Synchronization and random long time dynamics for noisy rotators in interaction

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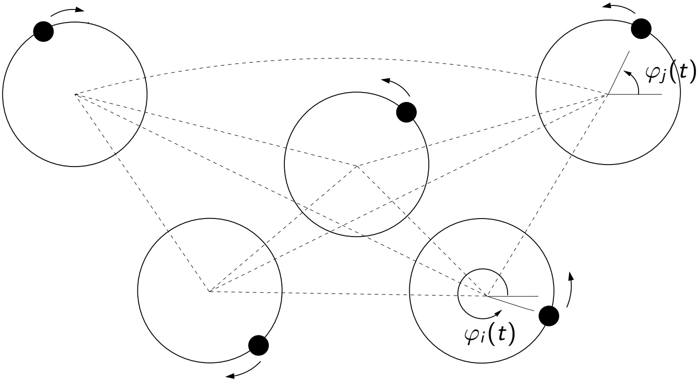
In collaboration with Lorenzo Bertini (Sapienza) and Giambattista Giacomin (LPMA)

Rotators in interaction



The model

We consider the system of N stochastic differential equations

$$\mathrm{d}\varphi_j(t) = -\frac{K}{N}\sum_{i=1}^{N}\sin\left(\varphi_j(t)-\varphi_i(t)\right)\,\mathrm{d}t + \sigma\,\mathrm{d}w_j(t)$$

for $j = 1, 2, \dots, N$, where

- $\{w_j(\cdot)\}_{j=1,2,\dots}$ are IID standard Brownian motions
- $K \geq 0$ and $\sigma > 0$

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The *real* parameter of the model is K/σ^2. In the following we take $\sigma = 1$.

Reversibility and Symmetry

System reversible with respect to the Gibbs measure :

$$\pi_{N,K}(\mathrm{d}\varphi) \propto \exp\left(\frac{K}{N}\sum_{i,j=1}^{N}\cos(\varphi_i - \varphi_j)\right)\lambda_N(\mathrm{d}\varphi),$$

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Rotation symmetry :

- Dynamical : if $\{\varphi_j(t)\}_{j=1\dots N}$ is solution, $\{\varphi_j(t)+\psi\}_{j=1\dots N}$ is solution for all $\psi\in\mathbb{S}$.
- Statical :

$$\pi_{N,K}\Theta_\psi\;=\;\pi_{N,K}\,,$$

where for all $\psi\in\mathbb{S}$

$$\Theta_\psi(\varphi)_j\;=\;\varphi_j+\psi\qquad\text{for all}\qquad j=1\dots N\,.$$

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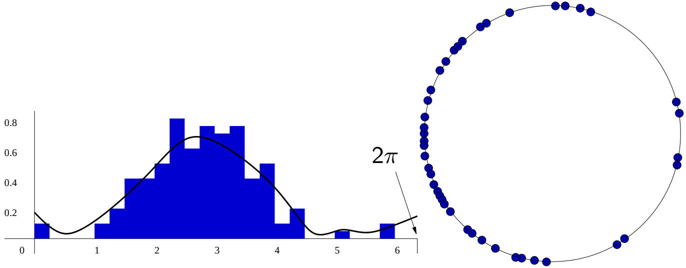

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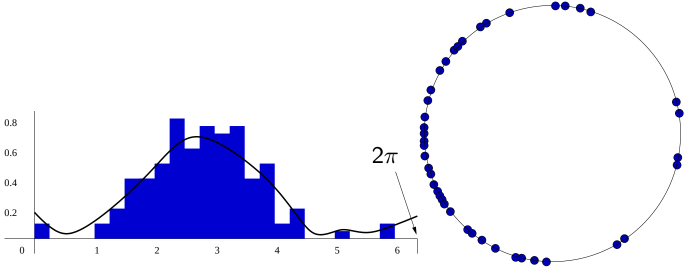

The model can be expressed as

$$\mathrm{d}\varphi_j(t) \;=\; \Big(J * \mu_{N,t}\Big)(\varphi_j(t))\,\mathrm{d}t + \mathrm{d}w_j(t)\,,$$

with $J(\theta) = -K\sin\theta$.

The empirical measure and the $N \to \infty$ limit

Recall

$$\mathrm{d}\varphi_j(t) = \frac{1}{N}\sum_{i=1}^{N} J\left(\varphi_j(t) - \varphi_i(t)\right)\,\mathrm{d}t + \mathrm{d}w_j(t)\,,$$

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Suppose $\lim_{N\to\infty}\mu_{N,0} = p_0(\theta)\,\mathrm{d}\theta$, and fix a time $T > 0$ independent from N. Then $\lim_{N\to\infty}\mu_{N,t}(\,\mathrm{d}\theta) = p_t(\theta)\,\mathrm{d}\theta$ in $C^0([0,T];\mathcal{M}_1)$, with

$$\partial_t p_t(\theta) = \frac{1}{2}\partial_\theta^2 p_t(\theta) - \partial_\theta\left[p_t(\theta)(J * p_t)(\theta)\right]\,.$$

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Important observations :

- No space and no time rescaling

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Important observations :

- No space and no time rescaling
- Rotation symmetry conserved.

Stationary solutions of the limit PDE

- $q(\theta) = \frac{1}{2\pi}$ is always a stationary solution.

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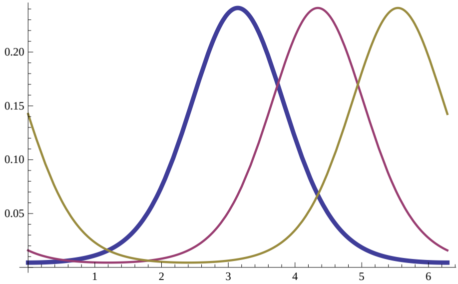
- If $K > 1$, the limit model admits moreover a manifold of synchronized stationary solutions

$$M_0 = \{q_\psi(\cdot) : \psi \in \mathbb{S}^1\},$$

where $q_\psi(\cdot) = q_0(\cdot - \psi)$.



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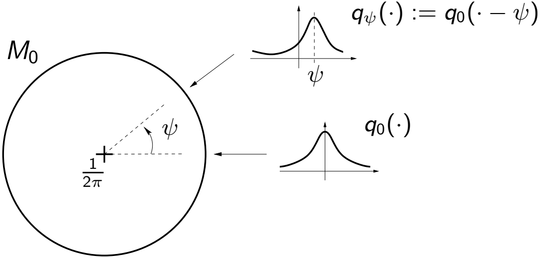

Local stability : $K > 1$

Define the operator of the linearized evolution at the neighborhood of a stationary profile q :

$$-L_q u(\theta) \; := \; \frac{1}{2}u'' - [uJ * q + qJ * u]' \, .$$

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- M_0 is locally stable :

Theorem [Bertini,Giacomin,Pakdaman,2010]

L_q is self-adjoint in $H_{-1,1/q}$ and has a discrete non-negative spectrum, has no effect on the tangent space of M_0 at q ($L_q q' = 0$), and admits a spectral gap on the normal space.

Global behaviour

Define

$$U = \left\{ p \in \mathcal{M}_1, \int_{\mathbb{S}} \exp(i\theta) p(\,\mathrm{d}\theta) = 0 \right\}.$$

If $p_0 \in U$, then $p_t \to 1/2\pi$ (heat equation).

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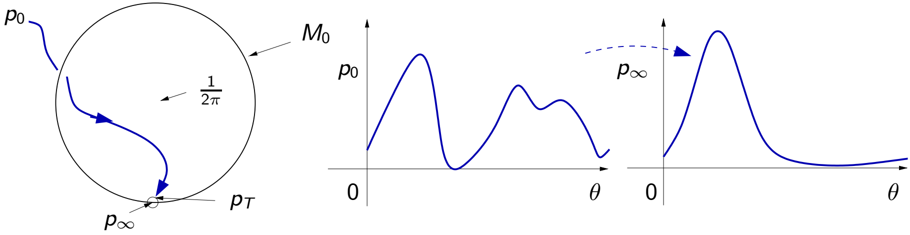
Theorem [Giacomin,Pakdaman,Pellegrin,2012]

If $p_0 \in \mathcal{M}_1 \setminus U$, then there exists $\psi \in \mathbb{S}$ such that $\lim_{t\to\infty} p_t =: p_\infty = q_\psi$ in $C^k(\mathbb{S}, \mathbb{R})$ (for all k)



Gradient flow point of view

The Fokker-Planck PDE can be rewritten in a *gradient form* :

$$\partial_t p_t(\theta) \;=\; \nabla \left[p_t(\theta) \nabla \left(\frac{\delta \mathcal{F}(p_t)}{\delta p_t}(\theta) \right) \right] \,,$$

where (with $\widetilde{J}(\cdot) = K \cos(\cdot)$)

$$\mathcal{F}(p) \;:=\; \frac{1}{2} \int_{\mathbb{S}} p(\theta) \log p(\theta) \, \mathrm{d}\theta + \frac{1}{2} \int_{\mathbb{S}^2} \widetilde{J}(\theta - \theta') p(\theta) p(\theta') \, \mathrm{d}\theta \, \mathrm{d}\theta' \,.$$

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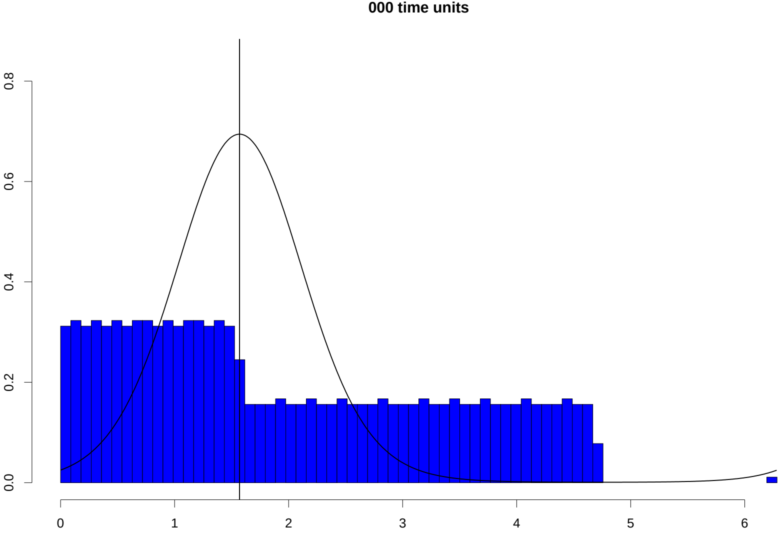
$\mathcal{F}$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}(p_t) \,=\, -\int_{\mathbb{S}} p_t(\theta)\left[\nabla\left(\frac{\delta\mathcal{F}(p_t)}{\delta p_t}(\theta)\right)\right]^2\mathrm{d}\theta\,.$$

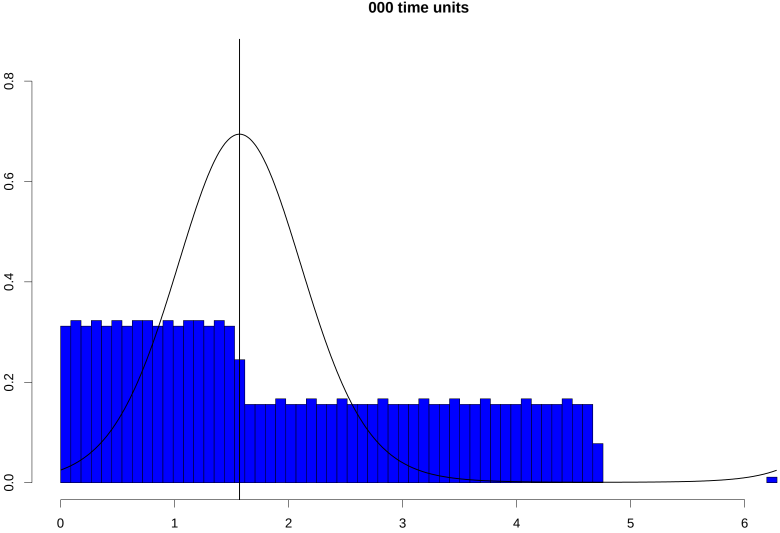
$N = 1000$, $K = 2$, $\sigma = 1$

Movie of the evolution of the empirical measure up to time $t = 15$



$N = 1000$, $K = 2$, $\sigma = 1$, but much faster

Movie of the evolution of the empirical measure up to time $t = 8000$



Long time fluctuations

Theorem [Bertini, Giacomin, P. (2013)]

Fix a constant τ_f and a phase $\psi_0 \in \mathbb{S}^1$. If for all $\varepsilon > 0$

$$\lim_{N\to\infty} \mathbb{P}\left(\|\mu_{N,0} - q_{\psi_0}\|_{-1} \leq \varepsilon \right) = 1 \,,$$

then

$$\lim_{N\to\infty} \mathbb{P}\left(\sup_{\tau\in[0,\tau_f]} \left\|\mu_{N,\tau N} - q_{\psi^N_\tau}\right\|_{-1} \leq \varepsilon \right) = 1 \,,$$

where

- $\psi^N_\tau = \psi_0 + D_K W^N_\tau$,
- W^N converges to a standard Brownian motion,
- $D_K = \|q'\|^{-1}_{-1,1/q}$.