Galois group of q-difference equations

Anton ELOY
University of Toulouse

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q-difference modules I

- Let $K := \mathbb{C}(\{z\})$ be the field of convergent Laurent series. We take a complex number q such that $|q| > 1$ and define the operator σ_q on K by $\sigma_q(f)(z) = f(qz)$.
- A q-difference module on K is a pair (V, ϕ) where V is a K-vector space of finite dimension and ϕ is a σ_q-linear automorphism on V.

q-difference modules II

- To each q-difference module we can associate a Newton polygon whose lower boundary is made of k vector (r_i, n_i). We call slopes of the q-difference module the rational numbers $\frac{n_i}{r_i}$.
- A module with one slope is said to be pure isoclinic and a module which is a direct sum of pure isoclinic modules is said to be pure.

Tannakian formalism I

Used categories :

- $\mathcal{E}_p$ the category of pure q-difference modules over K
- $\mathcal{E}_{p,0}$ the category of fuchsian modules over K, i.e. pure isoclinic q-difference modules of slope 0
- $\mathcal{E}_{p,r}$ the category of pure q-difference modules over K of slopes $\frac{k}{r}$

Tannakian formalism II

Let $z_0 \in \mathbb{C}^*$, we define the functor :

$$\begin{array}{rrcl} \omega_{z_0} : & \mathcal{E}_p & \longrightarrow & \mathbf{Vect}^f_{\mathbb{C}} \\ & (K^n, \phi_A) & \longmapsto & \mathbb{C}^n \\ & F & \longmapsto & F(z_0) \end{array}$$

It is a fiber functor for the now neutral tannakian category $\mathcal{E}_p$ and we define the Galois group associated with $\mathcal{E}_p$ by $G_p := \mathrm{Aut}^{\otimes}(\omega_{z_0})$.

Tannakian formalism III

$\mathcal{E}_{p,0}$ and $\mathcal{E}_{p,r}$ are tannakian subcategories of $\mathcal{E}_p$ and ω_{z_0} can be restricted to these categories. We define then their Galois groups by

$$G_{p,0} := \mathrm{Aut}^{\otimes}(\omega_{z_0}|_{\mathcal{E}_{p,0}})$$

and

$$G_{p,r} := \mathrm{Aut}^{\otimes}(\omega_{z_0}|_{\mathcal{E}_{p,r}})$$

Galois group for modules with integral slopes

Theorem (Baranovsky-Ginzburg)

$$G_{p,0} = \mathrm{Hom}_{\mathbf{Grp}}\left(\frac{\mathbb{C}^*}{q^{\mathbb{Z}}}, \mathbb{C}^*\right) \times \mathbb{C}$$

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Theorem (Ramis, Sauloy)

$$G_{p,1} = \mathbb{C}^* \times \mathrm{Hom}_{\mathbf{Grp}}\left(\frac{\mathbb{C}^*}{q^{\mathbb{Z}}}, \mathbb{C}^*\right) \times \mathbb{C}$$

Action of $G_{p,1}$

Given a pure isoclinic module of integral slope μ $M = (K^n, \Phi_{z^{-\mu}A})$ with $A \in GL_n(\mathbb{C})$, an element $\varphi = (t, \gamma, \lambda)$ acts on M by :

$$\varphi(A) = t^{\mu}\gamma(A_s)A_u^{\lambda}$$

Galois group for pure modules with slopes with fixed denominator

Virginie Bugeaud said in her thesis that an element $\varphi \in G_{p,r}$ can be seen as a triple $(t, \gamma, \lambda) \in \mathbb{C}^* \times \mathrm{Hom}_{\mathbf{Grp}}\left(\frac{\mathbb{C}^*}{q^{\mathbb{Z}}}, \mathbb{C}^*\right) \times \mathbb{C}$. Furthermore she showed that $G_{p,r} = H_r \times \mathbb{C}$ as a group where

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{C}^* & \longrightarrow & H_r & \longrightarrow & \mathrm{Hom}_{\mathbf{Grp}}\left(\frac{\mathbb{C}^*}{q^{\mathbb{Z}}}, \mathbb{C}^*\right) & \longrightarrow & 1 \\ & & t & \longmapsto & (t,1) & & & & \\ & & & & (t,\gamma) & \longmapsto & \gamma & & \end{array}$$

is a central extension

Universal Galois group for pure q-difference modules I

- If $r|s$ then $\mathcal{E}_{p,r}$ is a full subcategory of $\mathcal{E}_{p,s}$ and we have a group morphism which is onto :

$$\begin{array}{rccc} \varphi_{r,s} : & G_{p,s} & \longrightarrow & G_{p,r} \\ & (t,\gamma,\lambda) & \longmapsto & (t^{\frac{s}{r}},\gamma,\lambda) \end{array}$$

- $(G_{p,r},\varphi_{r,s})_{r|s}$ is a projective system and

$$G_p = \varprojlim_{r|s} G_{p,r}$$

Universal Galois group for pure q-difference modules II

Let $H = \varprojlim H_r$, then $G_p = H \times \mathbb{C}$ and

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathrm{Hom}_{\mathbf{Grp}}(\mathbb{Q}, \mathbb{C}^*) & \longrightarrow & H & \longrightarrow & \mathrm{Hom}_{\mathbf{Grp}}\left(\frac{\mathbb{C}^*}{q^{\mathbb{Z}}}, \mathbb{C}^*\right) & \longrightarrow & 1 \\ & & \alpha & \longmapsto & (\alpha, 1) & & & & \\ & & & & (\alpha, \gamma) & \longmapsto & \gamma & & \end{array}$$

is a central extension.

Galois group of q-difference equations I

Let $\mathcal{E}$ be the category of all q-difference modules over K. To each of these modules we can associate a unique graded module which will be a pure module : this defines a functor $gr : \mathcal{E} \longrightarrow \mathcal{E}_p$. The functor $\hat{\omega}_{z_0} := \omega_{z_0} \circ gr$ is a fiber functor for $\mathcal{E}$ and we denote by G the Galois group associated.

If we denote by i the inclusion of $\mathcal{E}_p$ in $\mathcal{E}$ by tannakian duality we have

$$G \underset{gr^*}{\overset{i^*}{\rightleftarrows}} G_p$$

with $i^* \circ gr^* = id_{G_p}$.

Galois group of q-difference equations II

Let $S := \ker i^*$ be the Stokes group. We have a split exact sequence

$$1 \quad S \quad G \quad G_p \quad 1$$

We deduce that $G = S \rtimes G_p$ where G_p acts on S by conjugation.

Density in G_1

Let $\mathcal{E}_r$ be the category of q-difference modules of slopes $\frac{k}{r}$, and G_r be the Galois group associated. We might want to add some explicit elements to $G_{p,r}$ in order to create a Zariski-dense subgroup of G_r.

In the case $r = 1$ Ramis and Sauloy has constructed q-alien derivations with the Stokes operators which generate the Lie algebra of S (the wild monodromy group) which led to a density theorem :

Theorem (Ramis, Sauloy)

$G_{p,1}$ and the group associated with the wild monodromy group generate a Zariski-dense subgroup of G_1.

Density in G_r

In her thesis Bugeaud generalized this theorem by creating an analogous to the q-alien derivation and associating a group.

Theorem (Bugeaud)

This group and $G_{p,r}$ generate a Zariski-dense subgroup of G_r.

Open problems

- The Stokes group of a module is known in the case of two slopes but not in the case of arbitrary number of slopes.
- For a module with three arbitrary slopes we don't know any explicit member of the Stokes group.
- Thus we cannot describe well G or the Galois group associated to a module with arbitrary slopes.

Thanks for your attention !