REAL CLOSED FIELDS AND PEANO ARITHMETIC

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A real closed field (RCF) is a totally ordered field R such that $R[\sqrt{-1}]$ is algebraically closed. An integer part (IP) of R is a discretely ordered subring Z such that for each $r \in R$, there exists some $z \in Z$ with $z \leq r < z + 1$. An important fragment of elementary number theory is Peano arithmetic (PA), the object of Gödel's celebrated incompleteness theorem (1930s), a theorem which provided a spectacular response to Hilbert's (1900s) program for the foundations of mathematics. We recall that PA is the axiom system for discretely ordered commutative unital rings, which satisfiy the induction axiom for each elementary statement. In turn, open induction (OI) denotes the fragment of PA where induction is restricted to quantifier free statements. Shepherdson (1970s) showed that a discretely ordered commutative unital ring is an IP for some RCF if and only if it satisfies the axioms of OI. Thus integer parts link *real algebra* to fragments of *number theory*.

In this talk, I will briefly introduce the class of RCFs and its valuation theory. I will show that every RCF admits an IP, a result obtained by a modification of Kaplansky' s embedding of a RCF into a field of generalized power series. Gödel's incompleteness theorem led to a great interest in studying the deductive power of recursive fragments of number theory. By the completeness theorem, this requires understanding their non-archimedean models. I will thus turn to models of OI, and using tools from commutative algebra, discuss the arithmetic properties of those rings (such as the existence of co-final sequences of prime elements). Finally, I will turn my attention to RCFs which admit an IP which is a model of full PA. I will show that such a RCF admits a real exponential function. I will close by a discussion of the peculiar valuation theoretic properties of real closed exponential fields, and present a few open questions.

References:

[1] Biljakovic, D. – Kotchetov, M. – Kuhlmann, S.: *Primes and Irreducibles in Truncation Integer Parts of Real Closed Fields*, Logic in Teheran, Lecture Notes in Logic, **26**, 42–65 (2006)

[2] Fornasiero, A. – Kuhlmann, F.-V. – Kuhlmann, S.: *Towers of complements and truncation closed embeddings of valued fields*, Journal of Algebra, **323**, 574-600 (2010)

[3] Carl, M. – D'Aquino, P. – Kuhlmann, S.: *On the value group of a model of Peano Arithmetic*, Forum Mathematicum, **29**, 951-957 (2017)

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Date: September 16, 2018.