Traveling Fronts of Allen-Cahn Equations and
Mean Curvature Flows

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joint work with

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Fronts and Nonlinear PDEs, a tribute to Professor Henri Berestycki
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At 30, I become a real man;

At 40, I have no doubts;

At 50, I know the mandates of God;

At 60, my ear was an obedient organ for the reception of truth.

Confucius (200BC)
Traveling wave problem for the bistable equation

We will consider the following equation:

\[ \Delta u + c\partial_{x_{N+1}} u + f(u) = 0, \quad \text{in } \mathbb{R}^{N+1}, \]  

which is the traveling wave problem for:

\[ u_t = \Delta u + f(u), \quad \text{in } \mathbb{R}^{N+1}, \quad t > 0, \]  

where \( u(x', x_{N+1}, t) = u(x', x_{N+1} - ct) \).
In this talk we mostly assume that

\[ f(u) = u(1 - u^2) \implies \int_{-1}^{1} f(u) \, du = 0, \]

i.e. \( f \) is a bistable, balanced nonlinearity. We will mention results of (0.1) with (bistable, unbalanced nonlinearity)

\[ f(u) = u(1 - u^2) + a(1 - u^2) \implies \int_{-1}^{1} f(u) \, du = \frac{4}{3} a \neq 0. \]

The potential corresponding to the two cases is of the form:

\[ W(u) = \frac{1}{2} (1 - u^2)^2 - au(1 - \frac{1}{3} u^2). \]

If \( a = 0 \) then \( W(-1) = 0 = W(1) \). If \( a > 0 \) then \( W(-1) > W(1) \), hence the phase \( u = 1 \) is more stable then \( u = -1 \).
We will first discuss the case $a > 0$. In one dimension we have:

$$\Phi'' + c\Phi' + f(\Phi) = 0.$$  

This problem has a unique solution such that $\Phi(\pm\infty) = \pm 1$, $\Phi' > 0$ and $c = \frac{4}{3}a$. This corresponds to a traveling wave that moves to the left; the more stable phase invades the less stable one.

When $a = 0$ there exists a unique (heteroclinic) solution:

$$H'' + H(1 - H^2) = 0,$$

such that $H(\pm\infty) = \pm 1$, $H$ is odd and $H' > 0$. Notice also that $-H$ is a solution connecting the two stable phases.
Planar fronts

When $c \neq 0$, $\Phi(x_{N+1})$ is a \textit{planar fronts}

- In the unbalanced case there are other solutions with asymptotically planar, V-shaped, fronts (Ninomiya-Taniguchi (in $N + 1 = 2$), Hamel-Monneau-Roquejoffre ($N + 1 > 2$)), pyramidal fronts (Taniguchi, $N = 3$).

- Stability of planar fronts (Levermore-Xin, Kapitula, Xin, Matano-Nara-Taniguchi).

- Related results in the monostable (KPP) case (Bonnet-Hamel, Hamel-Monneau-Roquejoffre, Hamel-Nadirashvili, Berestycki-Hamel-Nadirashvili).
Rotationally symmetric traveling waves

From now on we will consider only the balanced case in $\mathbb{R}^{N+1}$:

$$\Delta u + c \partial_{x_{N+1}} u + u - u^3 = 0. \quad (0.2)$$

In this problem, planar traveling waves do not exist. Thus the velocity $c$ is not determined by the difference in the heights of the potential wells, but rather determined by the curvature effect.

Traveling wave solution is an eternal solution of the parabolic Allen-Cahn equation

$$u_t = \Delta u + u - u^3, (x, t) \in \mathbb{R}^{N+1} \times \mathbb{R}.$$
Parabolic De Giorgi Conjecture

Consider eternal solutions of parabolic Allen-Cahn equation

\[ u_t = \Delta u + u - u^3, (x, t) \in \mathbb{R}^{N+1} \times \mathbb{R}. \]

Assuming their monotonicity in the \( x_{N+1} \) direction:

\[ \partial_{x_{N+1}} u > 0, \lim_{x_{N+1} \to \pm \infty} u(x', x_{N+1}, t) = \pm 1, \quad t \in \mathbb{R} \]

then the level set of \( u \) must be hyperplanes, at least when \( N \leq 7 \)

This is a natural generalization of the Elliptic De Giorgi Conjecture.
Elliptic De Giorgi Conjecture

Let $u$ be a bounded solution of Allen-Cahn equation in $\mathbb{R}^N$ such that

$$\frac{\partial u}{\partial x_N} > 0.$$  

Then the level sets $\{u = \lambda\}$ are all hyperplanes, at least for dimension $N \leq 8$. 

$N = 3$ by Ambrosio and Cabré (2000).

$4 \leq N \leq 8$ by Savin under an additional assumption

$$\lim_{x_N \to \pm \infty} u(x', x_N) = \pm 1$$

$N \geq 9$ counterexample by Del Pino, Kowalczyk and Wei

In Dimension $N=8$, Pacard-Wei 2010 constructed a stable solution whose level set is not hypersurface.

What about Parabolic DG Conjecture?
- $4 \leq N \leq 8$ by Savin under an additional assumption
  \[
  \lim_{x_N \to \pm \infty} u(x', x_N) = \pm 1
  \]
- $N \geq 9$ counterexample by Del Pino, Kowalczyk and Wei
- In Dimension $N=8$, Pacard-Wei 2010 constructed a stable solution whose level set is not hypersurface.

What about Parabolic DG Conjecture?
The parabolic DG conjecture is false even in dimension $N + 1 = 2$.

**Theorem** (Chen, Guo, Hamel, Ninomiya, Roquejoffre, 2007) For any speed $c > 0$, there exists solutions to

$$\Delta u + cu_{x_{N+1}} + u - u^3 = 0 \text{ in } \mathbb{R}^{N+1}$$

of the form $u(x', x_{N+1}) = U(r, x_{N+1})$, $r = |x'|$, $N \geq 1$. Functions $U$ have paraboloid-like profiles of their nodal sets $\Gamma$.

- when $N > 1$, the asymptotic profiles of the fronts are given:

  $$\lim_{x_{N+1} \to +\infty \atop (x', x_{N+1}) \in \Gamma} \frac{r^2}{2x_{N+1}} = \frac{N - 1}{c}, \quad \text{if } N > 1.$$

- When $N = 1$ the ends of the fronts diverges logarithmically.
Let $u$ be a bounded solution of equation

$$(AC)_{TW} \quad \Delta u + u - u^3 + cu_{x_{N+1}} = 0 \quad \text{in } \mathbb{R}^{N+1}$$

which satisfies

$$\partial_{x_{N+1}} u > 0$$

Then, $u$ must be radially symmetric in $x'$. 

Gui: partial result when $N = 2$
Parabolic Allen-Cahn Equation and Mean Curvature Flow

Consider the mean curvature flow for a hypersurface $\Sigma = \Sigma(t)$:

$$\frac{\partial \Sigma}{\partial t} = H_\Sigma \nu,$$  \hspace{1cm} (0.3)

where $\nu$ is the normal to the surface and $H_\Sigma$ is its mean curvature. It is known that the evolution of zero-level set of $\epsilon$—version of the Allen-Cahn equation

$$\epsilon u_t = \epsilon \Delta u + \frac{1}{\epsilon} (u - u^3) \hspace{1cm} (0.4)$$

can be reduced to (0.3)

Eternal Solutions of Mean Curvature Flow

Surfaces which are translated by the mean curvature (MC) flow with constant velocity (say 1) in a fixed direction satisfy:

\[ H_{\Sigma} = \nu_{N+1}, \quad (x_{N+1} \text{ direction}). \]  

(0.5)

Let \( \Sigma = \Sigma(t) \) be such a surface and consider its scaling \( \Sigma_{\varepsilon} \):

\[ y \in \Sigma_{\varepsilon}(t) \iff \varepsilon y \in \Sigma(t). \]

Then, denoting the mean curvatures of these surfaces by \( H_{\Sigma}, H_{\Sigma_{\varepsilon}} \):

\[ H_{\Sigma} = \nu_{N+1}, \quad H_{\Sigma_{\varepsilon}} = \varepsilon \nu_{N+1}. \]  

(0.6)
Translating solutions to the MC flow are called **eternal solutions** since they exist for all $t \in (-\infty, \infty)$. They play important roles in the analysis of Type II singularity of mean curvature flow.

Convex eternal solutions are important in the study of singularities for the MC flow (**Huisken-Sinestrari**, also **Wang, Wang-Sheng, B.White**).

Examples by **Altschuler-Wu, Clutterbuck-Schnurer-Schulze, Nguyen**.
We describe the result of Clutterbuck-Schnurer-Schulze (2003). When $\Sigma(t)$ is a graph $\{F(x') + t\}$, where $F: \mathbb{R}^N \to \mathbb{R}$, is a smooth function then translating graph satisfies

$$\nabla \left( \frac{\nabla F}{\sqrt{1 + |\nabla F|^2}} \right) = \frac{1}{\sqrt{1 + |\nabla F|^2}}. \quad (0.7)$$

There exists a unique radially symmetric solution $F$ of (0.7):

$$F(r) = \frac{r^2}{2(N - 1)} - \log r + 1 + O(r^{-1}), \quad r \gg 1. \quad (0.8)$$
\[ x_{N+1} = F(|x|) = \frac{|x|^2}{2(N-1)} - \log |x| + O(|x|^{-1}) \]
The surface $\Gamma$ moving with speed $c = 1$, and the surface $\Gamma_\varepsilon$ moving with speed $c = \varepsilon$. 
The first term in this asymptotic behavior coincides with the asymptotic behavior of the nodal set of solutions to (0.2) found by Chen, Guo, Hamel, Ninomiya, Roquejoffre.

The rotationally symmetric graphs are stable.

They find other solutions, which are still rotationally symmetric, have the same asymptotic behavior, but are not graphs. These are called traveling catenoids.
traveling catenoid $\Gamma$ and its rescaled version $\Gamma_\epsilon$
First Result: Single Convex Front

**Theorem 1 (del Pino-Kowalczyk-Wei)** For each sufficiently small $\varepsilon$, the traveling wave problem

$$(AC)_{TW} \quad \Delta u + u - u^3 + cu_{x_{N+1}} = 0 \quad \text{in } \mathbb{R}^{N+1}$$

has a solution $u_\varepsilon$ moving with speed $c = \varepsilon$, and with the following properties:

- The 0-level set of $u_\varepsilon$ is rotationally symmetric and smooth hypersurfaces $\Gamma_\varepsilon$.
- The nodal surface $d(\Gamma_\varepsilon, \{x_{N+1} = \varepsilon^{-1} F(\varepsilon |x'|)\}) = O(\varepsilon \log \frac{1}{\varepsilon})$ where $F$ is the rotationally symmetric MC Soliton.
- More precise asymptotic behavior can be found

This provides a another proof of Chen-Guo-Hamel-Ninomiya-Roquejoffre result in the case of small speed $c$. 
The surface $\Gamma$ moving with speed $c = 1$, and the surface $\Gamma_\varepsilon$ moving with speed $c = \varepsilon$. 
Traveling catenoids for MC Solitons

- There are other eternal, rotationally symmetric solutions to the mean curvature flow called traveling catenoids.
- The result of Clutterbuck, Schnurer and Schulze says that there exists a one parameter family of rotationally symmetric surfaces $\Sigma_R$, $R > 0$, of genus 0, translated with speed $c = 1$ by the mean curvature flow.
- The surfaces $\Sigma_R$ are given as a union of graphs of functions $x_{N+1} = W_R^\pm(r), r \geq R$, which satisfy asymptotically:
  \[ W_R^\pm(r) \approx F(r) + C^\pm. \]
- As a consequence the ends of the surfaces $\Sigma_R$ are parallel.
The traveling catenoid $\Sigma_R$, $c = 1$ and its rescaled version $\Sigma_{R,\varepsilon}$, $c = \varepsilon$. The surface $\Gamma$ is also represented for comparison.
Non-convex traveling fronts

**Theorem 2 del Pino-Kowalczyk-Wei** For each $R > 0$ and each $\varepsilon$ sufficiently small there exists a traveling wave solution of the problem (0.9), with the following properties:

(i) The level set $\tilde{\Sigma}_{R,\varepsilon} = \{u_\varepsilon = 0\}$ is a rotationally symmetric, two-ended, smooth surface of genus $0$.

(ii) The surface $\tilde{\Sigma}_{R,\varepsilon}$ divides the space into two disjoint, unbounded components $D_{R,\varepsilon}^{\pm}$ such that $u_\varepsilon > 0$ in $D_{R,\varepsilon}^{+}$ and $u_\varepsilon < 0$ in $D_{R,\varepsilon}^{-}$.

(iii) The ends of the surface $\tilde{\Sigma}_{R,\varepsilon}^{\pm}$ can be represented for all large $r$ as graphs of functions $\tilde{W}_{R,\varepsilon}^{\pm}(r)$, $\tilde{W}_{R,\varepsilon}^{+} > \tilde{W}_{R,\varepsilon}^{-}$ and we have:

$$\text{dist} \left( \tilde{\Sigma}_{R,\varepsilon}^{\pm}, \Gamma_\varepsilon \right) \leq C \log \left( \frac{1 + \varepsilon^2 r^2}{\varepsilon^2} \right).$$
Some Remarks

- The surface $\tilde{\Sigma}_{R,\varepsilon}$ has some characteristics common with a catenoid and also with the family of traveling catenoids $\Sigma_{R,\varepsilon}$.

- The ends of $\Sigma^\pm_{R,\varepsilon} = \{x_{N+1} = \varepsilon^{-1} W^\pm R(\varepsilon r)\}$ and that of $\tilde{\Sigma}_{R,\varepsilon}$ satisfy

  \[
  \text{dist} \left( \tilde{\Sigma}^\pm_{R,\varepsilon}, \Sigma^\pm_{R,\varepsilon} \right) \leq C \log \left( \frac{1 + \varepsilon^2 r^2}{\varepsilon^2} \right).
  \]

  The ends of $\Sigma_{R,\varepsilon}$ are parallel to the traveling paraboloid $\Gamma_\varepsilon$.

- Statement (iii) of Theorem 2 shows that the ends of $\tilde{\Sigma}_{R,\varepsilon}$ bend away and diverge logarithmically from $\Gamma_\varepsilon$.

- This is due to the interactions of the ends when $r > r_\varepsilon \sim e^{c\varepsilon^{-1}}$. 
The surface $\tilde{\Sigma}_{R,\varepsilon}$ is presented as well as the asymptotic values of the traveling wave solution $u_\varepsilon$. We include also the surfaces $\Gamma_\varepsilon$ and $\Sigma_{\varepsilon,R}$. 
Third Result: Multiple (double) component traveling wave fronts

- We want to find a solution to

\[ \Delta u + \varepsilon \partial_{x_{N+1}} u + u(1 - u^2) = 0, \quad \text{in } \mathbb{R}^{N+1}, \quad (0.9) \]

where \( \varepsilon \ll 1 \) and \( N \geq 2 \).

- For each \( \varepsilon \) by \( \Gamma_\varepsilon \) we denote the eternal graph:

\[ \Gamma_\varepsilon = \{ x_{N+1} = \varepsilon^{-1} F(\varepsilon r) \}. \]

- We look for solutions depending on just two variables \( r = |x'| \), \( x' \in \mathbb{R}^N \) and \( x_{N+1} \).

- We want the nodal set of such solution to have two components which should resemble in some sense the eternal graph.
Theorem 3 (del Pino-Kowalczyk-Wei) For each small $\varepsilon > 0$ there is a solution $u_\varepsilon$ to (0.9) whose 0-level set consists of two $N - 1$ dimensional hypersurfaces, with the following properties:

(i) Each of the two components $\Gamma^{\pm}_\varepsilon$ is a rotationally symmetric graph of a smooth function $\Gamma^{\pm}_\varepsilon = \{x_{N+1} = F^{\pm}_\varepsilon(r)\}$, where $F^{+}_\varepsilon > F^{-}_\varepsilon$.

(ii) There exists a constant $C > 0$ such that

$$\text{dist} (\Gamma_\varepsilon, \Gamma^{\pm}_\varepsilon) \leq C \log \left( \frac{1 + \varepsilon^2 r^2}{\varepsilon^2} \right).$$
Illustration of the results of Theorem 3. The surfaces $\Gamma_{\varepsilon}^{\pm}$ are presented as well as the asymptotic values of $u_{\varepsilon}$. 
With some extra (technical) effort a similar result can be proven in case of $k > 2$ foliating traveling waves.

This is a multi-component traveling front in the sense of Berestycki-Hamel 2005.

The travelling wave solution we construct connects the stable phase $-1$ (minimum of the potential $W(u) = \frac{1}{4}(1 - u^2)^2$) with itself, as it is common two distinct stable phases $-1$ and $1$. This is counterintuitive for this nonlinearity (bistable not monostable).

We refer to this phenomenon as foliation by traveling waves. This is motivated by the apparent analogy with the foliation by constant mean curvature submanifolds (Ye, Mazzeo-Pacard, Mahmoudi-Mazzeo-Pacard), and foliations by interfaces (del Pino, Kowalczyk, Wei, Yang).

These phenomena seem to be quite different at the end.
The mechanism of foliations

- To explain we observe that the ”single” traveling wave is stable (though no proofs yet) and so is the eternal solution.
- The speed of the eternal solutions is very sensitive to the asymptotic profiles of their ends. In fact there is a continuous family of eternal solutions parametrized by their speeds. They foliate the space.
- The middle parts of the two components of the multiple front traveling wave are ”attracted” by an eternal solution with the given speed $c = \varepsilon$, while their ends ”approach” the ends of eternal solutions with different speeds: the bottom one is slightly slower while the upper one is slightly faster.
- Foliating traveling waves ”lie” on the boundary of the basin of attraction of the wave whose speed is $\varepsilon$. 
Summary of the Results

- Theorems 1 and 2 provides an (almost) one-to-one correspondence between the traveling wave solutions of balanced Allen-Cahn equation with the translating solitons of mean curvature flows.

\[ \Delta u + u - u^3 + cu_{x_{N+1}} = 0 \]

- Theorem 2 provides a first non-convex front.

- Theorem 3 establishes the existence of multi-component fronts—a phenomena never found in the mean curvature flow. This shows that the structure of traveling wave solutions to Allen-Cahn equation is richer than the translating solitons of mean curvature flow.
Profiles of the multiple component traveling waves

We first introduce the *Jacobi operator* for an eternal solution of the MC flow $\Sigma$.

**Motivation:** expand $\Delta + \partial_{x_{N+1}}$ near $\Sigma$ in local Fermi coordinates

$$x = y + z\nu(y), \quad y \in \Sigma, \quad \Sigma_z = \Sigma + z\nu$$

to get

$$\Delta + \partial_{x_{N+1}} = \Delta_{\Sigma_z} + \partial_z^2 - (H_{\Sigma_z} - \nu_{N+1})\partial_z + \nabla_{\Sigma_z, x_{N+1}}$$

$$= \Delta_{\Sigma} + z|A_{\Sigma}|^2 + \nabla_{\Sigma, x_{N+1}} + \partial_z^2 - (H_{\Sigma} - \nu_{N+1})\partial_z + \ldots$$

**Term** denoted by $I$ eventually gives rise to:

$$J_\Sigma(\nu) = \Delta_{\Sigma}\nu + |A_{\Sigma}|^2\nu + \nabla_{\Sigma, x_{N+1}}\nu.$$
Next we define the *Toda system* (Pasta-Ulam, Toda). We consider an assembly of particles along a straight line each interacting with its (two) closest neighbors. The potential of the interactions decays with the exponential of their distances:

\[ f_j'' - [e^{-(f_j-f_{j-1})} - e^{-(f_{j+1}-f_j)}] = 0, \quad j = 1, \ldots, m, \]

where \(f_j\)'s (positions) are real functions (Kostant, Moser). Recent applications for semilinear equations (del Pino-Kowalczyk-Pacard-Wei).

To close this system we put \(f_0 = -\infty\) and \(f_{m+1} = \infty\). When \(m = 2\) this system reduces to a Liouville type equation:

\[ u'' + 2e^u = 0, \quad u = f_1 - f_2, \]

and the linear equation \(v'' = 0, \quad v = f_1 + f_2\).
The laminating profiles are normal graphs of functions $f_j: \Sigma \to \mathbb{R}$, $j = 1, 2$.

To determine them we need to solve the *Jacobi-Toda* system:

$$
\varepsilon^2 \alpha_0 \left( \Delta_\Sigma + \nabla_\Sigma, x_{N+1} + |A_\Sigma|^2 \right) u + e^u = 0, \quad u = \sqrt{2}(f_2 - f_1),
$$

(0.10)

and

$$
\left( \Delta_\Sigma + \nabla_\Sigma, x_{N+1} + |A_\Sigma|^2 \right) v = 0, \quad v = f_1 + f_2.
$$

This can be written also:

$$
\varepsilon^2 \alpha_0 \mathcal{J}_\Sigma(u) + e^u = 0,
$$

$$
\mathcal{J}_\Sigma(v) = 0.
$$

This equations are stated on the surface $\Sigma$ (scaling $\Sigma_\varepsilon = \frac{1}{\varepsilon} \Sigma$) with natural identification $v(\varepsilon \cdot): \Sigma_\varepsilon \to \mathbb{R}$.

In the problem of laminations by interfaces (del Pino-Kowalczyk-Wei-Yang 2009) a similar equations appear.
We introduce the Fermi coordinates around $\Gamma_\varepsilon$,
\[ x = p + z\nu_\varepsilon(p), \quad p \in \Gamma_\varepsilon, \]

Since we seek solutions that depend on $(r, x_{N+1})$ only, we can assume that the Fermi coordinates depend on $(r, z)$ only.

We build an approximate solution of the form:
\[ u_\varepsilon(r, z) = H(z - f_{\varepsilon,1}(\varepsilon r)) - H(z - f_{\varepsilon,2}(\varepsilon r)) - 1 \]
\[ \equiv H_{\varepsilon,1} - H_{\varepsilon,2} - 1, \]

where functions $f_{\varepsilon,j}$ are to be determined and $H$ is the heteroclinic:
\[ H'' + H(1 - H^2) = 0, \quad H(\pm\infty) = \pm1. \]
The error of the approximate solution

\[ S(u_\varepsilon) \sim \sum_{j=1}^{2} \left\{ \partial_{zz} H_{\varepsilon,j} + f(H_{\varepsilon,j}) \right\} \]

\[ + \sum_{j=1}^{2} \left\{ (\varepsilon \nu_{N+1} - H_{\Gamma_\varepsilon}) \partial_z H_{\varepsilon,j} \right\} \]

\[ + \sum_{j=1}^{2} \left\{ \left( \Delta_{\Gamma_\varepsilon} - z |A_{\Gamma_\varepsilon}|^2 \partial_z \right) H_{\varepsilon,j} + \varepsilon \nabla_{\Gamma_\varepsilon,x_{N+1}} H_{\varepsilon,j} \right\} \]

\[ + f(\sum_{j=1}^{2} H_{j,\varepsilon} - 1) - \sum_{j=1}^{2} f(H_{j,\varepsilon}). \]

Projection of the error onto \( \partial_z H_{\varepsilon,j} \) gives formally the Jacobi-Toda system, and infinite dimensional reduction is used to justify this rigorously.
Jacobi-Toda System

How to solve

\[(JT1)\quad \varepsilon^2 (\Delta_{\Sigma} u + \nabla_{\Sigma, x_{N+1}} u + |A_{\Sigma}|^2 u) + e^u = 0 \text{ on } \Gamma\]

A similar Jacobi-Toda system

\[(JT2)\quad \varepsilon^2 (\Delta_g u + (|A|^2 + \text{Ric}) u) + e^u = O(\varepsilon^{2+\sigma}) \text{ on } \Gamma\]

was derived for the laminations of interfaces of Allen-Cahn equation on a compact \(N\)-dimensional Riemannian manifold \((M, \tilde{g})\)

\[(AC)_M \quad \varepsilon^2 \Delta_{\tilde{g}} u + (1 - u^2) u = 0 \quad \text{in } M, \quad (0.11)\]

where \(\Delta_{\tilde{g}}\) is the Laplace-Beltrami operator on \(M\).
Pacard and Ritoré: single interface on non-degenerate minimal \((N - 1)\)-dimensional submanifold \(\Gamma\) of \(M\).

del Pino-Kowalczyk-Wei-Yang 2009: Assume that

\[
|A|^2 + \text{Ric} > 0
\]  

(0.12)

For any fixed integer \(K \geq 2\), there exists a positive sequence \((\varepsilon_i)\) approaching 0 such that problem \((AC)_M\) has a solution \(u_\varepsilon\) with \(K\) phase transition layers with mutual distance \(O(|\varepsilon| \ln |\varepsilon|)\).
The existence of multiple interfaces depend on the following Jacobi-Toda system

\[(JT2) \quad \varepsilon^2 (\Delta_g u + (|A|^2 + Ric) u) + e^u = O(\varepsilon^{2+\sigma}) \text{ on } \Gamma\]

What are the difficulties in solving (JT2)?

1. **variational methods**, if works, can only find a solution and there is no information on asymptotic behavior of the solutions, since we have to ask

\[u \ll -1\]

2. when \(N \geq 3\), \(e^u\) is supercritical, there is no way of using variational method.

3. A more difficult problem is the resonance phenomena.
Resonance Phenomena

Let us for simplicity we assume that

\[ |A|^2 + Ric \equiv \text{Constant} = 1 \]

Then equation (JT2) becomes

\[ (JT3) \quad \varepsilon^2 (\Delta_g u + u) + e^u = \varepsilon^{2+\sigma} h \text{ on } \Gamma \]

When \( h = 0 \), it has a constant solution

\[ \varepsilon^2 u_0 + e^{u_0} = 0 \]

\[ u_0 = \log \varepsilon^2 + \log \log \frac{1}{\varepsilon} + O(\log \log \log \frac{1}{\varepsilon}) \]

Thus we take \( u = u_0 + u_1 \), then we are reduced to solving

\[ (\Delta_g + (2 \log \frac{1}{\varepsilon} + 1))u_1 = \varepsilon^\sigma h \]

The left hand operator has eigenvalues \( \lambda_j - 2 \log \frac{1}{\varepsilon} \).

Weyl’s formula, \( \lambda_j \sim j^{\frac{2}{N-1}} \). As \( j \to +\infty \), \( \lambda_j - 2 \log \frac{1}{\varepsilon} \) may cross zero at large \( N \).
Gap Condition

The problem can still be solved under some gap condition: it is possible to obtain

$$|\lambda_j - 2 \log \frac{1}{\varepsilon}| \geq \delta^p$$

for $p$ large and hence

$$\|v\| \leq C\delta^{-p}\|(\Delta g + (2 \log \frac{1}{\varepsilon} - 1))v\|$$

But the right hand error is $O(\varepsilon^\sigma)$ which controls any power of $\delta^{-p}$.

More complicated proofs when $|A|^2 + Ric \neq Constant$.
A New Jacobi-Toda System for Traveling Waves

\[ (JT2) \quad \varepsilon^2 (\Delta_g u + (|A|^2 + \text{Ric})u) + e^u = O(\varepsilon^{2+\sigma}) \text{ on } \Gamma \]

We need to solve the following new Jacobi-Toda system:

\[ (JT1) \quad \varepsilon^2 (\Delta_{\Sigma} u + |A_{\Sigma}|^2 u + \nabla_{\Sigma, x_{N+1}} u) + e^u = 0 \text{ on } \Gamma \]

Main Result: For all \( \varepsilon \) small, the Jacobi-Toda system (JT1) can be solved.

NO Resonance Needed !!! Why?

The convection term \( \nabla_{\Sigma} \psi_{\varepsilon, j} \cdot \nabla_{\Sigma} (x_{N+1}) \) saves the day!!!
A New Jacobi-Toda System for Traveling Waves

\[(JT2)\quad \varepsilon^2(\Delta_g u + (|A|^2 + Ric)u) + e^u = \mathcal{O}(\varepsilon^{2+\sigma}) \text{ on } \Gamma\]

We need to solve the following new Jacobi-Toda system:

\[(JT1)\quad \varepsilon^2(\Delta_\Sigma u + |A_\Sigma|^2 u + \nabla_{\Sigma,x_{N+1}} u) + e^u = 0 \quad \text{on } \Gamma\]

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Traveling Fronts for Nonlinear Schrödinger Equation

Similar construction also works for

$$cu_{xN} + \Delta u - u + u^p = 0 \quad \text{in } \mathbb{R}^N \quad (\text{III}).$$

**Theorem 4 del Pino-Wei-Yao 2010** For each small $\varepsilon > 0$ and $N \geq 1, k \geq 1$ there is a solution $u_\varepsilon$ to (III) whose 0-level set consists of $k$ hypersurfaces given as normal graphs over $\Sigma_\varepsilon$ of smooth functions $f_{\varepsilon,j}(r), j = 1, \ldots, k$. Denoting

$$U_{j,\varepsilon} = f_{\varepsilon,j+1} - f_{\varepsilon,j} > 0, \quad V_\varepsilon = \sum_{j=1}^{k} f_{\varepsilon,j},$$

we have

$$U_\varepsilon(r) \sim \log \frac{1}{\varepsilon^2}, \quad V_\varepsilon = \mathcal{O}(\varepsilon^\tau)$$
this result is new even for a single traveling layer.

this result can not be proved by sub-super solutions. The profile itself is unstable:

\[ H'' - H + H^p = 0, \text{ in } \mathbb{R}, \ H(t) \to 0 \text{ as } |t| \to +\infty \]

It is well-known that resonance phenomena happen for spike-type front. (Dancer's solution.) Again the convection term avoids the resonance phenomena.

same result holds for the unbalanced nonlinearity

\[ cu_\times_{N+1} + \Delta u + u(u - a)(1 - u) = 0 \text{ in } \mathbb{R}^{N+1} \]

The traveling wave is connecting zero to zero. The profile function is the one-dimensional spike

\[ \psi'' + \psi(\psi - a)(1 - \psi) = 0, \psi(\pm\infty) = 0 \]
Bernstein Conjecture for Translating Graphs of Mean Curvature Flow

We return to the MC Soliton equation:

$$\nabla \cdot \left( \frac{\nabla F}{\sqrt{1 + |\nabla F|^2}} \right) = \frac{c}{\sqrt{1 + |\nabla F|^2}} \quad \text{in } \mathbb{R}^N. \quad (MC)_c$$

A Bernstein problem for $(MC)_c$:

**(B)** Is it true that entire solutions of $(MC)_c$ for $c \geq 0$ need to be convex?

This statement for $c = 0$ reduces to Bernstein’s problem: If $F$ solving $(MC)_c$ was necessarily convex, then so would be $-F$. Hence $F$ would be a linear affine function.
Convexity of Mean Curvature Flows

This statement is important in the following problem for mean curvature flow (asked by B. White 2003):

mean-convex graph by mean curvature flow $\implies$ becomes convex $t \gg 1$

**True** for $N = 2$ (X.-J. Wang 2010). Solutions are radial.
Examples of self-translating convex graphs:

- Altschuler and Wu, Clutterbuck, Schnürer and Schulze: A unique radially symmetric solution (for $c = 1$, $N \geq 2$)

$$F(|x|) = \frac{|x|^2}{2(N - 1)} - \log |x| + O(|x|^{-1}) \text{ as } |x| \to \infty.$$ 

- X.-J. Wang: Examples for $N \geq 3$ of convex, non-radial solutions.

**Note:** For $c \neq 0$ $F$ solves $(MC)_c$ iff $G(x) = cF(c^{-1}x)$ solves $(MC)_1$:

$$\nabla \cdot \left( \frac{\nabla G}{\sqrt{1 + |\nabla G|^2}} \right) = \frac{1}{\sqrt{1 + |\nabla G|^2}} \text{ in } \mathbb{R}^N.$$
The answer to (B) is negative for $c > 0$ and $N \geq 8$, in analogy to the result of Bombieri, De Giorgi and Giusti:

**Theorem 5** Daskalopoulos, Del Pino, Kowalczyk, Wei (2011)
Assume that $N \geq 8$. Then there exists a one-parameter family of non-convex entire solutions $F_\varepsilon(x)$, $\varepsilon > 0$ to Equation ($MCG$)

$$\nabla \cdot \left( \frac{\nabla F}{\sqrt{1 + |\nabla F|^2}} \right) = \frac{1}{\sqrt{1 + |\nabla F|^2}} \quad \text{in } \mathbb{R}^N.$$  \hspace{1cm} (MC)_1
Replacing $F_\varepsilon(x)$ with $\varepsilon^{-1}F_\varepsilon(\varepsilon x)$ we are reduced to finding a non-convex solution $F_\varepsilon$ of the equation

$$\nabla \cdot \left( \frac{\nabla F}{\sqrt{1 + |\nabla F|^2}} \right) = \frac{\varepsilon}{\sqrt{1 + |\nabla F|^2}} \quad \text{in } \mathbb{R}^N. \quad (MCG)_\varepsilon$$

When $\varepsilon = 0$ this is the equation of minimal graph:

$$\nabla \cdot \left( \frac{\nabla F}{\sqrt{1 + |\nabla F|^2}} \right) = 0 \quad \text{in } \mathbb{R}^{N-1}$$
Bernstein Conjecture: any minimal hypersurface in $\mathbb{R}^N$, which is also a graph of a function of $N - 1$ variables, must be a hyperplane.


Bombieri, De Giorgi, Giusti (1969) found an analytic minimal graph that is not a hyperplane for $N \geq 9$. 
The Bombieri-De Giorgi-Giusti minimal graph:

Explicit construction by super and sub-solutions. \( N = 9 \):

\[
\nabla \cdot \left( \frac{\nabla F}{\sqrt{1 + |\nabla F|^2}} \right) = 0 \quad \text{in} \quad \mathbb{R}^8.
\]

\( F : \mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{R}, \quad (u, v) \mapsto F(u, v), \quad u = |u|, \ v = |v|. \)

In addition, \( F(u, v) > 0 \) for \( v > u \) and

\[
F(u, v) = -F(v, u).
\]
Asymptotic behavior for BDG surface in polar coordinates $u = r \cos \theta$, $v = r \sin \theta$ (del Pino, Kowalczyk, Wei (2008)):

\[ F(u, v) = r^3 g(\theta) + O(r^{-\sigma}) \quad \text{as} \quad r \to +\infty \]

where

\[ g(\theta) > 0, \quad \theta \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right), \quad g\left(\frac{\pi}{4}\right) = 0 = g'\left(\frac{\pi}{2}\right). \]

\[ \frac{21g \sin^3 2\theta}{\sqrt{9g^2 + g'^2}} + \left(\frac{g' \sin^3 2\theta}{\sqrt{9g^2 + g'^2}}\right)' = 0 \quad \text{in} \quad \left(\frac{\pi}{4}, \frac{\pi}{2}\right) \quad (0.13) \]
solution (the BDG graph) of the form

\[ x = (u, v) \in \mathbb{R}^4 \times \mathbb{R}^4 \mapsto \bar{F}(u, v), \quad u = |u|, \quad v = |v|, \]

\[ \bar{F}(x) = O(|x|^3) \quad \text{as} \quad |x| \to \infty. \]

For small \( \varepsilon > 0 \) we find

\[ F_\varepsilon(x) = \bar{F}(x) + \varepsilon \phi_\varepsilon(x). \]

with

\[ |\phi_\varepsilon(x)| \leq C(|x|^2 + 1) \quad \text{in} \quad \mathbb{R}^8. \]
The method: construction of ordered sub and super solutions for the equation

\[ M[F] := \nabla \cdot \left( \frac{\nabla F}{\sqrt{1 + |\nabla G|^2}} \right) - \frac{\varepsilon}{\sqrt{1 + |\nabla F|^2}} = 0 \quad \text{in } \mathbb{R}^8, \]
The equation $M[\bar{F} + \varphi] = 0$ is at main order, for $r$ large,

$$L_{\bar{F}}[\varphi] = \frac{\varepsilon}{\sqrt{1 + |\nabla F_0|^2}} \approx \frac{\varepsilon p_1(\theta)}{r^2}$$

We can solve by barriers equations of the form

$$L_{\bar{F}}[\varphi] = g = O(r^{-4-\sigma}).$$

where $\sigma > 0$. The barrier procedure however does not work for decays $O(r^{-4})$ or slower, and the main error term only has decay $O(r^{-2})$. 
To overcome this difficulty, we need to improve the approximation: 

There is a smooth function \( \varphi_*(r, \theta) = O(\varepsilon r^2) \) as \( r \to \infty \) such that for some \( \sigma > 0 \)

\[
M[F + \varphi_*] = O(r^{-4-\sigma}).
\]

The function \( \varphi_*(r, \theta) \) is found by setting first

\[
\varphi_*(r, \theta) = \varepsilon \varphi_1(r, \theta) + \varepsilon^2 \varphi_2(r, \theta) + \varepsilon^3 \varphi_2(r, \theta) + \cdots
\]

and solving (explicitly, up to fast decaying terms) the linear equations for the first 3 coefficients (which at main order separate variables).

This and a refinement of the asymptotic behavior of \( \bar{F} - F_0 \) yields the result.
After the above is achieved, the second step is the following.

*There exists a smooth function $\phi$ with $\phi(r, \theta) = O(\varepsilon r^{-\sigma})$ as $r \to \infty$ for some $\sigma > 0$, such that globally*

$$M[\bar{F} + \varphi* + \phi] \leq 0, \quad M[F + \varphi* - \phi] \geq 0.$$  

In essence, $\phi$ is a positive supersolution for the equation

$$L_{\bar{F}}[\phi] = -M[\bar{F} + \varphi*] = O(\varepsilon r^{-4-\sigma})$$
Using the above fact, the proof of our main result can be concluded as follows.

We consider an arbitrary $R$ and the equation

$$\nabla \cdot \left( \frac{\nabla F}{\sqrt{1 + |\nabla F|^2}} \right) - \frac{\varepsilon}{\sqrt{1 + |\nabla F|^2}} = 0 \quad \text{in } B_R(0),$$

$$F = \bar{F} + \varphi_* - \phi \quad \text{on } \partial B_R(0).$$
By super-subsolutions, this problem has a smooth solution \( F_R \) with
\[
\bar{F} + \varphi_* - \phi \leq F_R \leq \bar{F} + \varphi_* + \phi \quad \text{in } B_R(0)
\]

\( G_R \) is increasing in \( R \) on each fixed ball \( B_{R_0}(0) \).

Regularity theory for the mean curvature operator yields
\[
|\nabla F_R| \leq C(R_0) \quad \text{on this ball.}
\]
Hence \( F_R \to F \) in local \( C^2 \)-sense. Thus \( F \) solves
\[
\nabla \cdot \left( \frac{\nabla F}{\sqrt{1 + |\nabla F|^2}} \right) - \frac{\varepsilon}{\sqrt{1 + |\nabla F|^2}} = 0 \quad \text{in } \mathbb{R}^8,
\]

\[
\bar{F} + \varphi_* - \phi \leq F \leq \bar{F} + \varphi_* + \phi \quad \text{in } \mathbb{R}^8.
\]
Summary

\[(AC)_{TW} \quad \Delta u + u - u^3 + cu_{x_N}^{N+1} = 0 \quad \text{in } \mathbb{R}^{N+1}.
\]

- the existence of a single convex radially symmetric front (Chen-Guo-Hamel-Ninomiya-Roquejoffre)
- the existence of a single nonconvex radially symmetric front (del Pino-Kowalczyk-Wei)
- the existence of multiple fronts (del Pino-Kowalczyk-Wei)
\[(MC)_{TW} \nabla \cdot \left( \frac{\nabla F}{\sqrt{1 + |\nabla F|^2}} \right) = \frac{1}{\sqrt{1 + |\nabla F|^2}} \text{ in } \mathbb{R}^N.\]

- the existence of radially symmetric (convex) solutions (Clutterbuck, Schnürer and Schulze)
- In dimension \( N = 2 \), all solutions are convex and radial (X-J Wang)
- the existence of convex, nonradial solutions for \( N \geq 3 \) (X-J Wang)
- the existence of nonconvex graphs for \( N \geq 8 \) (Daskalopoulos–Del Pino-Kowalczyk-Wei)
De Giorgi Type Conjecture: Let $u$ be a bounded solution of equation

$$(AC)_{TW} \quad \Delta u + u - u^3 + cu_{x_{N+1}} = 0 \quad \text{in } \mathbb{R}^{N+1}.$$ 

which satisfies

$$\partial_{x_{N+1}} u > 0$$

Then, $u$ must be radially symmetric in $x'$, at least when $N \leq 2$.

Reason for $N \leq 2$: X-J. Wang’s example of convex nonradial solutions to the translating graphs in $N \geq 3$. 
Bernstein Type Conjecture: Let $u$ be a bounded solution of equation

$$(AC)^T_W \quad \Delta u + u - u^3 + cu_{x_{N+1}} = 0 \quad \text{in } \mathbb{R}^{N+1}. $$

which satisfies

$$\partial_{x_{N+1}} u > 0$$

Then, the level set $\{u = c\}$ is convex, at least when $N \leq 8$.

Reason for $N \leq 8$: DDKW’s example of nonconvex solutions to the translating graphs in $N \geq 9$. 
Happy Birthday, Professor Berestycki!