Universal Galois groups of q-difference equations

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q-difference equations

- Let $K := \mathbb{C}(\{z\})$ be the field of convergent Laurent series. We take a complex number q such that $|q| > 1$ and define the operator σ_q on K by $\sigma_q(f)(z) = f(qz)$.
- A (linear) q-difference equation is an equation with coefficients in K of the form

$$f(q^n z) + a_{n-1} f(q^{n-1} z) + \cdots + a_0 f(z) = 0$$

$$\Leftrightarrow \sigma_q^n f + a_{n-1} \sigma_q^{n-1} f + \cdots + a_0 f = 0$$

q-difference modules I

- q-difference equations can be seen as systems $\sigma_q(X) = AX, A \in GL_n(K)$, left $K < \sigma_q, \sigma_q^{-1} >$-modules of finite length or as q-difference modules.
- A q-difference module on K is a pair (V, ϕ) where V is a K-vector space of finite dimension and ϕ is a σ_q-linear automorphism on V.

q-difference modules II

- The q-difference module associated to a q-difference system $\sigma_q X = AX$ is of the form (K^n, ϕ_A) with $\phi_A = A^{-1}\sigma_q$.
- Each q-difference system is isomorphic to one of these by choice of a basis of V.

Slopes of a q-difference module I

- Each q-difference module seen as a $K < \sigma_q, \sigma_q^{-1} >$ is isomorphic to a module of the form $\frac{K<\sigma_q,\sigma_q^{-1}>}{K<\sigma_q,\sigma_q^{-1}>L}$ where L is an operator of the form $\sum_{i=0}^{n} a_i \sigma_q^i$ (vector cyclic lemma).
- To such an operator L we associate a Newton polygon by taking the convex hull of $\{(i,j) | j \geq v_0(a_i)\}$ where v_0 is the valuation on K.

Slopes of a q-difference module II

- The slopes of the lower boundary of this Newton polygon are called slopes of the q-difference module; there constitue an invariant for q-difference modules.
- A module with one slope is said to be pure isoclinic and a module which is a direct sum of pure isoclinic modules is said to be pure.

Slopes of a q-difference module III

- To a q-difference module M of slopes $\mu_1 < \cdots < \mu_k$ we can associate a tower of submodules

$$0 \subset M_1 \subset \cdots \subset M_k = M$$

such that the $\frac{M_i}{M_{i-1}}$ are pure isoclinics of slope μ_i.

- Furthermore the associated graded module $\bigoplus \frac{M_i}{M_{i-1}}$ is unique up to isomorphism and define a functor gr from q-difference modules to pure q-difference modules.

Tannakian Galois group of a category I

Let k be an algebraic closed field of characteristic 0.

- A tannakian category over k is an abelian rigid symetric monoidal category such that $End(I) = k$ where I is the unit for the monoidal structure.
- A fiber functor over a tannakian category $\mathcal{C}$ is a functor $\mathcal{C} \to \mathbf{Vect}_k^f$ wich is exact, faithfull, k-linear and tensor-compatible. A tannakian category with a fiber functor is said to be neutral.

Tannakian Galois group of a category II

- We call tannakian Galois group of the tannakian category $\mathcal{C}$ neutralized by ω the affine group scheme $\underline{Aut}^{\otimes}(\omega)$.
- It is a proalgbraic group and an algebraic group if and only if $\mathcal{C}$ is generated as a tannakian category by one element.

Tannakian Galois group of a category III

- A neutral tannakian category is equivalent to the category of representations of its tannakian Galois group.
- Usual Galois groups (of an object) can be seen as tannakian Galois group of the tannakian category generated by the object or as the image of the representation associated.

Tannakian formalism I

Used categories :

- $\mathcal{E}_p$ the category of pure q-difference modules over K
- $\mathcal{E}_{p,0}$ the category of fuchsian modules over K, i.e. pure isoclinic q-difference modules of slope 0
- $\mathcal{E}_{p,r}$ the category of pure q-difference modules over K of slopes $\frac{k}{r}$

Tannakian formalism II

Let $z_0 \in \mathbb{C}^*$, we define the functor :

$$\begin{array}{rrcl} \omega_{z_0}: & \mathcal{E}_p & \longrightarrow & \mathbf{Vect}_{\mathbb{C}}^f \\ & (K^n, \phi_A) & \longmapsto & \mathbb{C}^n \\ & F & \longmapsto & F(z_0) \end{array}$$

It is a fiber functor for the now neutral tannakian category $\mathcal{E}_p$ and we define the Galois group associated with $\mathcal{E}_p$ by $G_p := \mathrm{Aut}^{\otimes}(\omega_{z_0})$.

Tannakian formalism III

$\mathcal{E}_{p,0}$ and $\mathcal{E}_{p,r}$ are tannakian subcategories of $\mathcal{E}_p$ and ω_{z_0} can be restricted to these categories. We define then their Galois groups by

$$G_{p,0} := \mathrm{Aut}^{\otimes}(\omega_{z_0}|_{\mathcal{E}_{p,0}})$$

and

$$G_{p,r} := \mathrm{Aut}^{\otimes}(\omega_{z_0}|_{\mathcal{E}_{p,r}})$$

Galois group for modules with integral slopes

Theorem (Baranovsky-Ginzburg)

$$G_{p,0} = \mathrm{Hom}_{\mathbf{Grp}}\left(\frac{\mathbb{C}^*}{q^{\mathbb{Z}}}, \mathbb{C}^*\right) \times \mathbb{C}$$

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Theorem (Ramis, Sauloy)

$$G_{p,1} = \mathbb{C}^* \times \mathrm{Hom}_{\mathbf{Grp}}\left(\frac{\mathbb{C}^*}{q^{\mathbb{Z}}}, \mathbb{C}^*\right) \times \mathbb{C}$$

Action of $G_{p,1}$

Given a pure isoclinic module of integral slope μ $M = (K^n, \phi_{z^{-\mu}A})$ with $A \in GL_n(\mathbb{C})$, an element $\varphi = (t, \gamma, \lambda)$ acts on M by :

$$\varphi(A) = t^{\mu}\gamma(A_s)A_u^{\lambda}$$

Galois group for pure modules with slopes with fixed denominator

Virginie Bugeaud said in her thesis that an element $\varphi \in G_{p,r}$ can be seen as a triple $(t, \gamma, \lambda) \in \mathbb{C}^* \times \mathrm{Hom}_{\mathbf{Grp}}\left(\frac{\mathbb{C}^*}{q^{\mathbb{Z}}}, \mathbb{C}^*\right) \times \mathbb{C}$. Furthermore she showed that $G_{p,r} = H_r \times \mathbb{C}$ as a group where

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{C}^* & \longrightarrow & H_r & \longrightarrow & \mathrm{Hom}_{\mathbf{Grp}}\left(\frac{\mathbb{C}^*}{q^{\mathbb{Z}}}, \mathbb{C}^*\right) & \longrightarrow & 1 \\ & & t & \longmapsto & (t, 1) & & & & \\ & & & & (t, \gamma) & \longmapsto & \gamma & & \end{array}$$

is a central extension

Universal Galois group for pure q-difference modules I

- If $r|s$ then $\mathcal{E}_{p,r}$ is a full subcategory of $\mathcal{E}_{p,s}$ and we have a group morphism which is onto :

$$\begin{array}{rrcl} \varphi_{r,s}: & G_{p,s} & \longrightarrow & G_{p,r} \\ & (t,\gamma,\lambda) & \longmapsto & (t^{\frac{s}{r}},\gamma,\lambda) \end{array}$$

- $(G_{p,r},\varphi_{r,s})_{r|s}$ is a projective system and

$$G_p = \varprojlim_{r|s} G_{p,r}$$

Universal Galois group for pure q-difference modules II

Let $H = \varprojlim H_r$, then $G_p = H \times \mathbb{C}$ and

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathrm{Hom}_{\mathbf{Grp}}(\mathbb{Q}, \mathbb{C}^*) & \longrightarrow & H & \longrightarrow & \mathrm{Hom}_{\mathbf{Grp}}\left(\frac{\mathbb{C}^*}{q^{\mathbb{Z}}}, \mathbb{C}^*\right) & \longrightarrow & 1 \\ & & \alpha & \longmapsto & (\alpha, 1) & & & & \\ & & & & (\alpha, \gamma) & \longmapsto & \gamma & & \end{array}$$

is a central extension.

Galois group of q-difference equations I

Let $\mathcal{E}$ be the category of all q-difference modules over K. To each of these modules we can associate a unique graded module which will be a pure module : this defines a functor $gr : \mathcal{E} \longrightarrow \mathcal{E}_p$. The functor $\hat{\omega}_{z_0} := \omega_{z_0} \circ gr$ is a fiber functor for $\mathcal{E}$ and we denote by G the Galois group associated.
If we denote by i the inclusion of $\mathcal{E}_p$ in $\mathcal{E}$ by tannakian duality we have

$$G \underset{gr^*}{\overset{i^*}{\rightleftarrows}} G_p$$

with $i^* \circ gr^* = id_{G_p}$.

Galois group of q-difference equations II

Let $S := \ker i^*$ be the Stokes group. We have a split exact sequence

$$1 \longrightarrow S \longrightarrow G \longrightarrow G_p \longrightarrow 1$$

We deduce that $G = S \rtimes G_p$ where G_p acts on S by conjugation.

Density in G_1

Let $\mathcal{E}_r$ be the category of q-difference modules of slopes $\frac{k}{r}$, and G_r be the Galois group associated. We might want to add some explicit elements to $G_{p,r}$ in order to create a Zariski-dense subgroup of G_r.

In the case $r = 1$ Ramis and Sauloy has constructed q-alien derivations with the Stokes operators which generate the Lie algebra of S (the wild monodromy group) which led to a density theorem :

Theorem (Ramis, Sauloy)

$G_{p,1}$ and the group associated with the wild monodromy group generate a Zariski-dense subgroup of G_1.

Density in G_r

In her thesis Bugeaud generalized this theorem by creating an analogous to the q-alien derivation and associating a group.

Theorem (Bugeaud)

This group and $G_{p,r}$ generate a Zariski-dense subgroup of G_r.

Open problems

- The Stokes group of a module is known in the case of two slopes but not in the case of arbitrary number of slopes.
- For a module with three arbitrary slopes we don't know any explicit member of the Stokes group.
- Thus we cannot describe well G or the Galois group associated to a module with arbitrary slopes.

Thank you for your attention !