Detection of non-constant long memory parameter

F. Lavancier[1], R. Leipus[2], A. Philippe[1], D. Surgailis[2]

[1]Laboratoire Jean Leray, Université de Nantes, France

[2]Institute of Mathematics and Informatics, Vilnius University, Lithuania

Journées MAS 2014, Toulouse.

1. Long memory processes

Long memory processes (stationary case)

Let $\{X_n, n \in \mathbb{N}\}$ be a second order stationary process with autocovariance function γ

$$\gamma(k) = cov(X_0, X_k).$$

Definition

$\{X_n\}$ is said to have long memory if $\sum_{k\in\mathbb{Z}} |\gamma(k)| = \infty$.

Asymptotic theory and statistical inference is possible if we specify the rate of convergence of γ

Long memory processes (stationary case)

Let $\{X_n, n \in \mathbb{N}\}$ be a second order stationary process with autocovariance function γ

$$\gamma(k) = cov(X_0, X_k).$$

Definition

$\{X_n\}$ is said to have long memory if $\sum_{k\in\mathbb{Z}} |\gamma(k)| = \infty$.

Asymptotic theory and statistical inference is possible if we specify the rate of convergence of γ

Definition

$d \in]0, 1/2[$ is the long memory parameter of $\{X_n\}$ if there exists a slowly varying function L at infinity such that

$$\gamma(k) = k^{2d-1} L(k).$$

Example : linear models $I(d)$

- $X_n \sim I(0)$ if

$$X_n = \sum_{j=0}^{\infty} a_j \zeta_{n-j}$$

where $\{\zeta_j\}$ i.i.d in L^2, $\sum |a_j| < \infty$, $\sum a_j \neq 0$.

$\longrightarrow$ short memory (i.e. the covariances are summable)

Example : linear models $I(d)$

- $X_n \sim I(0)$ if

$$X_n = \sum_{j=0}^{\infty} a_j \zeta_{n-j}$$

where $\{\zeta_j\}$ i.i.d in L^2, $\sum |a_j| < \infty$, $\sum a_j \neq 0$.
$\longrightarrow$ short memory (i.e. the covariances are summable)

- $X_n \sim I(d)$ for $|d| < 0.5$ if

$$X_n = (1 - L)^{-d} \epsilon_n = \sum_{j=0}^{\infty} \psi_j(d) \epsilon_{n-j} \qquad (1)$$

where $\epsilon_n \sim I(0)$ and $\{\psi_j(d)\}$ satisfy $(1 - z)^{-d} = \sum_{j=0}^{\infty} \psi_j(d) z^j$, $|z| < 1$.
$\longrightarrow$ long memory for $0 < d < 0.5$.

Example : linear models $I(d)$

- $X_n \sim I(0)$ if

$$X_n = \sum_{j=0}^{\infty} a_j \zeta_{n-j}$$

where $\{\zeta_j\}$ i.i.d in L^2, $\sum |a_j| < \infty$, $\sum a_j \neq 0$.
$\longrightarrow$ short memory (i.e. the covariances are summable)

- $X_n \sim I(d)$ for $|d| < 0.5$ if

$$X_n = (1-L)^{-d}\epsilon_n = \sum_{j=0}^{\infty} \psi_j(d)\epsilon_{n-j} \qquad (1)$$

where $\epsilon_n \sim I(0)$ and $\{\psi_j(d)\}$ satisfy $(1-z)^{-d} = \sum_{j=0}^{\infty} \psi_j(d) z^j$, $|z| < 1$.
$\longrightarrow$ long memory for $0 < d < 0.5$.

Example : If ϵ_n follows an ARMA(p,q) model then

$\epsilon_n \sim I(0)$

X_n in (1) follows a FARIMA(p,d,q) model and $X_n \sim I(d)$

Example : linear models $I(d)$, extension to $d > 0.5$

We can extend the definition of $I(d)$ processes to the non-stationary case, i.e. $d > 0.5$:

- $X_n \sim I(d)$ for $d > 0.5$ if

$$X_n = \sum_{j=1}^{n} Y_j$$

where $Y_n \sim I(d-1)$

Example : linear models $I(d)$, extension to $d > 0.5$

We can extend the definition of $I(d)$ processes to the non-stationary case, i.e. $d > 0.5$:

- $X_n \sim I(d)$ for $d > 0.5$ if

$$X_n = \sum_{j=1}^{n} Y_j$$

where $Y_n \sim I(d-1)$

Informally, for any $d > -0.5$, $X_n \sim I(d)$ if there exists $\epsilon_n \sim I(0)$ s.t.

$$(1-L)^d X_n = \epsilon_n.$$

Example : linear models $I(d)$, extension to $d > 0.5$

We can extend the definition of $I(d)$ processes to the non-stationary case, i.e. $d > 0.5$:

- $X_n \sim I(d)$ for $d > 0.5$ if

$$X_n = \sum_{j=1}^{n} Y_j$$

where $Y_n \sim I(d-1)$

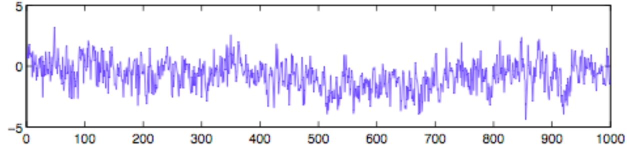
Informally, for any $d > -0.5$, $X_n \sim I(d)$ if there exists $\epsilon_n \sim I(0)$ s.t.

$$(1-L)^d X_n = \epsilon_n.$$

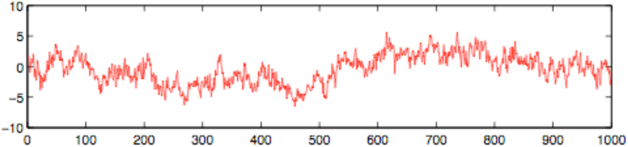
$\longrightarrow$ d is abusively called the long memory parameter in all cases.

Examples of $I(d)$ processes

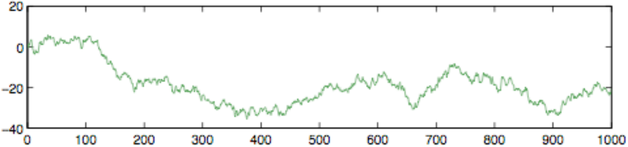
$d = 0.4$



$d = 0.6$



$d = 1$



Asymptotic of partial sums for $I(d)$ processes

Let $X_n \sim I(d)$ for some $d \in (-0.5, 1.5)$, $d \neq 0.5$.

If $d \leq 0$, assume in addition $\mathbb{E}|\zeta_1|^p < \infty$, for some $p > 1/(0.5 + d)$.

Then, as $n \to \infty$,

$$\left(n^{-d-0.5} \sum_{k=1}^{\lfloor nt \rfloor} X_k \right)_{t \in [0,1]} \underset{D[0,1]}{\longrightarrow} \kappa \Big(B_{d+0.5}(t) \Big)_{t \in [0,1]}$$

where $\kappa > 0$ and

- for $-0.5 < d < 0.5$, $B_{d+0.5}$ is the fractional Brownian motion.
- for $0.5 < d < 1.5$, $B_{d+0.5}(\tau) = \int_0^\tau B_{d-0.5}(u) \mathrm{d}u$.

If $d = 0$: this is the standard functional CLT

If $d \neq 0$: non-standard normalisation and the limiting process has dependent increments.

Applications

Application to the non-standard behaviour of the partial sums :

1. test for the presence of long memory :
   $$H_0 : d = 0 \quad \text{vs} \quad H_1 : d \neq 0$$
   [Giraitis Kokoszka Leipus Teyssière 2003]
2. test for long memory versus non-stationarity
   $$H_0 : d < 0.5 \quad \text{vs} \quad H_1 : d > 0.5$$
   [Giraitis Leipus Philippe 2006]
3. a two-sample test for comparison of long memory parameters between two time series
   $$H_0 : d_1 = d_2 \quad \text{vs} \quad H_1 : d_1 \neq d_2$$
   [Lavancier Philippe Surgailis 2010]
4. detection of non-constant long memory parameter...

2 Examples of non-constant long memory parameter models

Examples of non-constant long memory parameter models

Recall the FARIMA coefficients $\psi_j(d) = \frac{d}{1} \cdot \frac{d+1}{2} \cdots \frac{d-1+j}{j} \; = \; \frac{\Gamma(d+j)}{j!\Gamma(d)}$.
Let $\{\zeta_n\}$ be i.i.d. with zero mean and unit variance.

- Sudden change in a FARIMA process, for $d_1 < d_2$

$$X_t = \begin{cases} \sum_{j=1}^{t} \psi_{t-j}(d_1)\,\zeta_j & \text{if } t \leq n/2, \\ \sum_{j=1}^{t} \psi_{t-j}(d_2)\,\zeta_j & \text{if } t > n/2. \end{cases}$$

Examples of non-constant long memory parameter models

Recall the FARIMA coefficients $\psi_j(d) = \frac{d}{1} \cdot \frac{d+1}{2} \cdots \frac{d-1+j}{j} = \frac{\Gamma(d+j)}{j!\Gamma(d)}$.
Let $\{\zeta_n\}$ be i.i.d. with zero mean and unit variance.

- Sudden change in a FARIMA process, for $d_1 < d_2$

$$X_t = \begin{cases} \sum_{j=1}^{t} \psi_{t-j}(d_1)\,\zeta_j & \text{if } t \leq n/2, \\ \sum_{j=1}^{t} \psi_{t-j}(d_2)\,\zeta_j & \text{if } t > n/2. \end{cases}$$

Let $d(.)$ be a function from $[0,1]$ to $]-0.5,\infty[$
We consider time series $(X_t)_{t=1\dots n}$ with changing memory parameter $d(\frac{t}{n})$

Examples of non-constant long memory parameter models

Recall the FARIMA coefficients $\psi_j(d) = \frac{d}{1} \cdot \frac{d+1}{2} \cdots \frac{d-1+j}{j} = \frac{\Gamma(d+j)}{j!\Gamma(d)}$.
Let $\{\zeta_n\}$ be i.i.d. with zero mean and unit variance.

- Sudden change in a FARIMA process, for $d_1 < d_2$

$$X_t = \begin{cases} \sum_{j=1}^{t} \psi_{t-j}(d_1)\, \zeta_j & \text{if } t \leq n/2, \\ \sum_{j=1}^{t} \psi_{t-j}(d_2)\, \zeta_j & \text{if } t > n/2. \end{cases}$$

Let $d(.)$ be a function from $[0,1]$ to $]-0.5, \infty[$
We consider time series $(X_t)_{t=1\dots n}$ with changing memory parameter $d(\frac{t}{n})$

- Rapidly changing memory FARIMA process.

$$b_j(t) := \psi_j\big(d(\tfrac{t}{n})\big), \quad j = 0, 1, \dots,$$

$$X_t = \sum_{j=1}^{t} b_{t-j}(t)\, \zeta_j, \qquad t = 1, \dots, n.$$

Examples of non-constant long memory parameter models

Recall the FARIMA coefficients $\psi_j(d) = \frac{d}{1} \cdot \frac{d+1}{2} \cdots \frac{d-1+j}{j} = \frac{\Gamma(d+j)}{j!\Gamma(d)}$.
Let $\{\zeta_n\}$ be i.i.d. with zero mean and unit variance.

- Sudden change in a FARIMA process, for $d_1 < d_2$

$$X_t = \begin{cases} \sum_{j=1}^{t} \psi_{t-j}(d_1)\, \zeta_j & \text{if } t \leq n/2, \\ \sum_{j=1}^{t} \psi_{t-j}(d_2)\, \zeta_j & \text{if } t > n/2. \end{cases}$$

Let $d(.)$ be a function from $[0,1]$ to $]-0.5, \infty[$
We consider time series $(X_t)_{t=1\dots n}$ with changing memory parameter $d(\frac{t}{n})$

- Rapidly changing memory FARIMA process.

$$b_j(t) \;:=\; \psi_j\big(d(\tfrac{t}{n})\big), \quad j = 0, 1, \dots,$$

$$X_t \;=\; \sum_{j=1}^{t} b_{t-j}(t)\, \zeta_j, \qquad t = 1, \dots, n.$$

- Gradually changing memory FARIMA process

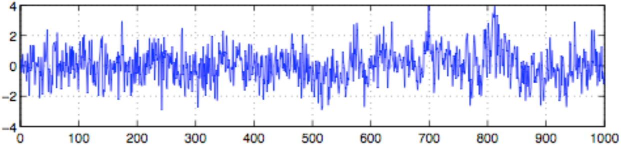
$$\tilde{b}_j(t) \;:=\; \frac{d(\frac{t}{n})}{1} \cdot \frac{d(\frac{t-1}{n}) + 1}{2} \cdots \frac{d(\frac{t-j+1}{n}) - 1 + j}{j}, \;\; j = 1, 2, \dots,$$

$$X_t \;=\; \sum_{j=1}^{t} \tilde{b}_{t-j}(t)\, \zeta_j, \quad t = 1, \dots, n.$$

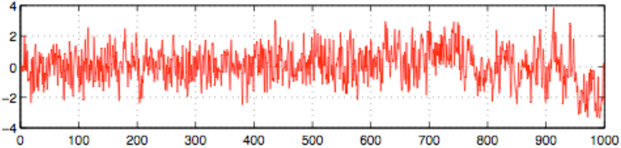
Examples of non-constant long memory parameter models

From $d = 0.1$ to $d = 0.4$

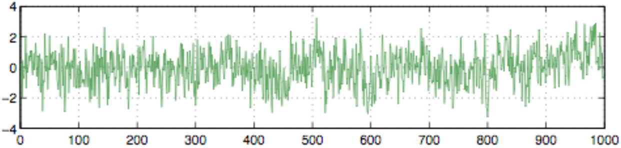
Sudden change
$d = 0.1$ for $k < \frac{n}{2}$
$d = 0.4$ for $k > \frac{n}{2}$



Rapidly changing
For $t \in [0, 1]$,
$d(t) = 0.1 + 0.3t$.



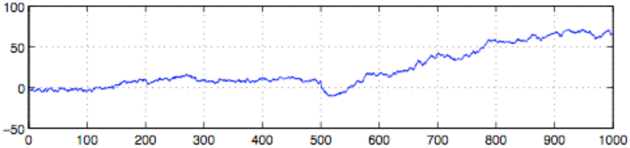
Gradually changing
For $t \in [0, 1]$,
$d(t) = 0.1 + 0.3t$.



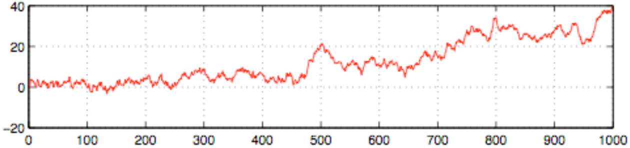
Examples of non-constant long memory parameter models

From $d = 0.8$ to $d = 1.1$

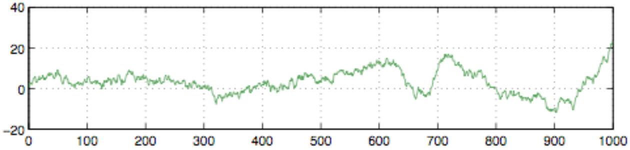
Sudden change
$d = 0.8$ for $k < \frac{n}{2}$
$d = 1.1$ for $k > \frac{n}{2}$



Rapidly changing
For $t \in [0, 1]$,
$d(t) = 0.8 + 0.3t$.



Gradually changing
For $t \in [0, 1]$,
$d(t) = 0.8 + 0.3t$.



Testing for an increase in persistence

Basically, we want to test

$$\mathbf{H_0} : \text{"no change in persistence of } X\text{"}$$

$$\text{against} \quad \mathbf{H_1} : \text{"an increase in persistence of } X\text{".}$$

Testing for an increase in persistence

Basically, we want to test

$$\mathbf{H_0} : \text{"no change in persistence of } X\text{"}$$

$$\text{against} \quad \mathbf{H_1} : \text{"an increase in persistence of } X\text{"}.$$

These hypotheses are specified through the asymptotic of **partial sums**.
Let us consider forward and backward partial sums of X :

$$S_k = \sum_{j=1}^{k} X_j, \qquad S^*_{n-k} = \sum_{j=k+1}^{n} X_j.$$

Testing for an increase in persistence

Basically, we want to test

$$\mathbf{H_0} : \text{"no change in persistence of } X\text{"}$$
$$\text{against} \quad \mathbf{H_1} : \text{"an increase in persistence of } X\text{"}.$$

These hypotheses are specified through the asymptotic of **partial sums**. Let us consider forward and backward partial sums of X :

$$S_k = \sum_{j=1}^{k} X_j, \qquad S^*_{n-k} = \sum_{j=k+1}^{n} X_j.$$

$\mathbf{H_0}$: There exist $d \in (-.5, 1.5)$, $d \neq .5$, $\kappa > 0$ such that

$$n^{-d-.5} S_{[n\tau]} \quad \longrightarrow_{D[0,1]} \quad \kappa B_{d+.5}(\tau).$$

$\longrightarrow$ $\mathbf{H_0}$ is satisfied for $I(d)$ processes.

Testing for an increase in persistence

Basically, we want to test

H₀ : "no change in persistence of X"

against **H₁** : "an increase in persistence of X".

These hypotheses are specified through the asymptotic of **partial sums**.
Let us consider forward and backward partial sums of X :

$$S_k = \sum_{j=1}^{k} X_j, \qquad S^*_{n-k} = \sum_{j=k+1}^{n} X_j.$$

$\mathbf{H_0}$: There exist $d \in (-.5, 1.5)$, $d \neq .5$, $\kappa > 0$ such that

$$n^{-d-.5} S_{[n\tau]} \quad \longrightarrow_{D[0,1]} \quad \kappa B_{d+.5}(\tau).$$

$\longrightarrow$ $\mathbf{H_0}$ is satisfied for $I(d)$ processes.

$\mathbf{H_1}$: There exist $0 \leq \upsilon_0 < \upsilon_1 \leq 1$, and $d > -.5$, such that

$$\left(n^{-d-.5} S_{[n\tau_1]},\, n^{-d-.5} S^*_{[n\tau_2]}\right) \quad \longrightarrow_{D[0,\upsilon_1]\times D[0,1-\upsilon_0]} \quad (0, Z(\tau_2)),$$

where $\{Z(\tau), \tau \in [1-\upsilon_1, 1-\upsilon_0]\}$ is a nondegenerate process.

$\longrightarrow$ $\mathbf{H_1}$ is satisfied for the changing memory models presented before, whenever $d(.)$ is a left-continuous nondecreasing function.

Testing for an increase in persistence

These hypotheses can be roughly understood as follows

- under $\mathbf{H}_0$: d is constant. For any t :

$$Var\left(S_{[nt]}\right) = Var\left(\sum_{k=1}^{[nt]} X(k)\right) \approx n^{2d+1}$$

Testing for an increase in persistence

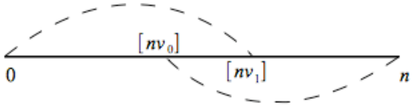
These hypotheses can be roughly understood as follows

- under $\mathbf{H}_0$: d is constant. For any t :

$$Var\left(S_{[nt]}\right) = Var\left(\sum_{k=1}^{[nt]} X(k)\right) \approx n^{2d+1}$$

- Under $\mathbf{H}_1$: d increases, for some $\upsilon_0 < \upsilon_1$

$$Var\left(S_{[n\upsilon_1]}\right) = Var\left(\sum_{k=1}^{[n\upsilon_1]} X(k)\right) \ll Var\left(\sum_{k=[n\upsilon_0]}^{n} X(k)\right) = Var\left(S^*_{[n(1-\upsilon_0)]}\right)$$



Testing for an increase in persistence

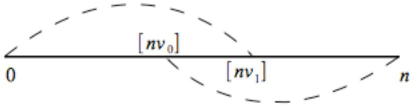
These hypotheses can be roughly understood as follows

- under $\mathbf{H}_0$: <u>d is constant</u>. For any t :

$$Var\left(S_{[nt]}\right) = Var\left(\sum_{k=1}^{[nt]} X(k)\right) \approx n^{2d+1}$$

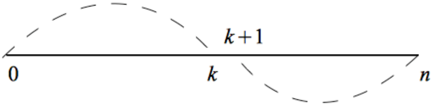
- Under $\mathbf{H}_1$: <u>d increases</u>, for some $\upsilon_0 < \upsilon_1$

$$Var\left(S_{[n\upsilon_1]}\right) = Var\left(\sum_{k=1}^{[n\upsilon_1]} X(k)\right) \ll Var\left(\sum_{k=[n\upsilon_0]}^{n} X(k)\right) = Var\left(S^*_{[n(1-\upsilon_0)]}\right)$$



⇒ To test for $\mathbf{H}_0$ against $\mathbf{H}_1$, we estimate and compare these variances.

Estimating the variances of partial sums



Let k be fixed in $\{1,\dots,n\}$ and consider for $j=1,\dots,k$:

$$\tilde{S}_j = \sum_{i=1}^{j}(X_i - \overline{X}_k), \qquad \tilde{S}^*_{n-j} = \sum_{i=j+1}^{n}(X_i - \overline{X}^*_{n-k}).$$

where $\overline{X}_k = \frac{1}{k}\sum_{i=1}^{k} X_i$ and $\overline{X}^*_{n-k} = \frac{1}{n-k}\sum_{i=k+1}^{n} X_i$

The variances of the partial sums (before and after k) are estimated by :

- the forward variance up to k : $V_k = \frac{1}{k}\widehat{Var}(\tilde{S}_1,\dots,\tilde{S}_k)$
- the backward variance back to $k+1$: $V^*_{n-k} = \frac{1}{n-k}\widehat{Var}(\tilde{S}^*_{n-k},\dots,\tilde{S}^*_1)$

More precisely

$$V_k = \frac{1}{k^2}\sum_{j=1}^{k}\tilde{S}_j^2 - \left(\frac{1}{k^{3/2}}\sum_{j=1}^{k}\tilde{S}_j\right)^2,$$

$$V^*_{n-k} = \frac{1}{(n-k)^2}\sum_{j=k+1}^{n}\tilde{S}^{*2}_{n-j+1} - \left(\frac{1}{(n-k)^{3/2}}\sum_{j=k+1}^{n}\tilde{S}^*_{n-j+1}\right)^2$$

Our test statistic

Basically : Under $\mathbf{H_0}$, for all k, $V_k \approx V^*_{n-k}$

Under $\mathbf{H_1}$, for some k, $V_k \ll V^*_{n-k}$

Test statistic

We compare the V_k's and V^*_{n-k}'s for $\frac{k}{n} \in \mathcal{T} := [\tau, 1-\tau]$ through

$$I_n = \int_{\mathcal{T}} \frac{V^*_{n-[nt]}}{V_{[nt]}} \mathrm{d}t$$

where τ has to be chosen (typically $\tau = 0.1$).

Our test statistic

Basically : Under $\mathbf{H_0}$, for all k, $V_k \approx V^*_{n-k}$

Under $\mathbf{H_1}$, for some k, $V_k \ll V^*_{n-k}$

Test statistic

We compare the V_k's and V^*_{n-k}'s for $\frac{k}{n} \in \mathcal{T} := [\tau, 1-\tau]$ through

$$I_n = \int_{\mathcal{T}} \frac{V^*_{n-[nt]}}{V_{[nt]}} \mathrm{d}t$$

where τ has to be chosen (typically $\tau = 0.1$).

Other possible choices :

$$W_n = \sup_{t \in \mathcal{T}} \frac{V^*_{n-[nt]}}{V_{[nt]}} \qquad R_n = \frac{\inf_{t \in \mathcal{T}} V^*_{n-[nt]}}{\inf_{t \in \mathcal{T}} V_{[nt]}}$$

Consistency of the test

Consistency of the test

- Under $\mathbf{H_0}$ (e.g. for $I(d)$ processes)

$$I_n \longrightarrow_{law} I(B_{d+.5})$$

where $I(B_{d+.5})$ is explicily given in terms of the fBm $B_{d+.5}$

- Under $\mathbf{H_1}$, if $\mathcal{T} \subset [v_0, v_1]$

$$I_n \longrightarrow_p +\infty$$

Remark

The same result holds for R_n and W_n statistics.

Sketch of proof

If for $0 \le v_0 \le v_1 \le 1$

$$\left(n^{-d_1-.5} S_{\lfloor n\tau_1 \rfloor},\, n^{-d_2-.5} S^*_{\lfloor n\tau_2 \rfloor}\right) \xrightarrow{D[0,v_1]\times D[0,1-v_0]} \left(B_{d_1+.5}(\tau_1), B^*_{d_2+.5}(\tau_2)\right) \tag{2}$$

where $B^*_{d+.5}(u) \;:=\; B_{d+.5}(1) - B_{d+.5}(1-u),\ u \in [0,1]$, then

- by the continuous mapping theorem

  $$\left(n^{-2d_1} V_{\lfloor n\tau_1 \rfloor},\, n^{-2d_2} V^*_{n-\lfloor n\tau_2 \rfloor}\right)$$

  converges in $D[0,v_1] \times D[v_0,1]$,
- which implies the convergence in law of

  $$n^{-2(d_2-d_1)} I_n = n^{-2(d_2-d_1)} \int_{\mathcal{T}} \frac{V^*_{n-[nt]}}{V_{[nt]}} \mathrm{d}t.$$

Sketch of proof

If for $0 \leq v_0 \leq v_1 \leq 1$

$$\left(n^{-d_1-.5} S_{\lfloor n\tau_1 \rfloor},\, n^{-d_2-.5} S^*_{\lfloor n\tau_2 \rfloor}\right) \xrightarrow{D[0,v_1]\times D[0,1-v_0]} \left(B_{d_1+.5}(\tau_1), B^*_{d_2+.5}(\tau_2)\right) \tag{2}$$

where $B^*_{d+.5}(u) \;:=\; B_{d+.5}(1) - B_{d+.5}(1-u),\ u \in [0,1]$, then

- by the continuous mapping theorem

  $$\left(n^{-2d_1} V_{\lfloor n\tau_1 \rfloor},\, n^{-2d_2} V^*_{n-\lfloor n\tau_2 \rfloor}\right)$$

  converges in $D[0, v_1] \times D[v_0, 1]$,
- which implies the convergence in law of

  $$n^{-2(d_2-d_1)} I_n = n^{-2(d_2-d_1)} \int_{\mathcal{T}} \frac{V^*_{n-[nt]}}{V_{[nt]}} \mathrm{d}t.$$

Under $\mathbf{H_0}$, $d_1 = d_2$ in (2) $\Longrightarrow$ I_n converges in law
Under $\mathbf{H_1}$, $d_1 < d_2$ in (2) $\Longrightarrow$ $I_n = O_P(n^{2(d_2-d_1)}) \to \infty$

Testing Procedure

1. Choose a type I error $\alpha \in (0,1)$
2. Reject $\mathbf{H_0}$ if
$$I_n > q(\alpha, d)$$
where $q(\alpha, d) := \inf\{x : P(I(B_{d+.5}) \leq x) \geq 1 - \alpha)\}$
3. The critical region of the test depends on unknown memory parameter d.
In practice : the unknown parameter d is replaced by a consistent estimator $\hat{d}$.

Testing Procedure

1. Choose a type I error $\alpha \in (0,1)$
2. Reject $\mathbf{H_0}$ if
$$I_n > q(\alpha, d)$$
where $q(\alpha, d) := \inf\{x : P(I(B_{d+.5}) \leq x) \geq 1 - \alpha)\}$
3. The critical region of the test depends on unknown memory parameter d.
In practice : the unknown parameter d is replaced by a consistent estimator $\hat{d}$.

Proposition

If $\hat{d} \longrightarrow_p d$ then $q(\alpha, \hat{d}) \longrightarrow_p q(\alpha, d)$, in other words the asymptotic significance level α is guaranteed

Testing Procedure

1. Choose a type I error $\alpha \in (0,1)$
2. Reject $\mathbf{H_0}$ if
$$I_n > q(\alpha, d)$$
where $q(\alpha, d) := \inf\{x : P(I(B_{d+.5}) \leq x) \geq 1 - \alpha)\}$
3. The critical region of the test depends on unknown memory parameter d.
In practice : the unknown parameter d is replaced by a consistent estimator $\hat{d}$.

Proposition

If $\hat{d} \longrightarrow_p d$ then $q(\alpha, \hat{d}) \longrightarrow_p q(\alpha, d)$, in other words the asymptotic significance level α is guaranteed

Our choice of $\hat{d}$: the Non-Stationarity Extended Local Whittle Estimator (NELWE) of Abadir et al. which applies to both the stationary ($|d| < .5$) and non-stationary ($d > .5$) case.

Comparison of different statistics (n=5000)

Percentages of rejection for a sudden change (at $n/2$) from $I(d_1)$ to $I(d_2)$ based on series of size $n = 5000$ and on 10^4 replications.

		$\tau = 0.05$					$\tau = 0.1$					$\tau = 0.2$				
τ	$d_2 \backslash d_1$	0	0.1	0.2	0.3	0.4	0	0.1	0.2	0.3	0.4	0	0.1	0.2	0.3	0.4
W_n	0	3.4					3.6					4.1				
	0.1	20.2	3.7				24.2	4.1				28.3	4.5			
	0.2	50.1	16	3.8			61.1	20.9	4.5			68.6	26	4.7		
	0.3	74.6	38.5	13.3	3.6		87.1	51.1	17.7	4.1		93.1	62	23.6	4.7	
	0.4	88.7	64.1	31.2	11.6	3.7	96.6	79.6	44.3	15	4	99.2	88.8	56.7	20	4.4
R_n	0	3.7					3.9					4.1				
	0.1	29	5.1				30.9	4.9				31.3	5			
	0.2	65	25.3	4.9			71	28.1	4.8			73.1	30.5	4.6		
	0.3	85.3	53.8	21.2	5		91.7	62.2	24.4	4.9		94.3	67	26.5	4.9	
	0.4	94.1	75.7	44.8	18.4	4.8	98	84.8	54.1	21.7	4.6	99.3	90.3	60.3	24.4	4.5
I_n	0	2.9					3.2					3.6				
	0.1	28.5	3.4				29.5	3.8				30.7	4.1			
	0.2	73.4	27.8	3.6			74	28.8	3.9			73.7	30.1	4.2		
	0.3	95.9	68	24.8	3.5		96.3	69.8	26.5	3.7		95.9	69.7	27.9	4.3	
	0.4	99.5	92.3	62.5	21.8	3.5	99.7	93.6	65.7	24.1	3.8	99.7	94.1	66.6	25.6	4
I_n^{Kim}	0	3.5					3.7					4.2				
	0.1	23.5	3.6				24.1	4				24.6	4.3			
	0.2	58.4	21.4	4.3			58.5	22	4.4			57.4	22.4	4.7		
	0.3	86.1	54.8	19.9	4		86.8	55.8	20.5	4.2		86.1	55.4	20.8	4.5	
	0.4	96.6	82.2	52.1	18.4	4	97.3	83.6	53.6	19.2	3.7	97.3	83.4	53.4	19.7	4

More simulations concerning I_n

Percentages of rejection for a sudden change (at $n/2$) from $I(d_1)$ to $I(d_2)$ based on series of size $n = 5000$ and on 10^4 replications.
The truncation parameter is $\tau = 10\%$.
The blue numbers correspond to the estimated level of the test.

d_2 \ d_1	0	0.1	0.2	0.3	0.4	0.6	0.8	1	1.2	1.4
0	3.2									
0.1	29.5	3.8								
0.2	74	28.8	3.9							
0.3	96.3	69.8	26.5	3.7						
0.4	99.7	93.6	65.7	24.1	3.8					
0.6	100	100	99.9	98.7	92.5	5.2				
0.8	100	100	99.9	99.4	96	11.8	4.9			
1	100	100	100	100	99.7	51.9	34	5.1		
1.2	100	100	100	100	100	91.7	84.2	44.6	5.1	
1.4	100	100	100	100	100	99.1	98.2	88.9	57.8	2.9

More simulations concerning I_n

Percentages of rejection for rapidly changing memory from d_1 to d_2

$$d(t) = d_1 + (d_2 - d_1)t, \quad t \in [0, 1]$$

based on series of size $n = 5000$ and on 10^4 replications.
The truncation parameter is $\tau = 10\%$.

d_2 \ d_1	0	0.1	0.2	0.3	0.4	0.6	0.8	1	1.2	1.4
0	.									
0.1	16.3	.								
0.2	42.3	15.7	.							
0.3	69.6	36.5	13.8	.						
0.4	86.7	61.8	32.8	11.8	.					
0.6	98.3	92.2	75.8	51.2	27.3	.				
0.8	99.8	98.8	94	82.7	62.3	18.5	.			
1	99.9	99.7	98.7	95	86.6	45.9	13.1	.		
1.2	99.9	99.9	99.7	98.2	94.7	72.1	29.9	8.4	.	
1.4	100	99.9	99.8	99.3	97.4	84.7	45.9	13.6	4.4	.

- GIRAITIS L., KOKOSZKA P., LEIPUS R. AND TEYSSIÈRE G. (2003) Rescaled variance and related tests for long memory in volatility and levels. *J. Econometrics* **112**, 265-294.
- GIRAITIS L., LEIPUS R. AND PHILIPPE A. (2006) A test for stationarity versus trends and unit roots for a wide class of dependent errors. *Econometric Th.* **22**, 989–1029.
- LAVANCIER F., PHILIPPE A. AND SURGAILIS D. (2010) A two-sample test for comparison of long memory parameters. *J. of Multivariate Analysis* **101**, 2118-2136
- LAVANCIER F., LEIPUS R., PHILIPPE A. AND SURGAILIS D. (2013) Detection of non-constant long memory parameter. *Econometric Theory* **29**, 1009-1056.
- PHILIPPE A., SURGAILIS D. AND VIANO M.-C. (2008) Time-Varying Fractionally Integrated Processes with Nonstationary Long Memory *Theory of Probability and its Applications*, **52**, 651-673.