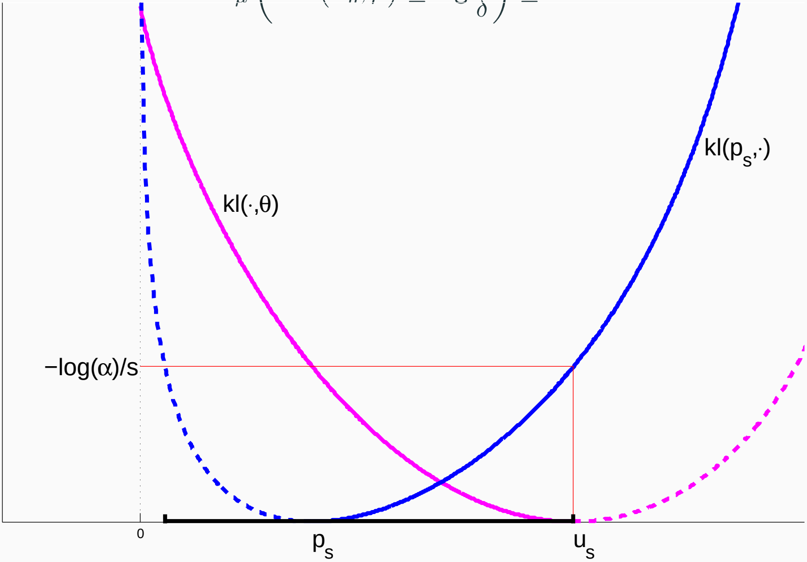
Corollary

For every $\delta > 0$,

$$\mathbb{P}_\mu\left(n\,\mathrm{kl}\left(\bar{X}_n,\mu\right) \ge \log\frac{1}{\delta}\right) \le 2\delta \ .$$

A Divergence on the Set of Possible Means

$$\mathbb{P}_\mu\left(n\,\mathrm{kl}\left(\bar{X}_n,\mu\right) \geq \log\frac{1}{\delta}\right) \leq 2\delta$$



Proof: Fenchel-Legendre transform of log-Laplace

For every $\lambda > 0$,

$$\begin{aligned}\mathbb{P}_\mu\left(\bar{X}_n \geq x\right) &= \mathbb{P}_\mu\left(e^{\lambda(X_1+\cdots+X_n)} \geq e^{\lambda n x}\right)\\ &\leq \frac{\mathbb{E}_\mu\left[e^{\lambda(X_1+\cdots+X_n)}\right]}{e^{\lambda n x}} \qquad \text{by Markov's inequality}\\ &= e^{-n\left(\lambda x - \log \mathbb{E}_\mu[\exp \lambda X_1]\right)} .\end{aligned}$$

Thus,
$$\begin{aligned}-\frac{1}{n}\log \mathbb{P}_\mu\left(\bar{X}_n \geq x\right) &\geq \sup_{\lambda>0}\left\{\lambda x - \log \mathbb{E}_\mu[\exp \lambda X_1]\right)\right\}\\ &= \sup_{\lambda>0}\left\{\lambda x - \log\left(1-\mu+\mu e^\lambda\right)\right\}\\ &= \mathrm{kl}(x,\mu) .\end{aligned}$$

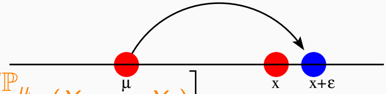
kl = binary Kullback-Leibler divergence: $\mathrm{kl}(x,\mu) = \mathrm{KL}\big(\mathcal{B}(x),\mathcal{B}(\mu)\big)$

where $$\mathrm{KL}(P,Q) = \begin{cases}\int \log\frac{dP}{dQ}dP & \text{if } P \ll Q ,\\ +\infty & \text{otherwise.}\end{cases}$$

Properties: $0 \leq \mathrm{KL}(P,Q) \leq +\infty$, and $\mathrm{KL}(P,Q) = 0$ iff $P = Q$.

Lower Bound: Change of Measure

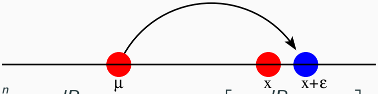
For all $\epsilon > 0$ and all $\alpha > 0$,



$$
\begin{aligned}
\mathbb{P}_\mu\left(\bar{X}_n \geq x\right) &= \mathbb{E}_\mu\left[\mathbb{1}\{\bar{X}_n \geq x\}\right] \\
&= \mathbb{E}_{x+\epsilon}\left[\mathbb{1}\{\bar{X}_n \geq x\} \times \frac{d\mathbb{P}_\mu}{d\mathbb{P}_{x+\epsilon}}(X_1, \ldots, X_n)\right] \\
&= \mathbb{E}_{x+\epsilon}\left[\mathbb{1}\{\bar{X}_n \geq x\} \times e^{-\sum_{i=1}^n \log \frac{dP_{x+\epsilon}}{dP_\mu}(X_i)}\right] \\
&\geq \mathbb{E}_{x+\epsilon}\left[\mathbb{1}\{\bar{X}_n \geq x\} \, \mathbb{1}\left\{\frac{1}{n}\sum_{i=1}^n \log \frac{dP_{x+\epsilon}}{dP_\mu}(X_i) \leq \mathbb{E}_{x+\epsilon}\left[\log \frac{dP_{x+\epsilon}}{dP_\mu}(X_1)\right] + \alpha\right\}\right. \\
&\qquad\qquad \left. \times \, e^{-\sum_{i=1}^n \log \frac{dP_{x+\epsilon}}{dP_\mu}(X_i)}\right] \\
&\geq e^{-n\left\{\mathbb{E}_{x+\epsilon}\left[\log \frac{dP_{x+\epsilon}}{dP_\mu}(X_i)\right] + \alpha\right\}}\left[1 - \mathbb{P}_{x+\epsilon}\left(\bar{X}_n < x\right)\right. \\
&\qquad \left. - \mathbb{P}_{x+\epsilon}\left(\frac{1}{n}\sum_{i=1}^n \log \frac{dP_{x+\epsilon}}{dP_\mu}(X_i) > \mathbb{E}_{x+\epsilon}\left[\log \frac{dP_{x+\epsilon}}{dP_\mu}(X_1)\right] + \alpha\right)\right] \\
&= e^{-n\left\{\mathrm{kl}(x+\epsilon, \mu) + \alpha\right\}}\left(1 - o_n(1)\right) .
\end{aligned}
$$

Lower Bound: Change of Measure

For all $\epsilon > 0$ and all $\alpha > 0$,



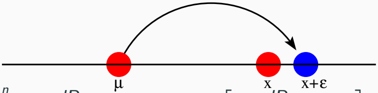
$$
\begin{aligned}
\mathbb{P}_\mu\left(\bar{X}_n \geq x\right) &= \mathbb{E}_\mu\left[\mathbb{1}\{\bar{X}_n \geq x\}\right] \\
&\geq \mathbb{E}_{x+\epsilon}\Bigg[\mathbb{1}\{\bar{X}_n \geq x\}\ \mathbb{1}\left\{\frac{1}{n}\sum_{i=1}^n \log\frac{dP_{x+\epsilon}}{dP_\mu}(X_i) \leq \mathbb{E}_{x+\epsilon}\left[\log\frac{dP_{x+\epsilon}}{dP_\mu}(X_1)\right] + \alpha\right\} \\
&\qquad \times e^{-\sum_{i=1}^n \log\frac{dP_{x+\epsilon}}{dP_\mu}(X_i)}\Bigg] \\
&\geq e^{-n\left\{\mathbb{E}_{x+\epsilon}\left[\log\frac{dP_{x+\epsilon}}{dP_\mu}(X_i)\right]+\alpha\right\}}\Bigg[1 - \mathbb{P}_{x+\epsilon}\left(\bar{X}_n < x\right) \\
&\qquad - \mathbb{P}_{x+\epsilon}\left(\frac{1}{n}\sum_{i=1}^n \log\frac{dP_{x+\epsilon}}{dP_\mu}(X_i) > \mathbb{E}_{x+\epsilon}\left[\log\frac{dP_{x+\epsilon}}{dP_\mu}(X_1)\right] + \alpha\right)\Bigg] \\
&= e^{-n\left\{\mathrm{kl}(x+\epsilon,\mu)+\alpha\right\}}\left(1 - o_n(1)\right) .
\end{aligned}
$$

Asymptotic Optimality (Large Deviation Lower Bound)

$$
\liminf_n \frac{1}{n}\log\mathbb{P}_\mu\left(\bar{X}_n \geq x\right) \geq -\mathrm{kl}(x,\mu) .
$$

Lower Bound: Change of Measure

For all $\epsilon > 0$ and all $\alpha > 0$,



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\begin{aligned}
\mathbb{P}_\mu\left(\bar{X}_n \geq x\right) &= \mathbb{E}_\mu\left[\mathbb{1}\{\bar{X}_n \geq x\}\right] \\
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&\qquad\qquad \times e^{-\sum_{i=1}^n \log\frac{dP_{x+\epsilon}}{dP_\mu}(X_i)}\Bigg] \\
&\geq e^{-n\left\{\mathbb{E}_{x+\epsilon}\left[\log\frac{dP_{x+\epsilon}}{dP_\mu}(X_i)\right]+\alpha\right\}}\Bigg[1 - \mathbb{P}_{x+\epsilon}\left(\bar{X}_n < x\right) \\
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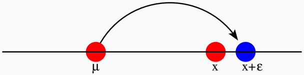
&= e^{-n\left\{\mathrm{kl}(x+\epsilon,\mu)+\alpha\right\}}\left(1 - o_n(1)\right).
\end{aligned}
$$

Asymptotic Optimality (Large Deviation Principle)

$$
\frac{1}{n}\log\mathbb{P}_\mu\left(\bar{X}_n \geq x\right) \underset{n\to\infty}{\longrightarrow} -\mathrm{kl}(x,\mu).
$$

Lower Bound: the Entropic Way



Let $\Omega = \{0,1\}^n$, $X_i(\omega) = \omega_i$

Probability laws on Ω: $\mathbb{P}_p = \mathcal{B}(p)^{\otimes n}$.

For all $\epsilon > 0$,

$$n\,\mathrm{kl}(x+\epsilon,\mu) = \mathrm{KL}\left(\mathbb{P}_{x+\epsilon}, \mathbb{P}_\mu\right)$$

$\mathrm{KL}(P \otimes P', Q \otimes Q') = \mathrm{KL}(P,Q) + \mathrm{KL}(P',Q')$

$$\geq \mathrm{KL}\left(\mathbb{P}_{x+\epsilon}^{\mathbb{1}\{\bar{X}_n \geq x\}}, \mathbb{P}_{\mu}^{\mathbb{1}\{\bar{X}_n \geq x\}}\right)$$

$\mathrm{KL}(P,Q) \geq \mathrm{KL}(P^X, Q^X)$
contraction of entropy
= data-processing inequality

$$= \mathrm{kl}\Big(\mathbb{P}_{x+\epsilon}\left(\bar{X}_n \geq x\right), \mathbb{P}_\mu\left(\bar{X}_n \geq x\right)\Big)$$

$$\geq \mathbb{P}_{x+\epsilon}\left(\bar{X}_n \geq x\right) \log \frac{1}{\mathbb{P}_\mu\left(\bar{X}_n \geq x\right)} - \log(2)$$

$\mathrm{kl}(p,q) \geq p \log \frac{1}{q} - \log 2$

A non-asymptotic lower bound

$$\forall \epsilon > 0, \qquad \mathbb{P}_\mu\left(\bar{X}_n \geq x\right) \geq e^{-\frac{n\,\mathrm{kl}(x+\epsilon,\mu)+\log(2)}{1-e^{-2n\epsilon^2}}}.$$

Intermediate: Sequential Test

Sequential Test : $\mu > 1/2$ versus $\mu < 1/2$

$X_1, X_2, \dots$ independent random variables with distribution $\mathcal{B}(\mu)$.

- Goal: say if $\mu > 1/2$ or $\mu < 1/2$.
- Sequential: stopping time τ, decision rule $\hat{a}_\tau = \in \{+, -\}$ is $\mathcal{F}_\tau$-measurable, where $\mathcal{F}_t = \sigma(X_1, \dots, X_t)$.
- δ-correctness constraint:

$$\forall \mu < 1/2,\ \mathbb{P}_\mu(\hat{a}_\tau = +) \leq \delta \quad \text{and} \quad \forall \mu > 1/2,\ \mathbb{P}_\mu(\hat{a}_\tau = -) \leq \delta \ .$$

- a good procedure minimizes $\mathbb{E}_\mu[\tau]$ for all $\mu \neq 1/2$.

Entropic Lower Bound on the Sample Complexity

If $\mu > 1/2$:

$$\mathbb{E}_\mu[\tau]\,\mathrm{kl}\left(\mu, \frac{1}{2} - \epsilon\right) = \mathrm{KL}\left(\mathbb{P}_\mu^{(X_1,\dots,X_\tau)}, \mathbb{P}_{\frac{1}{2}-\epsilon}^{(X_1,\dots,X_\tau)}\right)$$

tensorization works with stopping times

$$\geq \mathrm{KL}\left(\mathbb{P}_\mu^{\mathbb{1}\{\hat{a}_\tau = +\}}, \mathbb{P}_{\frac{1}{2}-\epsilon}^{\mathbb{1}\{\hat{a}_\tau = +\}}\right)$$

contraction just like before

$$= \mathrm{kl}\big(\mathbb{P}_\mu(\hat{a}_\tau = +), \mathbb{P}_{\frac{1}{2}-\epsilon}(\hat{a}_\tau = +)\big)$$

$$\geq \mathrm{kl}(1-\delta, \delta)$$

by δ-correctness.

Theorem

For every δ-correct stopping time τ and every $\mu \in [0,1]$,

$$\mathbb{E}_\mu[\tau] \geq \frac{1}{\mathrm{kl}\left(\mu, \frac{1}{2}\right)}\,\mathrm{kl}(1-\delta, \delta)\ .$$

Remark: $\mathrm{kl}(\delta, 1-\delta) \underset{\delta \to 0}{\sim} \log \frac{1}{\delta}$ and $\mathrm{kl}(\delta, 1-\delta) \geq \log \frac{1}{2.4\delta}$.

Tight: possible to choose τ s.t. $\mathbb{E}_\mu[\tau] \leq \dfrac{\log \frac{1}{\delta}}{\mathrm{kl}\left(\mu, \frac{1}{2}\right)}\Big(1 + o_\delta\big(\log \dfrac{1}{\delta}\big)\Big)$.

Active Learning: Finding the Distribution with Largest Mean

Best-Arm Identification with Fixed Confidence

K options $= \left(\mathcal{B}(\mu_a)\right)_{1\le a\le K}$. Parameter $\boldsymbol{\mu} = (\mu_1, \dots, \mu_K)$.

μ_1

μ_2

μ_3

μ_4

μ_5

At round t, you may:

- choose an option $A_t = \phi_t\left(A_1, X_1, \dots, A_{t-1}, X_{t-1})\right) \in \{1, \dots, K\}$
- observe a new independent sample $X_t \sim \mathcal{B}\left(\mu_{A_t}\right)$

so as to identify the best arm $a^*(\boldsymbol{\mu}) = \operatorname{argmax}_a \mu_a$ as fast as possible: stopping time τ, decision $\hat{a}_\tau$.

Goal: minimize $\mathbb{E}_{\boldsymbol{\mu}}[\tau]$

under δ-correctness constraint: $\forall \boldsymbol{\mu}, \mathbb{P}_{\boldsymbol{\mu}}\left(\hat{a}_\tau \neq a^*(\boldsymbol{\mu})\right) \le \delta$.

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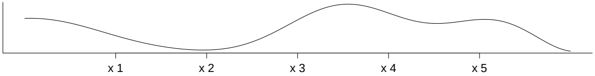

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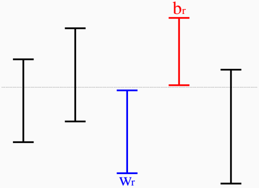
Goal: minimize $\mathbb{E}_{\boldsymbol{\mu}}[\tau]$
under δ-correctness constraint: $\forall \boldsymbol{\mu}, \mathbb{P}_{\boldsymbol{\mu}}\left(\hat{a}_\tau \neq a^*(\boldsymbol{\mu})\right) \leq \delta$.

Racing Strategy see [Kaufmann & Kalyanakrishnan '13]



$\mathcal{R} := \{1, \dots, K\}$ set of remaining arms.
$r := 0$ current round

while $|\mathcal{R}| > 1$

- $r := r + 1$
- draw each $a \in \mathcal{R}$, compute $\hat{\mu}_{a,r}$, the empirical mean of the r samples observed so far
- compute the empirical best and empirical worst arms:

$$b_r = \underset{a \in \mathcal{R}}{\text{argmax}} \ \hat{\mu}_{a,r} \qquad w_r = \underset{a \in \mathcal{R}}{\text{argmin}} \ \hat{\mu}_{a,r}$$

- Elimination step: if

$$\ell_{b_r}(r) > u_{w_r}(r),$$

then eliminate w_r : $\mathcal{R} := \mathcal{R} \backslash \{w_r\}$

end

Outpout: $\hat{a}$ the single element in $\mathcal{R}$.

Entropic Lower Bound

Let $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_K)$ and $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_K)$ be two parameters with different maxima: $a^*(\boldsymbol{\mu}) \neq a^*(\boldsymbol{\lambda})$. Then

$$\begin{aligned}\sum_{a=1}^{K} \mathbb{E}_{\boldsymbol{\mu}}\big[N_a(\tau)\big]\,\mathrm{kl}(\mu_a, \lambda_a) &= \mathrm{KL}\Big(\mathbb{P}_{\boldsymbol{\mu}}^{(X_1,\ldots,X_\tau)}, \mathbb{P}_{\boldsymbol{\lambda}}^{(X_1,\ldots,X_\tau)}\Big) \\ &\geq \mathrm{KL}\big(\mathbb{P}_{\boldsymbol{\mu}}^{\mathbb{1}\{\hat{a}_\tau = a^*(\boldsymbol{\mu})\}}, \mathbb{P}_{\boldsymbol{\lambda}}^{\mathbb{1}\{\hat{a}_\tau = a^*(\boldsymbol{\mu})\}}\big) \\ &\geq \mathrm{kl}\Big(\mathbb{P}_{\boldsymbol{\mu}}\big(\hat{a}_\tau = a^*(\boldsymbol{\mu})\big), \mathbb{P}_{\boldsymbol{\lambda}}\big(\hat{a}_\tau = a^*(\boldsymbol{\mu})\big)\Big) \\ &\geq \mathrm{kl}(1-\delta, \delta)\ .\end{aligned}$$

Entropic Lower Bound

[Kaufmann, Cappé, G.'15],[G., Ménard, Stoltz '16]

For every δ-correct procedure, if $a^*(\boldsymbol{\mu}) \neq a^*(\boldsymbol{\lambda})$ then

$$\sum_{a=1}^{K} \mathbb{E}_{\boldsymbol{\mu}}\big[N_a(\tau)\big]\,\mathrm{kl}(\mu_a, \lambda_a) \geq \mathrm{kl}\big(1-\delta, \delta\big)\ .$$

Using the Entropic Lower Bound

Let $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_K)$ and $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_K)$ be two bandit models.

Entropic Lower Bound

If $a^*(\boldsymbol{\mu}) \neq a^*(\boldsymbol{\lambda})$, any δ-correct algorithm satisfies

$$\sum_{a=1}^{K} \mathbb{E}_{\boldsymbol{\mu}}\left[N_a(\tau)\right] \mathrm{kl}(\mu_a, \lambda_a) \geq \mathrm{kl}(\delta, 1-\delta)\ .$$

Using it for each arm separately, one obtains:

Proposition [Kaufmann, Cappé, G.'15]

For any δ-correct algorithm,

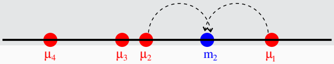
$$\mathbb{E}_{\boldsymbol{\mu}}[\tau] \geq \left(\frac{1}{\mathrm{kl}(\mu_1, \mu_2)} + \sum_{a=2}^{K} \frac{1}{\mathrm{kl}(\mu_a, \mu_1)}\right) \mathrm{kl}(\delta, 1-\delta)\ .$$

Combining the Entropic Lower Bounds

Entropic Lower Bound

If $a^*(\boldsymbol{\mu}) \neq a^*(\boldsymbol{\lambda})$, any δ-correct algorithm satisfies

$$\sum_{a=1}^{K} \mathbb{E}_{\boldsymbol{\mu}}\left[N_a(\tau)\right] \mathrm{kl}(\mu_a, \lambda_a) \geq \mathrm{kl}(\delta, 1-\delta).$$



Let $\mathrm{Alt}(\boldsymbol{\mu}) = \{\boldsymbol{\lambda} : a^*(\boldsymbol{\lambda}) \neq a^*(\boldsymbol{\mu})\}$.

$$\begin{aligned}
\inf_{\boldsymbol{\lambda} \in \mathrm{Alt}(\boldsymbol{\mu})} \sum_{a=1}^{K} \mathbb{E}_{\boldsymbol{\mu}}[N_a(\tau)] \, \mathrm{kl}(\mu_a, \lambda_a) &\geq \mathrm{kl}(\delta, 1-\delta) \\
\mathbb{E}_{\boldsymbol{\mu}}[\tau] \times \inf_{\boldsymbol{\lambda} \in \mathrm{Alt}(\boldsymbol{\mu})} \sum_{a=1}^{K} \frac{\mathbb{E}_{\boldsymbol{\mu}}[N_a(\tau)]}{\mathbb{E}_{\boldsymbol{\mu}}[\tau]} \mathrm{kl}(\mu_a, \lambda_a) &\geq \mathrm{kl}(\delta, 1-\delta) \\
\mathbb{E}_{\boldsymbol{\mu}}[\tau] \times \left(\sup_{w \in \Sigma_K} \inf_{\boldsymbol{\lambda} \in \mathrm{Alt}(\boldsymbol{\mu})} \sum_{a=1}^{K} w_a \, \mathrm{kl}(\mu_a, \lambda_a) \right) &\geq \mathrm{kl}(\delta, 1-\delta)
\end{aligned}$$

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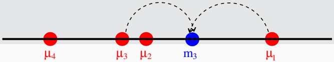

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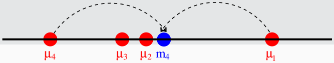

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$$\mathbb{E}_{\boldsymbol{\mu}}[\tau] \times \inf_{\boldsymbol{\lambda} \in \mathrm{Alt}(\boldsymbol{\mu})} \sum_{a=1}^{K} \frac{\mathbb{E}_{\boldsymbol{\mu}}[N_a(\tau)]}{\mathbb{E}_{\boldsymbol{\mu}}[\tau]} \mathrm{kl}(\mu_a, \lambda_a) \geq \mathrm{kl}(\delta, 1-\delta)$$

$$\mathbb{E}_{\boldsymbol{\mu}}[\tau] \times \left(\sup_{w \in \Sigma_K} \inf_{\boldsymbol{\lambda} \in \mathrm{Alt}(\boldsymbol{\mu})} \sum_{a=1}^{K} w_a \, \mathrm{kl}(\mu_a, \lambda_a) \right) \geq \mathrm{kl}(\delta, 1-\delta)$$

Combining the Entropic Lower Bounds

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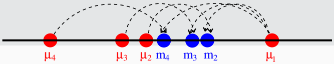

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Lower Bound: the Complexity of BAI

Theorem [G. and Kaufmann, 2016]

For any δ-correct algorithm,

$$\mathbb{E}_{\boldsymbol{\mu}}[\tau] \geq T^*(\boldsymbol{\mu}) \, \mathrm{kl}(\delta, 1-\delta) \,,$$

where

$$T^*(\boldsymbol{\mu})^{-1} = \sup_{w \in \Sigma_K} \inf_{\boldsymbol{\lambda} \in \mathrm{Alt}(\boldsymbol{\mu})} \sum_{a=1}^{K} w_a \, \mathrm{kl}(\mu_a, \lambda_a)$$

$$= \max_{w \in \Sigma_K} \min_{b \neq a^*} \inf_{\mu_b \leq \lambda \leq \mu_{a^*}} w_{a^*} \, \mathrm{kl}(\mu_{a^*}, \lambda) + w_b \, \mathrm{kl}(\mu_b, \lambda) \,.$$

- A kind of game : you choose the proportions of draws $(w_a)_a$, the opponent chooses the alternative.

➜ the optimal proportions of arm draws are

$$w^*(\boldsymbol{\mu}) = \underset{w \in \Sigma_K}{\mathrm{argmax}} \inf_{\lambda \in \mathrm{Alt}(\boldsymbol{\mu})} \sum_{a=1}^{K} w_a \, \mathrm{kl}(\mu_a, \lambda_a) \,.$$

What if I want only an ϵ-optimal distribution?

Now, at round t, you may:

- choose an option $A_t = \phi_t\left(A_1, X_1, \dots, A_{t-1}, X_{t-1)}\right) \in \{1, \dots, K\}$
- observe a new independent sample $X_t \sim \nu_{A_t}$

so as to identify any ϵ-optimal $a \in \mathcal{A}_\epsilon = \left\{a : \mu_a \geq \mu^* - \epsilon\right\}$where $\mu^* = \max_a \mu_a$ as fast as possible: stopping time $\tau_{\delta,\epsilon}$.

$\implies$ minimize $\mathbb{E}[\tau_{\delta,\epsilon}]$ under the PAC constraint $\mathbb{P}_{\boldsymbol{\mu}}\left(\mu_{\hat{a}_\tau} < \mu^* - \epsilon\right) \leq \delta$.

- PAC constraint: Probably Approximately Correct;
- more natural objective, especially in the context of discretized optimization;
- permits to avoid infinite loops in case of draws.

Apparently more complicated

Theorem [G. and Kaufmann, to be finished...]

For any ϵ, δ-PAC algorithm with converging proportions of draws,

$$\liminf_{\delta \to 0} \frac{\mathbb{E}_{\boldsymbol{\mu}}[\tau]}{\log \frac{1}{\delta}} \geq T^*_\epsilon(\boldsymbol{\mu}),$$

where

$$T^*_\epsilon(\boldsymbol{\mu})^{-1} = \sup_{w \in \Sigma_K} \max_{a \in \mathcal{A}_\epsilon} \min_{b \neq a} \inf_{(\lambda_a, \lambda_b) : \lambda_a \leq \lambda_b - \epsilon} w_a \operatorname{kl}(\mu_a, \lambda_a) + w_b \operatorname{kl}(\mu_b, \lambda_b) \, .$$

- Asymptotic result (only).
- Assumption on the algorithm (convergence of the proportions of draws).
- We do not manage to use the information-theoretic technique! We have to go back to the change of measure...
  even for the sequential test!