Actions de groupes et quotients toriques

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From here

21 juin 2016

1. $\mathbb{C}^* \hookrightarrow \mathbb{C}^3, \; t.(z_1, z_2, z_3) = (tz_1, tz_2, tz_3)$

2. $(\mathbb{C}^*)^2 \hookrightarrow \mathbb{C}^4, \; (t_1, t_2).(z_1, z_2, z_3, z_4) = (t_1 z_1, t_1 z_2, t_2 z_3, t_2 z_4)$

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   $A_2 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$, M_2 est engendré par $\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}$ et $\begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}$.

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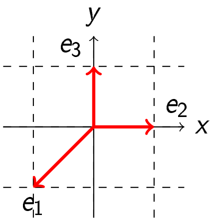
$B_1 = \begin{pmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$, on a alors les vecteurs



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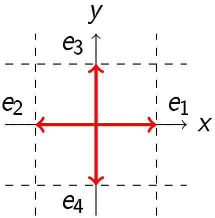
$B_2 = \begin{pmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{pmatrix}$, on a alors les vecteurs



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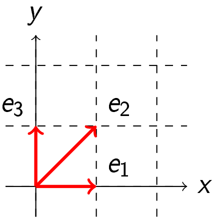


$\mathbb{C}^* \hookrightarrow \mathbb{C}^3, \ \ t.(z_1, z_2, z_3) = (tz_1, tz_2, tz_3)$

solutions $\begin{pmatrix} s \\ t \\ 1-s-t \end{pmatrix}$, $s, t \geq 0$,

$s + t \leq 1 : \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$

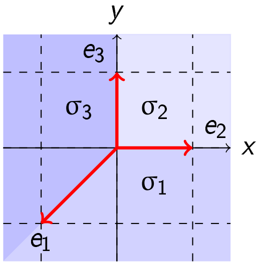
On a $I_1 = \{3\}, I_2 = \{1\}, I_3 = \{2\}$: 3 cônes

$\sigma_1 = Cone(e_1, e_2), \sigma_2 = Cone(e_2, e_3),$

$\sigma_3 = Cone(e_1, e_3)$

et

$U = \mathbb{C}^3 \setminus \{0\}$



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solutions $\begin{pmatrix} s \\ 1-s \\ t \\ 1-t \end{pmatrix}$, $0 \leq s, t \leq 1$:

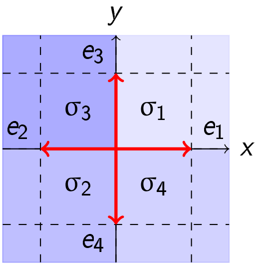
$\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}.$

On a $I_1 = \{2, 4\}, I_2 = \{1, 3\}, I_3 = \{1, 4\}, I_4 = \{2, 3\}$: 4 cônes

$\sigma_1 = Cone(e_1, e_3), \sigma_2 = Cone(e_2, e_4),$
$\sigma_3 = Cone(e_2, e_3), \sigma_4 = Cone(e_1, e_4)$

et

$U = \mathbb{C}^4 \setminus (\{z_1 = z_2 = 0\} \cup \{z_3 = z_4 = 0\})$



$\mathbb{C}^* \hookrightarrow \mathbb{C}^3$, $t.(z_1, z_2, z_3) = (tz_1, t^{-1}z_2, tz_3)$

solutions $\begin{pmatrix} s \\ s+t-1 \\ t \end{pmatrix}$, $s, t \geq 0$,

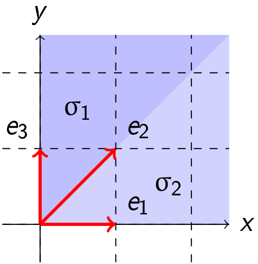
$s + t \geq 1$: $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

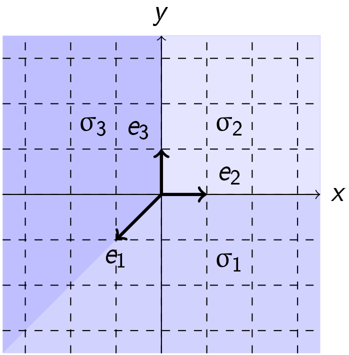
On a $I_1 = \{1\}, I_2 = \{3\}, I_3 = \{1, 2, 3\}$: 3 cônes

$\sigma_1 = Cone(e_2, e_3), \sigma_2 = Cone(e_1, e_2)$, $\sigma_3 = Cone(\emptyset) = \{0\}$

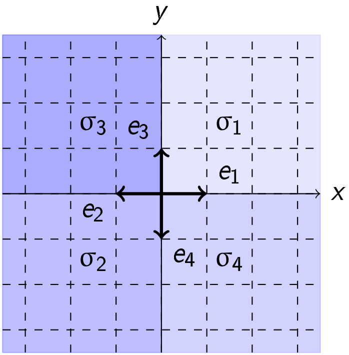
et

$U = \mathbb{C}^3 \setminus \{z_1 = z_3 = 0\}$

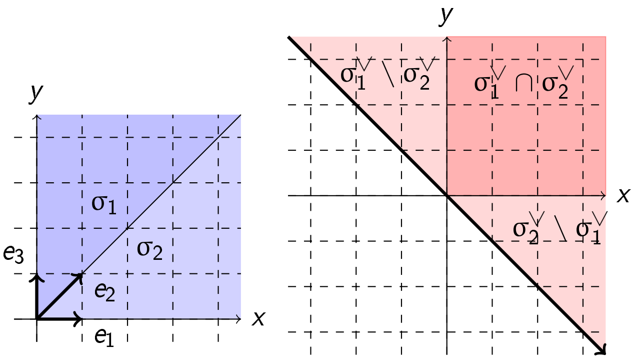




3 copies de $\mathbb{C}^2$ recollées : $\mathbb{C}[x,y], \mathbb{C}[xy^{-1}, y^{-1}], \mathbb{C}[x^{-1}, x^{-1}y]$:
on obtient $\mathbb{P}^2(\mathbb{C}) = \mathbb{C}^3 \setminus \{0\}/\mathbb{C}^*$.



4 copies de $\mathbb{C}^2$ recollées : $\mathbb{C}[x,y], \mathbb{C}[x^{-1},y], \mathbb{C}[x,y^{-1}], \mathbb{C}[x^{-1},y^{-1}]$: on obtient
$\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}) = \left(\mathbb{C}^4 \setminus (\{z_1 = z_2 = 0\} \cup \{z_3 = z_4 = 0\})\right) / (\mathbb{C}^*)^2.$



2 copies de $\mathbb{C}^2$ recollées : $\mathbb{C}[x, x^{-1}y], \mathbb{C}[y, xy^{-1}]$:
on obtient $\mathbb{C}^2$ éclaté en un point !

Merci de votre attention !

Je tiens à remercier également tous mes fans pour cet exposé, eux sans qui rien n'aurait été possible.