AN EXAMPLE OF LIMIT OF LEMPERT FUNCTIONS

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1. Introduction

Let Ω be a domain in $\mathbb{C}^n$, and $a_j \in \Omega$, $j = 0, ..., N$. Coman's Lempert function is defined by [2], [5]:

$$(1.1)\quad \ell(z) := \ell_{a_0,\dots,a_N}(z) := \inf \Big\{ \sum_{j=0}^{N} \log|\zeta_j| : \varphi(0) = z, \\ \varphi(\zeta_j) = a_j, j = 0, ..., N \text{ for some } \varphi \in \mathcal{O}(\mathbb{D}, \Omega) \Big\},$$

where $\mathbb{D}$ is the unit disc in $\mathbb{C}$.

For most of this paper, we will consider $\Omega = \mathbb{D}^2$, $|z| := \max(|z_1|, |z_2|)$, $a_0 = (0,0)$, $a_1 = (\varepsilon_1, 0)$, and $a_2 = (0, \varepsilon_2)$, where $\varepsilon_j \in \mathbb{D}$, $j = 1, 2$. We will write $\ell_\varepsilon(z)$ for the Lempert function with respect to the three poles a_0, a_1, a_2 evaluated at the point $z = (z_1, z_2) \in \mathbb{D}$. It is clear that the Lempert function approaches $-\infty$ near each of its poles a_j. When $\varepsilon_j \to 0$, all poles concentrate at the origin of $\mathbb{C}^2$, and the Lempert function may converge to some limit with its singularities concentrated at the origin.

Our goal is to understand in detail the aspect of this singularity. A first remark is that the Lempert function is always related to the corresponding Green function for the same poles,

$$g(z) := \sup\{u \in PSH(\Omega, \mathbb{R}_-) : u(z) \le \log|z - a_j| + C_j, j = 0, ..., N\},$$

where $PSH(\Omega, \mathbb{R}_-)$ stands for the set of all negative plurisubharmonic functions in Ω. The inequality $g(z) \le \ell(z)$ always holds, and it is known that it can be strict [1], [7], [6]. If ℓ ever turns out to be plurisubharmonic itself, then it must be equal to g [2].

In the special case that we are studying, the Green function (denoted by g_ε) is not known either, nor do we know whether it always admits a limit as $\varepsilon_j \to 0$, but it can be compared to the Green functions for the three following systems of points:

$$S_1 := \{a_0, a_1\}, \quad S_2 := \{a_0, a_2\}, \quad S_3 := \{a_0, a_1, a_2, (\varepsilon_1, \varepsilon_2)\}.$$

Those are all product sets, so their Green functions are explicitly known [3] as well as their limits when $\varepsilon_j \to 0$, which are respectively

$$g_1(z) := \max(2\log|z_1|, \log|z_2|), \quad g_2(z) := \max(\log|z_1|, 2\log|z_2|),$$
$$g_3(z) := \max(2\log|z_1|, 2\log|z_2|).$$

Nguyen Van Trao remarked that it follows from the definition of the Green function that

$$g_3 \le g_\varepsilon \le \min(g_1, g_2) \tag{1.2}$$

throughout the bidisk, and therefore when $|z_2| \le |z_1|^2$, $g_\varepsilon(z) = g_3(z) = g_1(z) = 2\log|z_1|$, and when $|z_1| \le |z_2|^2$, $g_\varepsilon(z) = g_3(z) = g_2(z) = 2\log|z_2|$. Also, for any z in the bidisk,

$$\liminf_{\varepsilon_1,\varepsilon_2 \to 0} \ell_\varepsilon(z) \ge g_\varepsilon(z) \ge g_3(z) = 2\log|z|.$$

We first give our result in a special case where the picture is more complete.

Theorem 1.1. *Suppose $\varepsilon_1 = \varepsilon_2 = \varepsilon$. Then*

(1) *If $z_1 = 0$ or $z_2 = 0$ or $z_1 + z_2 = 0$, there exists a constant $C > 0$ such that*
$$\limsup_{\varepsilon_1,\varepsilon_2 \to 0} \ell_\varepsilon(z) \le 2\log|z| + C.$$
(2) *For any $c_0 > 0$, there exists a constant $C = C(c_0) > 0$ such that for any $z = (z_1, z_2)$ verifying $c_0 \le \left|\frac{z_1}{z_2}\right| \le c_0^{-1}$, and $c_0 \le \left|1 + \frac{z_1}{z_2}\right|$, then*
$$\liminf_{\varepsilon \to 0} \ell_\varepsilon(z) \ge \frac{3}{2}\log|z| - C$$
and
$$\liminf_{\varepsilon \to 0} \ell_\varepsilon(z) \le \frac{3}{2}\log|z| + C.$$

Corollary 1.2. *If g is some cluster point of the family $\{g_\varepsilon\}$, then for any $z \in \mathbb{D}^2$,*
$$g(z) \le \frac{3}{2}\log|z|,$$
and $g(z_1, -z_1) = 2\log|z_1|$.

This is an improvement over the inequalities (1.2) when $|z_2|^2 < |z_1| < |z_2|^{1/2}$.

Proof. The function g must be plurisubharmonic, so subharmonic on any complex line going through the origin. It is also negative everywhere. The inequality that its restriction on a complex line satisfies near the origin implies that it is bounded above by the corresponding one-variable Green function, which gives us the required upper bound. The equality in the special case follows from (1.2). □

Theorem 1.1 will follow from the more detailed result below.

Theorem 1.3. (1) *If* $z_1 = 0$ *or* $z_2 = 0$ *or if* $z_1 \neq 0, z_2 \neq 0$ *and* $\lim_{\varepsilon_1,\varepsilon_2\to 0} \frac{\varepsilon_1}{\varepsilon_2} = -\frac{z_1}{z_2}$*, there exists a constant* $C > 0$ *such that*

$$\limsup_{\varepsilon_1,\varepsilon_2\to 0} \ell_\varepsilon(z) \le 2\log|z| + C.$$

(2) *For any* $c_0 > 0$*, there exists a constant* $C = C(c_0) > 0$ *such that for any* $z = (z_1, z_2)$ *verifying* $c_0 \le \left|\frac{z_1}{z_2}\right| \le c_0^{-1}$*, with the* $(\varepsilon_1, \varepsilon_2)$ *involved in the limes (inferior and superior) below always verifying*

$$\left| \frac{z_1}{|z_1|}\frac{|z_2|}{z_2} + \frac{\varepsilon_1}{|\varepsilon_1|}\frac{|\varepsilon_2|}{\varepsilon_2} \right| \ge c_0, \tag{1.3}$$

then

$$\liminf_{\varepsilon_1,\varepsilon_2\to 0} \ell_\varepsilon(z) \ge \frac{3}{2}\log|z| - C. \tag{1.4}$$

If, furthermore,

$$\lim_{\varepsilon_1,\varepsilon_2\to 0} \frac{\varepsilon_1^2}{\varepsilon_2} = \lim_{\varepsilon_1,\varepsilon_2\to 0} \frac{\varepsilon_2^2}{\varepsilon_1} = 0,$$

then

$$\limsup_{\varepsilon_1,\varepsilon_2\to 0} \ell_\varepsilon(z) \le \frac{3}{2}\log|z| + C. \tag{1.5}$$

2. Upper estimates

To prove the upper estimates in the above theorem, we shall need to construct appropriate maps φ from the disk to the bidisk. It will be useful to relax a little the condition that $\varphi(\mathbb{D}) \subset \mathbb{D}^2$.

Lemma 2.1. *Suppose that* $a_j(\varepsilon) \in \mathbb{D}^2$*,* $j = 0, ..., N$ *depend on some parameter* $\varepsilon \in \mathbb{C}^m$*, with* $\lim_{\varepsilon\to 0} a_j(\varepsilon) = a_j \in \Omega$*,* $j = 0, ..., N$*, and that there exists a function* $\gamma(\varepsilon)$ *such that* $\lim_{\varepsilon\to 0}\gamma(\varepsilon) = 0$ *and* $\lambda \in \mathbb{R}$ *with the following property: for any* $\varepsilon > 0$*,* $z \in \mathbb{D}^2\setminus\{a_1, \dots, a_N\}$*, there exists a holomorphic map* $\varphi : \mathbb{D} \longrightarrow D(0, 1 + \gamma(\varepsilon))^2$ *and* $\zeta_0, \dots, \zeta_N$ *such that*

$$\varphi(0) = z, \varphi(\zeta_j) = a_j(\varepsilon), j = 0, ..., N, \quad \text{and} \sum_{j=0}^{N} \log|\zeta_j| \le \lambda. \tag{2.1}$$

Then $\limsup_{\varepsilon\to 0} \ell_{a_0(\varepsilon),\dots,a_N(\varepsilon)}(z) \le \lambda$.

Proof. First note that by applying an automorphism ϕ of the bidisk exchanging z and $(0,0)$, we have a map $\tilde{\varphi} := \phi \circ \varphi$ such that

$$\tilde{\varphi}(0) = (0,0), \varphi(\zeta_j) = \phi(a_j(\varepsilon)), j = 0, ..., N, \text{ and } \tilde{\varphi} : \mathbb{D} \longrightarrow D(0, 1+\gamma(\varepsilon))^2,$$

for another function γ with the same property as the original one.

Likewise, to estimate $\ell_{a_1(\varepsilon),\dots,a_N(\varepsilon)}(z)$ it is equivalent to look for maps ψ and points $\zeta'_0, \dots, \zeta'_N$ such that $\psi(\mathbb{D}) \subset \mathbb{D}^2$, and

$$\psi(0) = (0,0), \varphi(\zeta'_j) = \phi(a_j(\varepsilon)), j = 0, ..., N, \text{and } \limsup_{\varepsilon \to 0} \sum_{j=0}^{N} \log |\zeta'_j| \le \lambda$$

Applying the usual Schwarz lemma to each coordinate of $\tilde{\varphi}$, we see that $\tilde{\varphi}(D(0, (1+\gamma(\varepsilon))^{-1}) \subset \mathbb{D}^2$, and therefore

$$\psi(\zeta) := \tilde{\varphi}(\frac{\zeta}{1+\gamma(\varepsilon)}) \quad \text{and } \zeta'_j := (1+\gamma(\varepsilon))\zeta_j$$

will satisfy our requirements. □

Notation: we always write $\phi_a(\zeta) := \frac{a-\zeta}{1-\zeta\bar{a}}$ for the involutive automorphism of the unit disk exchanging a and 0. Applying an automorphism of $\mathbb{D}$ exchanging ζ_0 and 0, the original problem of construction of maps as in (2.1) is equivalent to finding a holomorphic map $\varphi : \mathbb{D} \longrightarrow D(0, 1+\gamma(\varepsilon))^2$ and new points $\zeta_0, \dots, \zeta_N$ such that

(2.2)

$$\varphi(0) = (0,0), \varphi(\zeta_j) = a_j(\varepsilon), j = 1, ..., N, \quad \text{and } |\zeta_0| + \sum_{j=1}^{N} \log |\phi_{\zeta_0}(\zeta_j)| \le \lambda.$$

Proof of Theorem 1.3, part (1).

Assume $z_2 = 0$. The case $z_1 = 0$ would be treated in the same way.

By the formulation (2.2) of our problem, we need to construct a map φ, with an appropriate control of the image of $\mathbb{D}$, such that

$$\left\{ \begin{array}{rcl} \varphi(0) & = & (0,0) \\ \varphi(\zeta_1) & = & (\varepsilon_1, 0) \\ \varphi(\zeta_2) & = & (0, \varepsilon_2) \\ \varphi(\zeta_0) & = & (z_1, 0) \end{array} \right. \tag{2.3}$$

First we choose $\zeta_1 := \varepsilon_1$, $\zeta_0 := z_1$, and ζ_2 close to 1, to be specified later. An approximate solution of this interpolation problem, will be given by the following map from $\mathbb{D}$ to $\mathbb{D}^2$:

$$\varphi^0(\zeta) := \left(\zeta \phi_{\zeta_2}(\zeta), \zeta \frac{\phi_{\varepsilon_1}(\zeta)}{\phi_{\varepsilon_1}(1)} \frac{\phi_{z_1}(\zeta)}{\phi_{z_1}(1)} \varepsilon_2 \right).$$

The errors with respect to the requirements in (2.3) are now given by

$$(2.4)\quad \begin{cases} \varphi^0(0)-(0,0) &= (0,0) \\ \varphi^0(\varepsilon_1)-(\varepsilon_1,0) &= (\varepsilon_1(\phi_{\zeta_2}(\varepsilon_1)-1),0) & =: (E_1,0) \\ \varphi^0(\zeta_2)-(0,\varepsilon_2) &= \Big(0,\varepsilon_2(\zeta_2\frac{\phi_{\varepsilon_1}(\zeta_2)}{\phi_{\varepsilon_1}(1)}\frac{\phi_{z_1}(\zeta_2)}{\phi_{z_1}(1)}-1)\Big) & =: (0,E_2) \\ \varphi^0(z_1)-(z_1,0) &= (z_1(\phi_{\zeta_2}(z_1)-1),0) & =: (E_3,0) \end{cases}$$

A computation shows that

$$|\phi_{\zeta_2}(\zeta)-1| = \left|\frac{(1-\zeta_2)+\zeta(1-\bar\zeta_2)}{1-\zeta\bar\zeta_2}\right| \le \left(\frac{1+|\zeta|}{1-|\zeta|}\right)|1-\zeta_2|,$$

so that for $|z_1|\le\frac12$, $\varepsilon_1\le\frac12|z_1|$, which we may assume,

$$(2.5)\qquad |E_1|\le\frac53|1-\zeta_2||\varepsilon_1|, |E_3|\le 3|1-\zeta_2||z_1|.$$

In the same way, one sees that under the above assumption, $|E_2|\preceq |1-\zeta_2||\varepsilon_2|$.

To get a map satisfying (2.3), we subtract from φ^0 a correcting term $\varphi^1(\zeta)=(\varphi_1^1(\zeta),\varphi_2^1(\zeta))$ obtained by Lagrange interpolation. More precisely, we choose

$$\varphi_2^1(\zeta)=\frac{\zeta}{\zeta_2}\frac{\phi_{\varepsilon_1}(\zeta)}{\phi_{\varepsilon_1}(\zeta_2)}\frac{\phi_{z_1}(\zeta)}{\phi_{z_1}(\zeta_2)}E_2,$$

and $\varphi_1^1(\zeta)=\zeta\phi_{\zeta_2}(\zeta)h(\zeta)$, where h must satisfy

$$h(\varepsilon_1)=\frac{E_1}{\varepsilon_1\phi_{\zeta_2}(\varepsilon_1)} =: E_1',\quad h(z_1)=\frac{E_3}{z_1\phi_{\zeta_2}(z_1)} =: E_3'.$$

We can then set

$$h(\zeta):=E_1'\frac{\zeta-z_1}{\varepsilon_1-z_1}+E_3'\frac{\zeta-\varepsilon_1}{z_1-\varepsilon_1}.$$

It is easy to see that $|\varphi_2^1(\zeta)|\preceq|E_2|\le|\varepsilon|$ for ζ_2 close enough to 1. It follows from (2.5) that $|E_1'|,|E_3'|\preceq|1-\zeta_2|$, therefore $|\varphi_1^1(\zeta)|\preceq|1-\zeta_2||z_1|^{-1}\le|\varepsilon|$ for ζ_2 close enough to 1. So the map $\varphi^0-\varphi^1$ satisfies the hypotheses of Lemma 2.1 and since we have

$$|\zeta_0|+\sum_{j=1}^2\log|\phi_{\zeta_0}(\zeta_j)|\le\log|z_1|+\log|\phi_{z_1}(\varepsilon_1)|\le\log|z_1|+\log(|z_1|+|\varepsilon_1|),$$

we may take $\lambda=2\log|z_1|+\eta$, for any $\eta>0$, which implies the conclusion of Theorem 1.3, part (1), with $C=0$. □

Proof of Theorem 1.3, part (2), (1.5).

We will need a few notations. We set $\mu := z_2/z_1$, $\sigma := \varepsilon_2/\varepsilon_1$. Exchanging coordinates if needed, we may assume $|\mu| \leq 1$, and therefore $|z| = |z_1|$. The hypothesis in the theorem is that

$$c_0 \leq \left| \frac{|\mu|}{\mu} + \frac{|\sigma|}{\sigma} \right| = \left| 1 + \left(\frac{|\mu|}{\mu} \right)^{-1} \frac{|\sigma|}{\sigma} \right|,$$

so the complex number σ/μ lies outside of certain plane sector containing -1, and in particular $|1 + (\sigma/\mu)| \geq c_0$. We choose a complex number ν such that $\nu^2 := (1 + (\sigma/\mu))^{-1}$; this remains bounded.

We choose a complex number ζ_0 such that $\zeta_0^2 = z_1$. This means that $z = (\zeta_0^2, \mu\zeta_0^2)$. We also choose, for each ε_1, an ε_1' such that ${\varepsilon_1'}^2 = \varepsilon_1$. Now we set

(2.6)

$$\zeta_1 := \nu\varepsilon_1' = \left(1 + \frac{\varepsilon_2}{\varepsilon_1\mu} \right)^{-1/2} \varepsilon_1^{1/2}, \quad \zeta_2 := -\frac{\sigma}{\mu}\zeta_1 = -\frac{\varepsilon_2}{\varepsilon_1^{1/2}} \left(1 + \frac{\varepsilon_2}{\varepsilon_1\mu} \right)^{-1/2}.$$

We will follow the pattern of the previous proof. In order to apply Lemma 2.1, we need to produce a map satisfying

$$\text{(2.7)} \qquad \begin{cases} \varphi(0) &= (0,0) \\ \varphi(\zeta_1) &= (\varepsilon_1, 0) \\ \varphi(\zeta_2) &= (0, \varepsilon_2) \\ \varphi(\zeta_0) &= (z_1, z_2) \end{cases}$$

We choose $\zeta_0, \zeta_1, \zeta_2$ as above and set

$$\varphi^0(\zeta) = \left(\zeta \frac{\zeta - \zeta_2}{1 - \zeta\bar{\zeta}_2}, \mu\zeta \frac{\zeta - \zeta_1}{1 - \zeta\bar{\zeta}_1} \right).$$

We remark that our ζ_j have been chosen so that

$$\zeta_1(\zeta_1 - \zeta_2) = \varepsilon_1, \mu\zeta_2(\zeta_2 - \zeta_1) = \varepsilon_2, \text{ and } |\zeta_1\zeta_2| = |\nu|^2|\mu|^{-1}|\varepsilon_2| \leq c_0^{-2}|\varepsilon_2|.$$

Elementary computations yield

(2.8)

$$\begin{cases} \varphi^0(0) - (0,0) &= (0,0) \\ \varphi^0(\zeta_1) - (\varepsilon_1, 0) &= \left(\varepsilon_1 \frac{\zeta_1\bar{\zeta}_2}{1 - \zeta_1\bar{\zeta}_2}, 0 \right) & =: (E_1, 0) \\ \varphi^0(\zeta_2) - (0, \varepsilon_2) &= \left(0, \varepsilon_2 \frac{\bar{\zeta}_1\zeta_2}{1 - \bar{\zeta}_1\zeta_2} \right) & =: (0, E_2) \\ \varphi^0(\zeta_0) - z &= \left(\zeta_0 \frac{\zeta_0^2\bar{\zeta}_2 - \zeta_2}{1 - \zeta_0\bar{\zeta}_2}, \mu\zeta_0 \frac{\zeta_0^2\bar{\zeta}_1 - \zeta_1}{1 - \zeta_0\bar{\zeta}_1} \right) & =: (E_3, E_4) \end{cases}.$$

We construct a correcting term φ^1 by Lagrange interpolation:

$$\begin{aligned}\varphi_1^1(\zeta) &= E_1\tfrac{\zeta(\zeta-\zeta_2)(\zeta-\zeta_0)}{\zeta_1(\zeta_1-\zeta_2)(\zeta_1-\zeta_0)} + E_3\tfrac{\zeta(\zeta-\zeta_1)(\zeta-\zeta_2)}{\zeta_0(\zeta_0-\zeta_1)(\zeta_0-\zeta_2)},\\ \varphi_2^1(\zeta) &= E_2\tfrac{\zeta(\zeta-\zeta_1)(\zeta-\zeta_0)}{\zeta_2(\zeta_2-\zeta_1)(\zeta_2-\zeta_0)} + E_4\tfrac{\zeta(\zeta-\zeta_1)(\zeta-\zeta_2)}{\zeta_0(\zeta_0-\zeta_1)(\zeta_0-\zeta_2)}.\end{aligned}$$

The map $\varphi^0 - \varphi^1$ will assume the correct values, now we need to see that it sends $\mathbb{D}$ to a neighborhood of the bidisk by estimating the size of φ^1.

First note that for $|\varepsilon|$ small enough, $|E_3| \le 3|z_1|^{1/2}|\zeta_2| \le 3c_0^{-3/2}|z_1|^{1/2}|\varepsilon_1|^{-1/2}|\varepsilon_2|$, and $|E_4| \le 3|z_1|^{1/2}|\zeta_1| \le 3c_0^{-1/2}|z_1|^{1/2}|\varepsilon_1|^{1/2}$, therefore, using the last hypothesis of the theorem, for $|\varepsilon|$ small enough (depending on $|z_1|$), the second terms in $\varphi_1^1(\zeta)$ and $\varphi_2^1(\zeta)$ can be made arbitrarily small.

On the other hand, for $|\varepsilon|$ small enough,

$$\left|\frac{E_1}{\zeta_1(\zeta_1-\zeta_2)}\right| = \left|\frac{\zeta_1\bar\zeta_2}{1-\zeta_1\bar\zeta_2}\right| \le 2c_0^{-2}|\varepsilon_2|,$$

$$\left|\frac{E_2}{\zeta_2(\zeta_2-\zeta_1)}\right| = \left|\frac{\mu\bar\zeta_1\zeta_2}{1-\bar\zeta_1\zeta_2}\right| \le 2c_0^{-3}|\varepsilon_2|.$$

So the first terms in $\varphi_1^1(\zeta)$ and $\varphi_2^1(\zeta)$ are bounded above by $2c_0^{-3}|\varepsilon_2||z_1|^{-1/2}$, which once again can be made arbitrarily small for $|\varepsilon|$ small enough.

Finally, the relevant sum

$$\log|\zeta_0| + \log|\phi_{\zeta_0}(\zeta_1)| + \log|\phi_{\zeta_0}(\zeta_2)| \le \log|\zeta_0| + \log(|\zeta_0|+|\zeta_1|) + \log(|\zeta_0|+|\zeta_2|),$$

so that $\lambda = 3\log|\zeta_0| + \eta = \frac{3}{2}\log|z_1| + \eta \le \frac{3}{2}\log|z| + \eta$ may be used to apply Lemma 2.1, for any $\eta > 0$. □

3. Lower Estimates

Proof of Theorem 1.3, part (2), (1.4).

We will assume that the conclusion fails, i.e. that for any $C_H > 0$ there exist arbitrarily small values of $\varepsilon = (\varepsilon_1, \varepsilon_2)$ such that

$$\ell_\varepsilon(z) \le \frac{3}{2}\log|z| - C_H, \tag{3.1}$$

which means (we change the value of the constant slightly while keeping the same notation) that there exists a holomorphic map φ from $\mathbb{D}$ to $\mathbb{D}^2$ and points $\zeta_j \in \mathbb{D}$ satisfying the conditions in (2.7) with

$$\log|\zeta_0| + \log|\phi_{\zeta_0}(\zeta_1)| + \log|\phi_{\zeta_0}(\zeta_2)| \le \frac{3}{2}\log|z| - C_H. \tag{3.2}$$

The interpolation conditions in (2.7) are equivalent to the existence of two holomorphic functions h_1, h_2 from $\mathbb{D}$ to itself such that

$$\varphi(\zeta) = \left(\zeta\phi_{\zeta_2}(\zeta)h_1(\zeta), \zeta\phi_{\zeta_1}(\zeta)h_2(\zeta)\right),$$

such that furthermore

$$h_1(\zeta_1) = \frac{\varepsilon_1}{\zeta_1 \phi_{\zeta_2}(\zeta_1)} =: w_1, \tag{3.3}$$

$$h_1(\zeta_0) = \frac{z_1}{\zeta_0 \phi_{\zeta_2}(\zeta_0)} =: w_2, \tag{3.4}$$

$$h_2(\zeta_2) = \frac{\varepsilon_2}{\zeta_2 \phi_{\zeta_1}(\zeta_2)} =: w_4, \tag{3.5}$$

$$h_2(\zeta_0) = \frac{z_2}{\zeta_0 \phi_{\zeta_1}(\zeta_0)} =: w_3. \tag{3.6}$$

For convenience, we will use the invariant (pseudohyperbolic) distance between points of the unit disk given by

$$d_G(a,b) := |\phi_a(b)| = |\phi_b(a)|\,.$$

By the invariant Schwarz Lemma, the existence of a holomorphic function h_1 mapping $\mathbb{D}$ to itself and satisfying (3.3) and (3.4) is equivalent to

$$|w_1| < 1, |w_2| < 1, \text{ and } d_G(w_1, w_2) < d_G(\zeta_1, \zeta_0) = |\phi_{\zeta_1}(\zeta_0)|\,. \tag{3.7}$$

In the same way, the existence of h_2 is equivalent to

$$|w_3| < 1, |w_4| < 1, \text{ and } d_G(w_3, w_4) < d_G(\zeta_2, \zeta_0) = |\phi_{\zeta_2}(\zeta_0)|\,. \tag{3.8}$$

The proof will proceed as follows: one main elementary tool is the fact that if the invariant distance of two points in the unit disk is small with respect to one of their moduli, then the difference of their arguments (or rather, the distance between their projections to the unit circle) must be small (Lemma 3.1).

Then, using (3.1) and the fact that $|w_2|$ and $|w_3|$ are in the unit disk, we will show that $|\phi_{\zeta_1}(\zeta_0)|$ and $|\phi_{\zeta_2}(\zeta_0)|$ must both be relatively small. This will imply that $|w_2|$ and $|w_3|$ are also relatively big, and because of (3.7), (3.8), so will be $|w_1|$ and $|w_4|$, and because ε is small, this will force $|\phi_{\zeta_2}(\zeta_1)|$ to be small too, and therefore $|\arg(\zeta_1/\zeta_2)|$. This will allow us to show that $\arg(w_1/w_4)$ is close to $\arg(-\varepsilon_1/\varepsilon_2)$, because $\phi_{\zeta_2}(\zeta_1)$ is almost opposite to $\phi_{\zeta_1}(\zeta_2)$. On the other hand, use of the triangle inequality on the unit circle will show that $\arg(w_1/w_4) = \arg[(w_1/w_2)(w_2/w_3)(w_3/w_4)]$ is close to $\arg(z_1/z_2)$. Hypothesis (1.3) will lead to a contradiction when C_H is big enough.

Lemma 3.1. *Suppose that $a, b \in \mathbb{D}$, and that $d_G(a,b) = \delta < |a|$, $\delta \leq \frac{1}{2}$. Then*

$$\left|\arg\left(\frac{a}{b}\right)\right| \leq \arcsin\left(\frac{\delta(1-|a|^2)}{|a|(1-\delta^2)}\right) \leq \frac{3\delta}{|a|}.$$

Note that in this result, and the computations that follow, we implicitly only consider arguments in the range $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

Proof. The set of all points $b \in \mathbb{D}$ such that $d_G(a,b) \leq \delta$ is a disk of center $\gamma := \frac{a(1-\delta^2)}{1-|a|^2\delta^2}$ and of radius $\rho := \frac{\delta(1-|a|^2)}{1-|a|^2\delta^2}$ [4, Chapter I, Section 1, p. 3]. Elementary geometry shows that the absolute value of the sine of the angle between a and b is bounded by $\rho/|\gamma|$. □

Our starting remark is that (3.2) can be rewritten as

$$
\begin{aligned}
(3.9)\quad -\log|w_2| - \log|w_3| &= \log\left|\frac{\zeta_0\phi_{\zeta_1}(\zeta_0)}{z_2}\right| + \log\left|\frac{\zeta_0\phi_{\zeta_0}(\zeta_2)}{z_1}\right| \\
&\leq \log|\zeta_0| + \frac{3}{2}\log|z| - \log|z_1| - \log|z_2| - C_H \\
&\leq \log|\zeta_0| - \frac{1}{2}\log|z| - \log c_0 - C_H.
\end{aligned}
$$

Since both terms on the left hand side of the first inequality sign are positive by (3.7), (3.8), they are each bounded by the right hand side, and as a consequence, writing $C'_H := \exp(C_H)$,

$$
(3.10)\qquad |\phi_{\zeta_1}(\zeta_0)| < \frac{|z|^{3/2}}{|z_1|C'_H} \leq \frac{|z|^{1/2}}{c_0 C'_H},
$$

$$
(3.11)\qquad |\phi_{\zeta_2}(\zeta_0)| < \frac{|z|^{3/2}}{|z_2|C'_H} \leq \frac{|z|^{1/2}}{c_0 C'_H}.
$$

Furthermore, since the right hand side must also be positive, we have

$$
(3.12)\qquad |\zeta_0| > c_0 C'_H |z|^{1/2} \geq (c_0 C'_H)^2 \max(|\phi_{\zeta_1}(\zeta_0)|, |\phi_{\zeta_2}(\zeta_0)|).
$$

Lower bounds for $|w_2|$ and $|w_3|$, and $|w_1|$ and $|w_4|$.

By (3.11),

$$
(3.13)\qquad |w_2| = \frac{|z_1|}{|\zeta_0||\phi_{\zeta_2}(\zeta_0)|} > \frac{c_0 C'_H |z|^{1/2}}{|\zeta_0|},
$$

and since $d_G(w_2, w_1) < |\phi_{\zeta_1}(\zeta_0)|$, classical facts about the invariant distance [4, Lemma 1.4, p. 4] imply

$$
(3.14)\quad |w_1| \geq \frac{|w_2| - |\phi_{\zeta_1}(\zeta_0)|}{1 - |w_2||\phi_{\zeta_1}(\zeta_0)|} \geq \frac{c_0 C'_H |z|^{1/2}}{|\zeta_0|} - \frac{|z|^{1/2}}{c_0 C'_H} \geq \frac{c_0 C'_H |z|^{1/2}}{2|\zeta_0|}
$$

when C_H is big enough (depending on c_0). The same computations go through for w_3 and w_4 respectively.

Upper bound for $d_G(\zeta_1, \zeta_2) = |\phi_{\zeta_2}(\zeta_1)|$.

By definition of w_1, we have $\phi_{\zeta_2}(\zeta_1) = \varepsilon_1/(\zeta_1 w_1)$. We estimate the modulus of $|\zeta_1|$ from below by noting that ζ_1 is close to ζ_0, by (3.10) and using (3.12):

$$|\zeta_1| \geq \frac{|\zeta_0| - |\phi_{\zeta_1}(\zeta_0)|}{1 - |\zeta_0||\phi_{\zeta_1}(\zeta_0)|} \geq \frac{1}{2}|\zeta_0|, \tag{3.15}$$

when C_H is big enough (depending on c_0). So by (3.14)

$$|\phi_{\zeta_2}(\zeta_1)| \leq 4\frac{|\varepsilon_1|}{c_0 C'_H |z|^{1/2}}. \tag{3.16}$$

Since $|\varepsilon|$ will tend to 0 as z remains fixed, this estimate is much stronger than the previous (3.10) and (3.11). From this, (3.15) and Lemma 3.1, we deduce

$$\left|\arg\left(\frac{\zeta_1}{\zeta_2}\right)\right| \leq 24\frac{|\varepsilon_1|}{(c_0 C'_H)^2 |z|}. \tag{3.17}$$

Argument estimates for w_1/w_4.
First compute

$$\frac{w_1}{w_4} = \frac{\varepsilon_1}{\varepsilon_2}\frac{\zeta_2}{\zeta_1}\frac{\phi_{\zeta_1}(\zeta_2)}{\phi_{\zeta_2}(\zeta_1)} = -\frac{\varepsilon_1}{\varepsilon_2}\frac{\zeta_2}{\zeta_1}\frac{1-\bar{\zeta}_2\zeta_1}{1-\bar{\zeta}_1\zeta_2}.$$

We need to bound $\arg(1-\bar{\zeta}_2\zeta_1)$. We use the fact that ζ_1 and ζ_2 are close to each other :

$$1-\bar{\zeta}_2\zeta_1 = (1-|\zeta_1|^2)\left(1 - \frac{(\bar{\zeta}_2 - \bar{\zeta}_1)}{1-|\zeta_1|^2}\zeta_1\right),$$

and we know that $\frac{a-b}{1-|a|^2} = O(d_G(a,b))$, so that (3.16) implies that

$$\left|\arg(1-\bar{\zeta}_2\zeta_1)\right| \leq c_1 \frac{|\varepsilon_1|}{c_0 C'_H |z|^{1/2}},$$

where c_1 is some absolute constant. To avoid problems of definition of arguments, use the notation $w^* := w/|w|$ for any nonzero complex number. We have proved that

$$\left|\left(\frac{w_1}{w_4}\right)^* - \left(-\frac{\varepsilon_1}{\varepsilon_2}\right)^*\right| \leq c_1 \frac{|\varepsilon_1|}{c_0 C'_H |z|^{1/2}}, \tag{3.18}$$

where c_1 is some absolute constant (not the same as above).

Argument estimates for w_2/w_3.
First note that

$$\frac{w_2}{w_3} = \frac{z_1}{z_2}\frac{\phi_{\zeta_1}(\zeta_0)}{\phi_{\zeta_2}(\zeta_0)} = \frac{z_1}{z_2}\frac{\phi_{\zeta_1}(\zeta_0)}{\phi_{\zeta_0}(\zeta_1)}\frac{\phi_{\zeta_0}(\zeta_1)}{\phi_{\zeta_0}(\zeta_2)}\frac{\phi_{\zeta_0}(\zeta_2)}{\phi_{\zeta_2}(\zeta_0)}.$$

Since ϕ_{ζ_0} is an automorphism of the unit disk, it preserves the invariant distances, so $d_G(\phi_{\zeta_0}(\zeta_1), \phi_{\zeta_0}(\zeta_2)) = d_G(\zeta_1, \zeta_2)$. To apply Lemma 3.1, we now need a *lower* bound for $|\phi_{\zeta_0}(\zeta_1)|$, say. But the fact that $|w_3| < 1$ (see (3.8)) already implies that $|\phi_{\zeta_0}(\zeta_1)| > |z_2| > c_0^{-1}|z|$. Finally, using the Lemma with (3.16),

$$\left|\arg\left(\frac{\phi_{\zeta_0}(\zeta_1)}{\phi_{\zeta_0}(\zeta_2)}\right)\right| \leq 12\frac{|\varepsilon_1|}{C'_H|z|^{3/2}}, \tag{3.19}$$

provided that $|\varepsilon|$ is small enough.

Now computations as in the previous paragraph yield

$$\frac{\phi_{\zeta_1}(\zeta_0)}{\phi_{\zeta_0}(\zeta_1)}\frac{\phi_{\zeta_0}(\zeta_2)}{\phi_{\zeta_2}(\zeta_0)} = \frac{1-\bar\zeta_0\zeta_1}{1-\bar\zeta_1\zeta_0}\frac{1-\bar\zeta_2\zeta_0}{1-\bar\zeta_0\zeta_2}.$$

Since

$$(1-\bar\zeta_0\zeta_1)(1-\bar\zeta_2\zeta_0) = (1-\bar\zeta_0\zeta_1)(1-(\bar\zeta_1+\bar\zeta_2-\bar\zeta_1)\zeta_0) = |1-\bar\zeta_0\zeta_1|^2\left(1+\frac{(\bar\zeta_2-\bar\zeta_1)\zeta_0}{1-\bar\zeta_1\zeta_0}\right),$$

we can estimate $\arg\left(\frac{1-\bar\zeta_0\zeta_1}{1-\bar\zeta_1\zeta_0}\frac{1-\bar\zeta_2\zeta_0}{1-\bar\zeta_0\zeta_2}\right)$ by a constant multiple of $d_G(\zeta_1, \zeta_2)$, and as in the previous paragraph,

$$\left|\arg\left(\frac{\phi_{\zeta_1}(\zeta_0)}{\phi_{\zeta_0}(\zeta_1)}\frac{\phi_{\zeta_0}(\zeta_2)}{\phi_{\zeta_2}(\zeta_0)}\right)\right| \leq c_1\frac{|\varepsilon_1|}{c_0C'_H|z|^{1/2}},$$

changing the value of c_1 as needed.

Using this and (3.19), the final result of this paragraph is then that

$$\left|\left(\frac{w_2}{w_3}\right)^* - \left(\frac{z_1}{z_2}\right)^*\right| \leq c_1\frac{|\varepsilon_1|}{c_0C'_H|z|^{1/2}} + 12\frac{|\varepsilon_1|}{C'_H|z|^{3/2}} \leq c_2\frac{|\varepsilon_1|}{c_0C'_H|z|^{3/2}}. \tag{3.20}$$

Triangle inequality, and contradiction.

The conditions about invariant distances in (3.7) and (3.8) express the fact that w_1 is close to w_2, and that w_4 is close to w_3. In fact, applying Lemma 3.1 and the estimates (3.13) and (3.10), we obtain

$$\left|\arg\left(\frac{w_1}{w_2}\right)\right| \leq \arcsin\left(\frac{\frac{|z|^{1/2}}{c_0C'_H}\left(1-\frac{(c_0C'_H)^2|z|}{|\zeta_0|^2}\right)}{\frac{c_0C'_H|z|^{1/2}}{|\zeta_0|}\left(1-\frac{|z|}{(c_0C'_H)^2}\right)}\right) \leq \frac{3}{(c_0C'_H)^2},$$

provided that C'_H is large enough (depending on c_0). As before, the same reasoning applies to w_4 and w_3, and we finally have

$$\left|\left(\frac{w_1}{w_2}\right)^* - 1\right| \leq \frac{3}{(c_0C'_H)^2}, \quad \left|\left(\frac{w_3}{w_4}\right)^* - 1\right| \leq \frac{3}{(c_0C'_H)^2}.$$

Now we combine this with (3.20) (and the fact that rotations are isometries) to obtain

$$\begin{aligned}\left|\left(\frac{w_1}{w_4}\right)^* - \left(\frac{z_1}{z_2}\right)^*\right| &= \left|\left(\frac{w_1}{w_2}\right)^*\left(\frac{w_3}{w_4}\right)^*\left(\frac{w_2}{w_3}\right)^* - \left(\frac{z_1}{z_2}\right)^*\right| \\ &\leq \left|\left(\frac{w_1}{w_2}\right)^* - 1\right| + \left|\left(\frac{w_3}{w_4}\right)^* - 1\right| + \left|\left(\frac{w_2}{w_3}\right)^* - \left(\frac{z_1}{z_2}\right)^*\right| \\ &\leq \frac{6}{(c_0 C'_H)^2} + c_2 \frac{|\varepsilon_1|}{c_0 C'_H |z|^{3/2}}.\end{aligned}$$

Combining this with (3.18), we see that, for $|\varepsilon|$ small enough (depending on $|z|$),

$$\left|\left(\frac{z_1}{z_2}\right)^* - \left(-\frac{\varepsilon_1}{\varepsilon_2}\right)^*\right| \leq \frac{6}{(c_0 C'_H)^2} + c_1 \frac{|\varepsilon_1|}{c_0 C'_H |z|^{1/2}} + c_2 \frac{|\varepsilon_1|}{c_0 C'_H |z|^{3/2}}.$$

However, our hypothesis (1.3) precisely says that the left hand side of this must be greater than c_0. Taking C'_H large enough (depending on c_0 only), we may assume $\frac{6}{(c_0 C'_H)^2} < \frac{c_0}{2}$, so that by taking $|\varepsilon|$ small enough (depending on $|z|$), we obtain a contradiction, q.e.d.

□

4. The remaining case

Proof of Theorem 1.3, part (1), when $\lim_{\varepsilon_1,\varepsilon_2\to 0} \frac{\varepsilon_1}{\varepsilon_2} = -\frac{z_1}{z_2}$.

We shall reuse the notation $\mu = z_1/z_2$, and reduce ourselves to the case $|\mu| \leq 1$. The hypothesis implies that $\varepsilon_2 = -(\mu - \gamma)\varepsilon_1$, where $\gamma = \gamma(\varepsilon)$ tends to 0 as ε tends to 0.

This time, instead of constructing an explicit map satisfying an approximate version of our interpolation problem, we will provide values of ζ_0, ζ_1 and ζ_2 such that the conditions (3.7) and (3.8) are satisfied. From now on we take $\zeta_0 = 1/2$, $\zeta_1 = \zeta_0 + \xi$, $\zeta_2 = \zeta_1 + \xi'$, where

$$\xi = C_1 z_1, \quad \xi' = \frac{\varepsilon_1}{z_1}\frac{\zeta_0}{\zeta_1}\frac{1 - |\zeta_1|^2}{1 - \zeta_0\bar{\zeta}_1}\xi,$$

and $C_1 = 40$.

Standing assumptions.

Throughout, we will assume that $|z_1|$ is small enough so that $|\xi| \leq 1/4$, so that $1/4 \leq \mathrm{Re}\zeta_1 \leq |\zeta_1| \leq 3/4$. This implies that $1/2 \leq |1 - \zeta_0\bar{\zeta}_1| \leq 1$, and therefore

$$|\xi'| \leq 4|\varepsilon_1|\frac{|\xi|}{|z_1|} \leq \frac{1}{2}|\xi|, \tag{4.1}$$

for ε small enough (depending on z_1). Also note that $|\xi'| \leq |\varepsilon_1|/10$. The estimate (4.1) implies that $|\xi + \xi'| \geq \frac{1}{2}|\xi|$, and also that

$$\frac{1}{2} \leq |1 - \zeta_0\bar{\zeta}_2| \leq 1, \quad \frac{1}{4}|1 - \zeta_2\bar{\zeta}_1| \leq 1.$$

Now we compute the invariant distances between the ζ_j's. We have

$$\begin{aligned} (4.2) \quad \phi_{\zeta_1}(\zeta_0) = \frac{\xi}{1 - \zeta_0\bar{\zeta}_1}, \quad & \phi_{\zeta_2}(\zeta_0) = \frac{\xi + \xi'}{1 - \zeta_0\bar{\zeta}_2}, \\ & \phi_{\zeta_2}(\zeta_1) = \frac{\xi'}{1 - \zeta_1\bar{\zeta}_2}, \quad \phi_{\zeta_1}(\zeta_2) = \frac{-\xi'}{1 - \zeta_2\bar{\zeta}_1}. \end{aligned}$$

This implies in particular that

$$(4.3) \qquad |\phi_{\zeta_1}(\zeta_0)| \geq |\xi| = C_1|z_1|, |\phi_{\zeta_2}(\zeta_0)| \geq |\xi'| \geq \frac{1}{2}|\xi| = \frac{C_1}{2}|z_1|.$$

Then

$$\begin{aligned} &\log|\zeta_0| + \log|\phi_{\zeta_0}(\zeta_1)| + \log|\phi_{\zeta_0}(\zeta_2)| \\ &\quad = -\log 2 + \log 2|\xi| + \log 4|\xi + \xi'| \leq 2\log|z_1| + C = 2\log|z| + C. \end{aligned}$$

To prove that the above choice of ζ_j's allows for a map passing through the required points, we will show that, for small enough $|\varepsilon|$, the conditions (3.7) and (3.8) are satisfied. To see this, we need to compute the quantities w_j, using (4.2).

$$\begin{aligned}
(4.4) \quad w_1 &= \frac{\varepsilon_1}{\zeta_1\phi_{\zeta_2}(\zeta_1)} = \frac{\varepsilon_1}{\zeta_1}\frac{1 - \zeta_1\bar{\zeta}_2}{\xi'} = \frac{z_1}{\zeta_0}\frac{1 - \zeta_1\bar{\zeta}_2}{1 - |\zeta_1|^2}\frac{1 - \zeta_0\bar{\zeta}_1}{\xi}, \\
(4.5) \quad w_2 &= \frac{z_1}{\zeta_0\phi_{\zeta_2}(\zeta_0)} = \frac{z_1}{\zeta_0}\frac{1 - \zeta_0\bar{\zeta}_2}{\xi + \xi'}, \\
(4.6) \quad w_3 &= \frac{z_2}{\zeta_0\phi_{\zeta_1}(\zeta_0)} = \mu\frac{z_1}{\zeta_0}\frac{1 - \zeta_0\bar{\zeta}_1}{\xi}, \\
(4.7) \quad w_4 &= \frac{\varepsilon_2}{\zeta_2\phi_{\zeta_1}(\zeta_2)} = \frac{(\mu - \gamma)\varepsilon_1}{\zeta_2}\frac{1 - \zeta_2\bar{\zeta}_1}{\xi'} \\
&= (\mu - \gamma)\frac{z_1}{\zeta_0}\frac{\zeta_1}{\zeta_2}\frac{1 - \zeta_2\bar{\zeta}_1}{1 - |\zeta_1|^2}\frac{1 - \zeta_0\bar{\zeta}_1}{\xi}.
\end{aligned}$$

We need to see that all those w_j lie in the unit disk. It follows from the Standing assumptions above that

$$|w_1| \le \left|\frac{z_1}{\xi}\right| \frac{1}{(1-|\zeta_1|^2)\zeta_0} \le \frac{5}{C_1} = \frac{1}{8}, \tag{4.8}$$

$$|w_2| \le \left|\frac{z_1}{\xi}\right| \frac{2}{\zeta_0} \le \frac{4}{C_1} = \frac{1}{10}, \tag{4.9}$$

$$|w_3| \le \left|\frac{z_1}{\xi}\right| \frac{1}{\zeta_0} \le \frac{2}{C_1} = \frac{1}{20}, \tag{4.10}$$

$$|w_4| \le |\mu-\gamma| \left|\frac{z_1}{\xi}\right| \frac{2}{(1-|\zeta_1|^2)\zeta_0} \le \frac{20}{C_1} = \frac{1}{2}, \tag{4.11}$$

where the last inequality holds when we assume ε small enough so that $|\gamma| \le 1$. It also follows from the above estimates that $|1-w_1\bar{w}_2| \ge 1/2$, $|1-w_3\bar{w}_4| \ge 1/2$, so that $d_G(w_1,w_2) \le 2|w_1-w_2|$ and $d_G(w_3,w_4) \le 2|w_3-w_4|$. We proceed to the computation of those Euclidean distances, which we will compare to the estimates (4.3).

$$w_2 - w_1 = \frac{z_1}{\zeta_0}\left[(1-\zeta_0\bar{\zeta}_2)\left(\frac{1}{\xi+\xi'} - \frac{1}{\xi}\right) + \right.$$
$$\left. \frac{1}{\xi}\left((1-\zeta_0\bar{\zeta}_2)-(1-\zeta_0\bar{\zeta}_1)\right) + \frac{1-\zeta_0\bar{\zeta}_1}{\xi}\left(1-\frac{1-\zeta_1\bar{\zeta}_2}{1-|\zeta_1|^2}\right)\right].$$

Each term is estimated by

$$\left|\frac{1}{\xi+\xi'} - \frac{1}{\xi}\right| = \left|\frac{\xi'}{\xi(\xi+\xi')}\right| \le \frac{8|\varepsilon_1|}{C_1|z_1|^2},$$
$$\left|(1-\zeta_0\bar{\zeta}_2)-(1-\zeta_0\bar{\zeta}_1)\right| = |\xi'\zeta_0| \le 2|\varepsilon_1|C_1,$$
$$\left|1-\frac{1-\zeta_1\bar{\zeta}_2}{1-|\zeta_1|^2}\right| = \frac{|\xi'\zeta_1|}{1-|\zeta_1|^2} \le 2|\xi'| \le 8|\varepsilon_1|C_1,$$

so $|w_2-w_1| \le 40|\varepsilon_1|C_1/|z_1|$.

Similarly,

$$w_3 - w_4 = \mu\frac{z_1}{\zeta_0}\frac{1-\zeta_0\bar{\zeta}_1}{\xi}\left[1-\frac{\zeta_1}{\zeta_2}+\frac{\zeta_1}{\zeta_2}\left(1-\frac{1-\zeta_1\bar{\zeta}_2}{1-|\zeta_1|^2}\right)\right]$$
$$+\gamma\frac{z_1}{\zeta_0}\frac{\zeta_1}{\zeta_2}\frac{1-\zeta_1\bar{\zeta}_2}{1-|\zeta_1|^2}\frac{1-\zeta_0\bar{\zeta}_1}{\xi}.$$

Using the fact that $|1-\frac{\zeta_1}{\zeta_2}| \le 8|\xi'|$, we find

$$|w_3-w_4| \le 96|\varepsilon_1| + 12|\gamma|/C_1.$$

It is then easy to see, using (4.3), that (3.7) and (3.8) can be verified whenever we choose ε_1 small enough (and therefore $|\gamma|$ small enough), depending on $|z_1|$. $\square$

References

1. M. Carlehed, J. Wiegerinck, *Le cône des fonctions plurisousharmoniques négatives et une conjecture de Coman,* Ann. Pol. Math. **80** (2003), 93–108.
2. D. Coman, *The pluricomplex Green function with two poles of the unit ball of* $\mathbb{C}^n$*,* Pacific J. Math. **194,** no 2, 257–283 (2000).
3. A. Edigarian, *Remarks on the pluricomplex Green function,* Univ. Iagel. Acta Math. **37** (1999), 159–164.
4. J. B. Garnett, *Bounded Analytic Functions,* Academic Press, Inc. New York-London, 1981.
5. L. Lempert, *La métrique de Kobayashi et la représentation des domains sur la boule,* Bull. Soc. Soc. Math. France **109**, 427-474 (1981).
6. N. Nikolov, W. Zwonek, Some remarks on the Lempert function, preprint, 12 pp. (2004).
7. P. J. Thomas, N. V. Trao, *Pluricomplex Green and Lempert functions for equally weighted poles.* Ark. Mat. 41, no. 2, 381–400 (2003).

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