

ON THE LOCUS OF PRYM CURVES WHERE THE PRYM-CANONICAL MAP IS NOT AN EMBEDDING

CIRO CILIBERTO, THOMAS DEDIEU, CONCETTINA GALATI,
AND ANDREAS LEOPOLD KNUTSEN

ABSTRACT. We prove that the locus of Prym curves (C, η) of genus $g \geq 5$ for which the Prym-canonical system $|\omega_C(\eta)|$ is base point free but the Prym-canonical map is not an embedding is irreducible and unirational of dimension $2g + 1$.

1. INTRODUCTION

Let $g \geq 2$ and \mathcal{R}_g be the moduli space of *Prym curves*, that is, of pairs (C, η) , with C a smooth complex projective genus g curve and η a non-zero 2-torsion point of $\text{Pic}^0(C)$. It is well-known that \mathcal{R}_g is irreducible of dimension $3g - 3$ and that the natural forgetful map $\mathcal{R}_g \rightarrow \mathcal{M}_g$, where \mathcal{M}_g denotes the moduli space of smooth genus g curves, is finite of degree $2^{2g} - 1$. The complete linear system $|\omega_C(\eta)|$ is of dimension $g - 2$ and it is base point free unless C is hyperelliptic and $\eta \simeq \mathcal{O}_C(p - q)$, with p and q ramification points of the g_2^1 (cf. Lemma 2.1 below).

In this note we study the locally closed locus \mathcal{R}_g^0 in \mathcal{R}_g of Prym curves (C, η) such that the *Prym-canonical system* $|\omega_C(\eta)|$ is base point free but the morphism $C \rightarrow \mathbb{P}^{g-2}$ it defines (the so-called *Prym-canonical map*) is not an embedding. Note that \mathcal{R}_g^0 is clearly dense in \mathcal{R}_g for $g \leq 4$. Our main result is the following:

Theorem 1.1. *Let $g \geq 5$. The locus \mathcal{R}_g^0 is irreducible and unirational of dimension $2g + 1$ and lies in the tetragonal locus.*

By the tetragonal locus $\mathcal{R}_{g,4}^1$ in \mathcal{R}_g we mean the inverse image via $\mathcal{R}_g \rightarrow \mathcal{M}_g$ of the tetragonal locus $\mathcal{M}_{g,4}^1$ of \mathcal{M}_g .

We also show:

Proposition 1.2. *For general $(C, \eta) \in \mathcal{R}_g^0$, $g \geq 5$, the Prym-canonical map is birational onto its image, and its image has precisely two nodes.*

Although we believe that these results are of independent interest, our main motivation for studying the locus \mathcal{R}_g^0 is that it naturally contains pairs (C, η) where C is a smooth curve lying on an Enriques surface S such that

$$\phi(C) = \min\{E \cdot C \mid E \in \text{Pic}(S), E > 0, E^2 = 0\} = 2,$$

and $\eta = \mathcal{O}_C(K_S)$, cf. Examples 5.1 and 5.2 and Remark 5.5, in which case the Prym-canonical map associated to η is the restriction to C of the map defined by the complete linear system $|C|$ on S . The locus \mathcal{R}_g^0 indeed naturally shows up in our recent work [6] concerning the moduli of smooth curves lying on an Enriques surface, in which we use the results in this note. Besides, we show in [6] that \mathcal{R}_g^0 is dominated by curves on Enriques surfaces for $5 \leq g \leq 8$.

The paper is organized as follows. Section 2 is devoted to recalling some preliminary results. The irreducibility and unirationality of \mathcal{R}_g^0 is proved in §3, whereas its dimension is computed in §4. We conclude with the proof of Proposition 1.2 together with the mentioned examples on Enriques surfaces.

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2. PRELIMINARY RESULTS

2.1. A basic lemma on Prym curves. The following is an immediate consequence of the Riemann-Roch theorem (see also [8, §0.1] or [12, Pf. of Lemma 2.1]). We include the proof for the reader's convenience.

Lemma 2.1. *Let (C, η) be any Prym curve of genus $g \geq 3$. Then:*

(i) *p is a base point of $|\omega_C(\eta)|$ if and only if $|p + \eta| \neq \emptyset$. This happens if and only if C is hyperelliptic and $\eta \sim \mathcal{O}_C(p - q)$, with p and q ramification points of the g_2^1 . In particular, p and q are the only base points;*

(ii) *if $|\omega_C(\eta)|$ is base point free, then it does not separate p and q (possibly infinitely near) if and only if $|p + q + \eta| \neq \emptyset$. This happens if and only if C has a g_4^1 and $\eta \sim \mathcal{O}_C(p + q - x - y)$, where $2(p + q)$ and $2(x + y)$ are members of the g_4^1 . In particular, also x and y are not separated by $|\omega_C(\eta)|$.*

Proof. We prove only (ii) and leave (i) to the reader. Assume that $|\omega_C(\eta)|$ is base point free. Then p and q are not separated by the linear system $|\omega_C(\eta)|$ if and only if $h^0(\omega_C(\eta) - p) = h^0(\omega_C(\eta) - p - q)$. By Riemann-Roch and Serre duality, this is equivalent to $h^0(\eta + p) + 1 = h^0(\eta + p + q)$. By (i), we have $h^0(\eta + p) = 0$, whence the latter condition is $h^0(\eta + p + q) = 1$. This is equivalent to $h^0(\eta + p + q) > 0$, because if $h^0(\eta + p + q) > 1$, then we would have $h^0(\eta + p) > 0$, a contradiction. This proves the first assertion.

We have $|p + q + \eta| \neq \emptyset$ if and only if $p + q + \eta \sim x + y$, for $x, y \in C$. This implies $2(p + q) \sim 2(x + y)$, whence C has a g_4^1 with $2(p + q)$ and $2(x + y)$ as its members. Conversely, if $2(p + q)$ and $2(x + y)$ are distinct members of a g_4^1 on C , then $\eta := \mathcal{O}_C(p + q - x - y)$ is a 2-torsion element of $\text{Pic}^0(C)$ and satisfies the condition that $|p + q + \eta| \neq \emptyset$. \square

The lemma says in particular that the locus in \mathcal{R}_g of pairs (C, η) for which the Prym-canonical system $|\omega_C(\eta)|$ is not base-point free dominates the hyperelliptic locus via the forgetful map $\mathcal{R}_g \rightarrow \mathcal{M}_g$.

Recall that the tetragonal locus $\mathcal{R}_{g,4}^1$ is irreducible of dimension $2g + 3$ if $g \geq 7$ and coincides with \mathcal{R}_g if $g \leq 6$. Lemma 2.1 implies that $\mathcal{R}_g^0 \subseteq \mathcal{R}_{g,4}^1$, thus proving the last statement in Theorem 1.1.

The lemma also enables us to detect the locus $\mathcal{R}_g^{0,\text{nb}}$ in \mathcal{R}_g^0 where the Prym-canonical morphism is not birational onto its image:

Corollary 2.2. *Let (C, η) be any Prym curve of genus $g \geq 4$ such that the Prym-canonical system $|\omega_C(\eta)|$ is base point free. If the Prym-canonical map is not birational onto its image, then it is of degree two onto a smooth elliptic curve.*

The locus $\mathcal{R}_g^{0,\text{nb}}$ is irreducible of dimension $2g - 2$ and dominates the bielliptic locus in \mathcal{M}_g . More precisely, $\mathcal{R}_g^{0,\text{nb}}$ consists of pairs (C, η) , with C bielliptic and $\eta := \varphi^\eta'$, where $\varphi : C \rightarrow E$ is a bielliptic map and η' is a nontrivial 2-torsion element in $\text{Pic}^0(E)$.*

Proof. Let (C, η) be as in the statement. Denote by C' the image of the Prym-canonical morphism $\varphi : C \rightarrow \mathbb{P}^{g-2}$. Let μ be the degree of φ and d the degree of C' . Then $d\mu = 2g - 2$ and, since C' is non-degenerate in \mathbb{P}^{g-2} , we must have $d \geq g - 2$. Since $g \geq 4$, then $2 \leq \mu \leq 3$; moreover $\mu = 3$ implies that $g = 4$ and φ maps C three-to-one to a conic. The latter case cannot happen: indeed, we would have $\omega_C(\eta) = 2\mathcal{L}$, where $|\mathcal{L}|$ is a g_3^1 . Then $4\mathcal{L} = 2\omega_C$. Since $|2\omega_C|$ is cut out by quadrics on the canonical image of C in \mathbb{P}^3 , it follows that the only quadric containing the canonical model is a cone. Then $|\mathcal{L}|$ is the unique g_3^1 on C and $2\mathcal{L} = \omega_C$, thus η is trivial, a contradiction.

Hence $\mu = 2$, and then $d = g - 1$, so that C' is a curve of almost minimal degree. It is easy to see, using the fact that $|\omega_C(\eta)|$ is complete, that C' is a smooth elliptic curve (alternatively, apply [5, Thm. 1.2]). Hence C is bielliptic and any pair of points p and q identified by φ satisfy $p+q \sim \varphi^*(r)$ for a point $r \in C'$. Thus $2p+2q \sim \varphi^*(2r)$ is a g_4^1 . By Lemma 2.1(ii) we have $\eta \sim \mathcal{O}_C(p+q-x-y)$, where also $\varphi(x) = \varphi(y)$, whence $x+y \sim \varphi^*(z)$, for a $z \in C'$. Hence, again by Lemma 2.1(ii), we have $\eta \sim p+q-x-y \sim \varphi^*(r-z)$ and $r-z$ is a nontrivial 2-torsion element in $\text{Pic}^0(C')$, because $\varphi^* : \text{Pic}^0(C') \rightarrow \text{Pic}^0(C)$ is injective.

Conversely, if C is a bielliptic curve, it admits at most finitely many double covers $\varphi : C \rightarrow E$ onto an elliptic curve (cf. e.g., [2]; in fact, for $g \geq 6$, it admits a unique such map), and for any such φ and any nontrivial 2-torsion element η' in $\text{Pic}^0(E)$, we have $\eta' \sim r-z$, for $r, z \in E$. Letting $\varphi^*(r) = p+q$ and $\varphi^*(z) = x+y$, we see that $2(p+q) \sim 2(x+y)$ and $\eta = \varphi^*\eta'$ satisfies the conditions of Lemma 2.1(ii).

We have therefore proved that $\mathcal{R}_g^{0,\text{nb}}$ consists of pairs (C, η) , with C bielliptic and $\eta := \varphi^*\eta'$, where $\varphi : C \rightarrow E$ is a bielliptic map and η' is a nontrivial 2-torsion element in $\text{Pic}^0(E)$.

The statement about the dimension of $\mathcal{R}_g^{0,\text{nb}}$ follows since the bielliptic locus has dimension $2g-2$. To prove its irreducibility, consider the map $f : \mathcal{R}_g^{0,\text{nb}} \rightarrow \mathcal{R}_1$ associating to (C, η) the pair (E, η') as above. We study the fibres of this map. Consider the following obvious cartesian diagram defining \mathcal{H} , where $U \subset \text{Sym}^{2g-2}(E)$ is the open subset consisting of reduced divisors:

$$\begin{array}{ccc} \mathcal{H} & \longrightarrow & \text{Pic}^{g-1}(E) \\ \downarrow & & \downarrow \otimes 2 \\ U & \longrightarrow & \text{Pic}^{2g-2}(E) \end{array}$$

By Riemann's existence theorem, $\mathcal{H}/\text{Aut}(E)$ is in one-to-one correspondence with the two-to-one covers of E branched at $2g-2$ points. Then the fibre of f over (E, η') is isomorphic to $\mathcal{H}/\text{Aut}(E)$ by what we said above. Now note that \mathcal{H} is irreducible, since it fibres over (an open subset of) $\text{Pic}^{g-1}(E)$ with fibres that are projective spaces of dimension $2g-3$. Hence also $\mathcal{H}/\text{Aut}(E)$ is irreducible.

The irreducibility of $\mathcal{R}_g^{0,\text{nb}}$ now follows from the irreducibility of \mathcal{R}_1 . Actually \mathcal{R}_1 is irreducible and rational. To see this consider the irreducible family of elliptic curves $y^2 = x(x-1)(x-\lambda)$, where $\lambda \in \mathbb{C} \setminus \{0, 1\}$. The three non-trivial points of order two of the fibre \mathcal{C}_λ over λ may be identified with the points $(0, 0)$, $(1, 0)$ and $(\lambda, 0)$. Moreover, the j -invariant of the fibres defines a six-to-one map $j : \mathbb{C} \setminus \{0, 1\} \rightarrow \mathcal{M}_1$. Now consider on this family the two sections defined by the points $(0, 0)$, $(1, 0)$ which stay fixed as λ varies. It is an exercise to prove that the irreducible family of two-marked elliptic curves we obtain in this way is isomorphic to the moduli space of pairs $(C, (\eta_1, \eta_2))$ where C is a smooth elliptic curve and (η_1, η_2) is an ordered pair of distinct non-trivial 2-torsion points of $\text{Pic}^0(C)$. This moduli space is, in turn, isomorphic to the moduli space $\mathcal{M}_1^{(2)}$ of elliptic curves with a level 2 structure [9, Ex. 2.2.1]. Finally $\mathcal{M}_1^{(2)} \simeq \mathbb{C} \setminus \{0, 1\}$ maps two-to-one dominantly to \mathcal{R}_1 , via the map $(C, (\eta_1, \eta_2)) \mapsto (C, (\eta_1 + \eta_2))$. This proves the statement. \square

2.2. A result on linear systems on rational surfaces. We will need the following:

Theorem 2.3 (cf. [1, Cor. (4.6)]). *Let X be a smooth projective rational surface and δ a non-negative integer. Let \mathcal{L} be a complete linear system on X such that:*

- (i) *the general curve in \mathcal{L} is smooth and irreducible;*
- (ii) *the genus $p_a(\mathcal{L})$ of the general curve in \mathcal{L} satisfies $p_a(\mathcal{L}) \geq \delta$;*
- (iii) *$\dim(\mathcal{L}) > 3\delta$;*
- (iv) *if p_1, \dots, p_δ are general points of X , there is an element C of \mathcal{L} singular at p_1, \dots, p_δ such that for each irreducible component C' of C one has $K_X \cdot C' < 0$.*

Then, if p_1, \dots, p_δ are general points of X and $\mathcal{L}(p_1^2, \dots, p_\delta^2)$ is the subsystem of \mathcal{L} formed by the curves singular at p_1, \dots, p_δ , one has:

- (a) *the general curve in $\mathcal{L}(p_1^2, \dots, p_\delta^2)$ is irreducible, has nodes at p_1, \dots, p_δ and no other singularity;*
- (b) *$\dim(\mathcal{L}(p_1^2, \dots, p_\delta^2)) = \dim(\mathcal{L}) - 3\delta$.*

Proof. The proof of (a) is in [1]. As for (b), one has $\dim(\mathcal{L}(p_1^2, \dots, p_\delta^2)) = \dim(\mathcal{L}) - 3\delta + \varepsilon$, with $\varepsilon \geq 0$. Consider the locally closed family of curves in \mathcal{L} given by

$$\mathcal{F} := \bigcup_{p_1, \dots, p_\delta} \mathcal{L}(p_1^2, \dots, p_\delta^2),$$

where the union is made by varying p_1, \dots, p_δ among all the δ -tuples of sufficiently general points of X . Of course

$$\dim(\mathcal{F}) = 2\delta + \dim(\mathcal{L}(p_1^2, \dots, p_\delta^2)) = \dim(\mathcal{L}) - \delta + \varepsilon.$$

On the other hand, if C is a general element in \mathcal{F} , it has nodes at p_1, \dots, p_δ and no other singularity by (a), hence the Zariski tangent space to \mathcal{F} at C is the linear system $\mathcal{L}(p_1, \dots, p_\delta)$ of curves in \mathcal{L} containing p_1, \dots, p_δ . Since p_1, \dots, p_δ are general, we have $\dim(\mathcal{L}(p_1, \dots, p_\delta)) = \dim(\mathcal{L}) - \delta$, which proves that $\varepsilon = 0$. \square

3. IRREDUCIBILITY AND UNIRATIONALITY OF \mathcal{R}_g^0

In this section we prove a first part of Theorem 1.1, namely:

Proposition 3.1. *The locus \mathcal{R}_g^0 is irreducible and unirational for $g \geq 5$.*

The proof is inspired by the arguments in [1] and requires some preliminary considerations. In [1, Theorem (5.3)] the authors prove that some Hurwitz schemes $\mathcal{H}_{g,d}$ are unirational. Here we focus on the case $d = 4$ and recall their construction.

Fix $g = 2h + \epsilon \geq 3$, with $0 \leq \epsilon \leq 1$. Then set $n = h + 3 + \epsilon$ and

$$\delta = \binom{n-1}{2} - \binom{n-4}{2} - g = h + 2\epsilon.$$

Fix now p, p_1, \dots, p_δ general points in the projective plane and consider the linear system $\mathcal{L}_n(p^{n-4}, p_1^2, \dots, p_\delta^2)$ of plane curves of degree n having multiplicity at least $n-4$ at p and multiplicity at least 2 at p_1, \dots, p_δ . As an application of Theorem 2.3, in [1, Cor. (4.7)] one proves that the dimension of $\mathcal{L}_n(p^{n-4}, p_1^2, \dots, p_\delta^2)$ is the expected one, i.e.,

$$\dim(\mathcal{L}_n(p^{n-4}, p_1^2, \dots, p_\delta^2)) = \frac{n(n+3)}{2} - \frac{(n-4)(n-3)}{2} - 3\delta = 2h + 9 - \epsilon,$$

and the general curve Γ in $\mathcal{L}_n(p^{n-4}, p_1^2, \dots, p_\delta^2)$ is irreducible, has an ordinary $(n-4)$ -tuple point at p , nodes at p_1, \dots, p_δ , and no other singularity. The normalization C of Γ has genus g and it has a g_4^1 , which is the pull-back to C of the linear series cut out on Γ by the pencil of lines through p .

Consider then the locally closed family of curves

$$\mathcal{H} := \bigcup_{p_1, \dots, p_\delta} \mathcal{L}_n(p^{n-4}, p_1^2, \dots, p_\delta^2),$$

where the union is made by varying p_1, \dots, p_δ among all the δ -tuples of sufficiently general points of the plane. Then \mathcal{H} is clearly irreducible, rational, of dimension $\dim(\mathcal{L}_n(p^{n-4}, p_1^2, \dots, p_\delta^2)) + 2\delta = 4h + 9 + 3\epsilon$, and in [1] it is proved that the natural map $\mathcal{H} \dashrightarrow \mathcal{M}_{g,4}^1$ is dominant, so that $\mathcal{M}_{g,4}^1$ is unirational.

Proof of Proposition 3.1. To prove our result, we slightly modify the above argument from [1]. Let us fix $g \geq 5, n, \delta$ as above. Let p, p_1, \dots, p_δ be general points in the plane.

Claim 3.2. *Consider the linear system $\mathcal{L}_{n-2}(p^{n-6}, p_1^2, \dots, p_\delta^2)$ of plane curves of degree $n-2$, having a point of multiplicity at least $n-6$ at p , and singular at p_1, \dots, p_δ . Then the dimension of $\mathcal{L}_{n-2}(p^{n-6}, p_1^2, \dots, p_\delta^2)$ is the expected one, i.e.,*

$$\dim(\mathcal{L}_{n-2}(p^{n-6}, p_1^2, \dots, p_\delta^2)) = \frac{(n-2)(n+1)}{2} - \frac{(n-6)(n-5)}{2} - 3\delta = 2h - 1 - \epsilon.$$

Proof of Claim A.1. Assume first $g = 5$, which implies $(h, \epsilon, n, \delta) = (2, 1, 6, 4)$. Then one has $\mathcal{L}_{n-2}(p^{n-6}, p_1^2, \dots, p_\delta^2) = \mathcal{L}_4(p_1^2, \dots, p_4^2)$, which consists of all pairs of conics through p_1, \dots, p_4 , and has dimension 2 as desired. We can assume next that $g \geq 6$, hence $h \geq 3$ and $n \geq 6$.

Let X be the blow-up of \mathbb{P}^2 at p . Note that the anticanonical system of X is very ample. Consider the linear system \mathcal{L} proper transform on X of $\mathcal{L}_{n-2}(p^{n-6})$. One checks that X and \mathcal{L} verify the hypotheses (i)–(iv) of Theorem 2.3. Indeed, (i) and (iv) are immediate, whereas (ii) and (iii) follow by standard computations and the fact that $h \geq 3$. Then the assertion follows by Theorem 2.3(b). \square

Next fix two distinct lines r_1, r_2 through p and, for $1 \leq i \leq 2$, two distinct points q_{ij} , both different from p , on the line r_i , with $1 \leq j \leq 2$. Consider then the linear system $\mathcal{L}_n(p^{n-4}, p_1^2, \dots, p_\delta^2; [q_{11}, q_{12}, q_{21}, q_{22}])$ consisting of all curves in $\mathcal{L}_n(p^{n-4}, p_1^2, \dots, p_\delta^2)$ whose intersection multiplicity with r_i at q_{ij} is at least 2, for $1 \leq i, j \leq 2$.

Claim 3.3. *The linear system $\mathcal{L}_n(p^{n-4}, p_1^2, \dots, p_\delta^2; [q_{11}, q_{12}, q_{21}, q_{22}])$ has the expected dimension, i.e.,*

$$\begin{aligned} \dim(\mathcal{L}_n(p^{n-4}, p_1^2, \dots, p_\delta^2; [q_{11}, q_{12}, q_{21}, q_{22}])) &= \frac{n(n+3)}{2} - \frac{(n-4)(n-3)}{2} - 3\delta - 8 \\ &= 2h + 1 - \epsilon, \end{aligned}$$

and the general curve in $\mathcal{L}_n(p^{n-4}, p_1^2, \dots, p_\delta^2; [q_{11}, q_{12}, q_{21}, q_{22}])$ is irreducible, has a point of multiplicity $n-4$ at p , has nodes at p_1, \dots, p_δ and no other singularity, and is tangent at r_i in q_{ij} , for $1 \leq i, j \leq 2$.

Proof of Claim A.2. Let X be the blow-up of the plane at p , at the points $q_{i,j}$ and at the infinitely near points to q_{ij} along the line r_i , for $1 \leq i, j \leq 2$. Note that the anticanonical system of X has a fixed part consisting of the strict transforms R_1, R_2 of r_1, r_2 plus the exceptional divisor E over p , and a movable part consisting of the pull back to X of the linear system of the lines in the plane.

Let \mathcal{L} be the strict transform on X of $\mathcal{L}_n(p^{n-4}; [q_{11}, q_{12}, q_{21}, q_{22}])$, the linear system of curves of degree n with multiplicity at least $n-4$ at p and whose intersection multiplicity with r_i at q_{ij} is at least 2, for $1 \leq i, j \leq 2$. One has

$$\dim(\mathcal{L}) = \frac{n(n+3)}{2} - \frac{(n-4)(n-3)}{2} - 8$$

and an application of Bertini's theorem shows that the general curve in \mathcal{L} is smooth and irreducible and its genus is

$$p_a(\mathcal{L}) = \binom{n-1}{2} - \binom{n-4}{2} \geq \delta.$$

Moreover

$$\dim(\mathcal{L}) - 3\delta = 2h + 1 - \epsilon > 0$$

Hence the linear system $\mathcal{L}(p_1^2, \dots, p_\delta^2)$ of curves in \mathcal{L} singular at p_1, \dots, p_δ has dimension

$$\dim(\mathcal{L}(p_1^2, \dots, p_\delta^2)) \geq 2h + 1 - \epsilon.$$

We claim that $\mathcal{L}(p_1^2, \dots, p_\delta^2)$ does not have R_1, R_2 or E in its fixed locus. Indeed, if E is in this fixed locus, then clearly also R_1 and R_2 split off $\mathcal{L}(p_1^2, \dots, p_\delta^2)$. If R_1 is in the fixed locus, then by symmetry, also R_2 is in the fixed locus. So, suppose by contradiction that R_1, R_2 are in the fixed locus. Then, after removing them from $\mathcal{L}(p_1^2, \dots, p_\delta^2)$ we would remain with \mathcal{L}' , the pull-back to X of $\mathcal{L}_{n-2}(p^{n-6}, p_1^2, \dots, p_\delta^2)$, which, by Claim A.1, has dimension $2h - 1 - \epsilon$. Hence we would have

$$2h - 1 - \epsilon = \dim(\mathcal{L}_{n-2}(p^{n-6}, p_1^2, \dots, p_\delta^2)) = \dim(\mathcal{L}(p_1^2, \dots, p_\delta^2)) \geq 2h + 1 - \epsilon,$$

a contradiction.

Let now C be a general curve in $\mathcal{L}(p_1^2, \dots, p_\delta^2)$. The above argument implies that no component of C is a fixed curve of the anticanonical system of X . Then for any irreducible component C' of C one has $K_X \cdot C' < 0$. In conclusion, \mathcal{L} verifies the hypotheses (i)–(iv) of Theorem 2.3, and Claim A.2 follows by the latter theorem. \square

We now end the proof of Proposition 3.1. Consider the locally closed family of curves

$$\mathcal{G} := \bigcup_{p_1, \dots, p_\delta, r_1, r_2, q_{11}, q_{12}, q_{21}, q_{22}} \mathcal{L}_n(p^{n-4}, p_1^2, \dots, p_\delta^2; [q_{11}, q_{12}, q_{21}, q_{22}])$$

where the union is made by varying p_1, \dots, p_δ among all δ -tuples of general distinct points of X , r_1, r_2 among all pairs of distinct lines through p and $q_{ij} \neq p$ among all pairs of distinct points of r_i , for $1 \leq i, j \leq 2$.

Of course \mathcal{G} is irreducible and rational, and we have a map $\alpha : \mathcal{G} \dashrightarrow \mathcal{R}_g^0$ which sends a general curve $\Gamma \in \mathcal{G}$ to (C, η) , where C is the normalization of Γ , and $\eta = \mathcal{O}_C(q_{11} + q_{12} - q_{21} - q_{22})$, where, by abusing notation, we denote by q_{ij} their inverse images in C , for $1 \leq i, j \leq 2$. Since $\mathcal{H} \dashrightarrow \mathcal{M}_{g,4}^1$ is dominant by [1, §5], then α is also dominant by Lemma 2.1.¹ This proves the proposition. \square

4. DIMENSION OF \mathcal{R}_g^0

In this section we finish the proof of Theorem 1.1 with the:

Proposition 4.1. *The irreducible locus \mathcal{R}_g^0 has dimension $2g + 1$ if $g \geq 5$.*

Proof. Let $\mathcal{H}_{g,4}$ denote the Hurwitz scheme parametrizing isomorphism classes of genus g degree 4 covers of \mathbb{P}^1 . We have a commutative diagram

$$\begin{array}{ccc} \mathcal{G} & \xrightarrow{\varphi} & \mathcal{H}_{g,4} \\ \alpha \downarrow & & \downarrow \pi \\ \mathcal{R}_g^0 & \xrightarrow{\psi} & \mathcal{M}_{g,4}^1 \end{array}$$

where π and ψ are the forgetful maps, α is the dominant map from the last part of the proof of Proposition 3.1 and φ maps a general curve $\Gamma \in \mathcal{G}$ to the degree 4 cover defined by $2(q_{11} + q_{12}) \sim 2(q_{21} + q_{22})$, using the notation of the proof of Proposition 3.1. Note that ψ is finite, whence the dimension of \mathcal{R}_g^0 equals the dimension of the image of $\pi \circ \varphi$.

The image of φ coincides with the locus $\mathcal{D} \subset \mathcal{H}_{g,4}$ parametrizing covers with two pairs of distinct ramification points each over the same branch point. By Riemann's existence theorem, \mathcal{D} has codimension 2 in $\mathcal{H}_{g,4}$ (whence $\dim(\mathcal{D}) = 2g + 1$). Since \mathcal{G} is irreducible (cf. the proof of Proposition 3.1), so is \mathcal{D} . Moreover, as the bielliptic locus in \mathcal{M}_g has dimension $2g - 2$ and each bielliptic curve has a one-dimensional family of g_4^1 s, the locus in $\mathcal{H}_{g,4}$ with bielliptic domain curve has dimension $2g - 1$. Thus, the general element in the image of $\pi \circ \varphi$ is not bielliptic, whence the general element (C, η) in the image of α has Prym-canonical image birational to C , by Corollary 2.2 (and necessarily singular, by Lemma 2.1(ii)). It follows that the fibre over C of the restriction of π to \mathcal{D} is finite. Indeed, C has finitely many preimages (C, η) in the image of α , and the Prym-canonical model of each of those has finitely many singular points, determining by Lemma 2.1(ii) only finitely many covers in $\mathcal{D} \subset \mathcal{H}_{g,4}$ mapping to C by π . Thus, the restriction of π to \mathcal{D} is generically finite, whence the image of $\pi \circ \varphi$ has dimension $2g + 1$. \square

5. PROOF OF PROPOSITION 1.2 AND SOME EXAMPLES

Consider again the locus $\mathcal{D} \subset \mathcal{H}_{g,4}$ from the proof of Proposition 4.1 parametrizing isomorphism classes of covers with two pairs of distinct ramification points each over a single branch point. By Riemann's existence theorem again, the general point in \mathcal{D} corresponds to a cover with only two such branch points. By Lemma 2.1(ii), if the

¹ In fact this statement requires additional arguments. We provide those in an Addendum (Section A).

domain curve has only one g_4^1 , which is automatic if $g \geq 10$, then the Prym–canonical model of such a curve has precisely two nodes. It cannot have fewer singularities by Lemma 2.1. Thus, Proposition 1.2 is proved for $g \geq 10$.

Instead of embarking in a more refined treatment for $g \leq 9$, we note that certain curves on Enriques surfaces provide examples, for any genus $g \geq 5$, of curves with two–nodal Prym–canonical models, thus finishing the proof of Proposition 1.2:

Example 5.1. The general Enriques surface S contains no smooth rational curves [3] and contains smooth elliptic curves E_1, E_2, E_3 with $E_i \cdot E_j = 1$ for $i \neq j$ (and $E_i^2 = 0$ by adjunction), for $1 \leq i, j \leq 3$, cf. e.g. [7, Thm. 3.2] or [8, IV.9.E, p. 273]. It also contains a smooth elliptic curve $E_{1,2}$ such that $E_{1,2} \cdot E_1 = E_{1,2} \cdot E_2 = 2$, and $E_{1,2} \cdot E_3 = 1$, cf. e.g. [7, Thm. 3.2] or [8, IV.9.B, p. 270]. In particular, none of the numerical equivalence classes of $E_1, E_2, E_3, E_{1,2}$ are divisible in $\text{Num}(S)$.

Consider, for any $g \geq 5$, the line bundle

$$H_g := \begin{cases} \mathcal{O}_S(\frac{g-2}{2}E_1 + E_2 + E_3), & g \text{ even} \\ \mathcal{O}_S(\frac{g-1}{2}E_1 + E_{1,2}), & g \text{ odd.} \end{cases}$$

The absence of smooth rational curves yields that H_g is nef. As $H_g^2 = 2g - 2$, all curves in $|H_g|$ have arithmetic genus g . Moreover, we claim that $\phi(H_g) = E_1 \cdot H_g = 2$ (see the introduction for the definition of ϕ) and that the only numerical class computing $\phi(H_g)$ is E_1 . Indeed, if g is even (respectively, odd), then $E_1 \cdot H_g = 2$, $E_2 \cdot H_g = E_3 \cdot H_g = \frac{g}{2} \geq 3$ (resp., $E_1 \cdot H_g = 2$, $E_{1,2} \cdot H_g = g - 1 \geq 4$), and if E is any nonzero effective divisor not numerically equivalent to any of E_1, E_2, E_3 (resp., $E_1, E_{1,2}$), then $E \cdot E_1 > 0$, $E \cdot E_2 > 0$ and $E \cdot E_3 > 0$ (resp., $E \cdot E_1 > 0$ and $E \cdot E_{1,2} > 0$) by [10, Lemma 2.1], so that $E \cdot H_g \geq \frac{g-2}{2} + 2 = \frac{g}{2} + 1 \geq 4$ (resp., $E \cdot H_g \geq \frac{g-1}{2} + 1 = \frac{g+1}{2} \geq 3$).

By [8, Prop. 4.5.1, Thm. 4.6.3, Prop. 4.7.1, Thm. 4.7.1] the complete linear system $|H_g|$ is therefore base point free and defines a morphism φ_{H_g} that is birational onto a surface with only double lines as singularities; the double lines are the images of curves computing $\phi(H_g)$, which, by what we said above, are E_1 and E'_1 , the only member of $|E_1 + K_S|$. Thus, the image of φ_{H_g} is a surface with precisely two double lines $\varphi_{H_g}(E_1)$ and $\varphi_{H_g}(E'_1)$ as singularities. Therefore, φ_{H_g} maps a general smooth $C \in |H|$ to a curve with precisely two nodes. Since φ_{H_g} restricted to C is the Prym–canonical map associated to $\eta := \mathcal{O}_C(K_S)$ by [8, Cor. 4.1.2], a general smooth curve C in $|H_g|$ together with η is an example of a Prym curve of any genus $g \geq 5$ with two–nodal Prym–canonical model.

We prove in [6, Thm. 2] that the general element in \mathcal{R}_g^0 is obtained in this way precisely for $5 \leq g \leq 8$.

Similar examples for odd $g \geq 7$ are obtained from the line bundle $H_g := \mathcal{O}_S(\frac{g-1}{2}E_1 + 2E_2)$ or $H_g := \mathcal{O}_S(\frac{g-1}{2}E_1 + 2E_2 + K_S)$, but (again by [6, Thm. 2]) the general element in \mathcal{R}_g^0 is not obtained in this way.

We conclude with an example of curves of genus 5 on an Enriques surface with 4-nodal Prym–canonical models and a result that will be used in [6]:

Example 5.2. With the same notation as in the previous example, set $H := \mathcal{O}_S(2E_1 + 2E_2 + K_S)$. Then $H^2 = 8$, so that any curve in $|H|$ has arithmetic genus 5. Moreover, $\phi(H) = 2$ and one easily checks that E_1 and E_2 are the only numerical equivalence classes computing $\phi(H)$. As in the previous example, the complete linear system $|H|$

is base point free and defines a morphism φ_H that is birational onto a surface with precisely four double lines as singularities, namely the images of E_1, E_2, E'_1 and E'_2 , where E'_i is the only member of $|E_i + K_S|$, $i = 1, 2$. Thus φ_H maps a general smooth $C \in |H|$ to a curve with precisely four nodes, so that, again by [8, Cor. 4.1.2], the pairs $(C, \mathcal{O}_C(K_S))$ are genus 5 Prym curves with 4-nodal Prym-canonical models.

Also note that for any smooth $C \in |H|$, we have

$$\omega_C \simeq \mathcal{O}_C(E_1 + E_2)^{\otimes 2} \simeq \mathcal{O}_C(E_1 + E_2 + K_S)^{\otimes 2},$$

whence C has two autoresidual g_4^1 s, namely $|\mathcal{O}_C(E_1 + E_2)|$ and $|\mathcal{O}_C(E_1 + E_2 + K_S)|$, and their difference is $\mathcal{O}_C(K_S)$. (A complete linear system $|D|$ is called *autoresidual* if D is a theta-characteristic, that is, $2D \sim \omega_D$.) Thus, $(C, \mathcal{O}_C(K_S))$ belongs to the locus in \mathcal{R}_5 consisting of Prym curves (C, η) carrying a theta-characteristic θ such that $h^0(\theta) = h^0(\theta + \eta) = 2$. The next result shows that this is a general phenomenon in \mathcal{R}_5^0 .

Proposition 5.3. *The locus in \mathcal{R}_5^0 of curves with 4-nodal Prym-canonical model is an irreducible unirational divisor whose closure in \mathcal{R}_5 coincides with the closure of the locus of Prym curves (C, η) carrying a theta-characteristic θ with $h^0(\theta) = h^0(\theta + \eta) = 2$.*

Proof. Let us denote by \mathcal{D}_5^0 the locus of curves in \mathcal{R}_5^0 with 4-nodal Prym-canonical model, which is nonempty by the previous example. Let \mathcal{V} denote the locus of curves of type $(4, 4)$ on $\mathbb{P}^1 \times \mathbb{P}^1$ with 4 nodes lying on the 4 nodes of a “square” configuration of two fibres of each projection to \mathbb{P}^1 . We will prove that \mathcal{V} is irreducible of dimension 16 and that there is a birational morphism

$$f : \mathcal{D}_5^0 \longrightarrow \mathcal{V}' := \mathcal{V} / \text{Aut}(\mathbb{P}^1 \times \mathbb{P}^1).$$

To define f , let $(C, \eta) \in \mathcal{D}_5^0$. By Lemma 2.1 there are four pairs of distinct points $(p, q), (x, y), (p', q')$ and (x', y') on C , each identified by the Prym-canonical map $\varphi : C \rightarrow \mathbb{P}^3$, such that

$$\begin{aligned} (1) \quad & 2(p + q) \sim 2(x + y), \quad 2(p' + q') \sim 2(x' + y') \quad \text{and} \\ (2) \quad & \eta \sim p + q - x - y \sim x' + y' - p' - q'. \end{aligned}$$

In particular, we get that

$$(3) \quad p + q + p' + q' \sim x + y + x' + y',$$

thus defining a base point free g_4^1 on C , which we call ℓ_1 . We let \mathcal{L}_1 on C be the corresponding line bundle. Since there exists a pencil of hyperplanes in \mathbb{P}^3 through any two of the four nodes of $\Gamma := \varphi(C)$, we see that

$$(4) \quad h^0(\omega_C(\eta) - \mathcal{L}_1) = h^0(\omega_C(\eta)(-p - q - p' - q')) = 2.$$

We claim that

$$(5) \quad h^0(\omega_C(\eta) - 2\mathcal{L}_1) = 0.$$

Indeed, if not, we would have $\omega_C(\eta) \simeq 2\mathcal{L}_1$, which together with (4) would yield that $\Gamma \subset \mathbb{P}^3$ is contained in a quadric cone Q , with the pullback of the ruling of the cone cutting ℓ_1 on C . Let \tilde{Q} be the desingularization of Q . Then $\tilde{Q} \simeq \mathbb{F}_2$. Since ℓ_1 is base point free, Γ does not pass through the vertex of Q , so that we may consider Γ as a curve in \tilde{Q} . Denote by σ the minimal section of \mathbb{F}_2 (thus, $\sigma^2 = -2$), which is contracted to the vertex of Q , and by \mathfrak{f} the class of the fibre of the ruling. Then, since $\Gamma \cdot \mathfrak{f} = 4$ and $\Gamma \cdot \sigma = 0$,

we get that $\Gamma \sim 4\sigma + 8f$. In particular, $\omega_\Gamma \simeq \mathcal{O}_\Gamma(K_{\tilde{Q}} + \Gamma) \simeq \mathcal{O}_\Gamma(2\sigma + 4f) \simeq \mathcal{O}_\Gamma(4f)$. Thus, from (3) we obtain

$$\omega_C \simeq \varphi^*(\omega_\Gamma)(-p - q - x - y - p' - q' - x' - y') \simeq \mathcal{O}_C(4\mathcal{L}_1 - 2\mathcal{L}_1) \simeq \mathcal{O}_C(2\mathcal{L}_1),$$

yielding $\eta = 0$, a contradiction. This proves (5).

The relations (4) and (5) imply that $\Gamma \subset \mathbb{P}^3$ is contained in a smooth quadric surface $Q \simeq \mathbb{P}^1 \times \mathbb{P}^1$. The first ruling is defined by the pencil ℓ_1 , whereas the second is defined by the pencil $\ell_2 = |\mathcal{L}_2|$, where $\mathcal{L}_2 := \omega_C(\eta) - \mathcal{L}_1 = \omega_C(\eta)(-p - q - p' - q')$ by (4). The curve Γ is of type (4, 4) on Q , with four nodes. Since $\omega_\Gamma \simeq \omega_{\mathbb{P}^1 \times \mathbb{P}^1}(C) \simeq \mathcal{O}_\Gamma(2, 2)$, we see that $\varphi^*(\omega_\Gamma) \simeq (\omega_C(\eta))^{\otimes 2} \simeq \omega_C^{\otimes 2}$. Thus,

$$\omega_C \simeq \omega_C^{\otimes 2}(-p - q - x - y - p' - q' - x' - y'),$$

whence

$$(6) \quad \omega_C \simeq \mathcal{O}_C(p + q + x + y + p' + q' + x' + y').$$

Combining with (2), we find that

$$(7) \quad \mathcal{L}_2 \simeq \omega_C(\eta)(-p - q - p' - q') \simeq \mathcal{O}_C(p + q + x' + y') \simeq \mathcal{O}_C(p' + q' + x + y).$$

The relations (3) and (7) tell us that the four nodes of Γ lie on two pairs of fibres of each ruling of $\mathbb{P}^1 \times \mathbb{P}^1$, thus showing that $\Gamma \in \mathcal{V}$. Of course this is all well-defined up to automorphisms of $\mathbb{P}^1 \times \mathbb{P}^1$, so we see that the construction associates to (C, η) an element in \mathcal{V}' , which we define to be the image of (C, η) by f .

This defines the map f , and in particular shows that \mathcal{V} is nonempty. We also note for later use that $\omega_C \simeq 2\mathcal{L}_1 \simeq 2\mathcal{L}_2$, so that \mathcal{D}_5^0 is contained in the locus of Prym curves (C, η) carrying a theta-characteristic θ with $h^0(\theta) = h^0(\theta + \eta) = 2$, which we henceforth call T_5 . Moreover, via the forgetful map $\mathcal{R}_5 \rightarrow \mathcal{M}_5$, the locus T_5 maps to the locus of curves with two (complete) autoresidual g_4^1 s, which we call \mathcal{B}_5 .

We next prove that \mathcal{V} is irreducible rational of dimension 16.

For any $X \in \mathcal{V}$, let $\nu : C \rightarrow X$ be the normalization; C has genus 5. If $z_i, i = 1, 2, 3, 4$, are the nodes of X , then the complete linear system $|\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(X) \otimes \mathcal{J}_{z_1}^2 \otimes \mathcal{J}_{z_2}^2 \otimes \mathcal{J}_{z_3}^2 \otimes \mathcal{J}_{z_4}^2|$ has dimension 12, as expected. Indeed, letting r be its dimension, we clearly have $r \geq 12$; on the other hand, this complete linear system induces a g_{16}^{r-1} on C , whence $r - 1 \leq 11$ by Riemann-Roch. It follows that \mathcal{V} is birational to $\mathbb{P}^{12} \times (\text{Sym}^2(\mathbb{P}^1))^2$ (because of the freedom of varying the four lines in the square configuration), in particular it is irreducible rational of dimension $12 + 4 = 16$.

We now define the inverse of f . Given a curve $X \in \mathcal{V}$, let \mathcal{L}_1 and \mathcal{L}_2 be the line bundles of degree 4 on C defined by the pullbacks of the two rulings on $\mathbb{P}^1 \times \mathbb{P}^1$. By the special position of the 4 nodes of X , the four pairs of points C lying above the four nodes of X , say $(p, q), (x, y), (p', q')$ and (x', y') , satisfy

$$\begin{aligned} \mathcal{L}_1 &\simeq \mathcal{O}_C(p + q + p' + q') \simeq \mathcal{O}_C(x + y + x' + y'), \\ \mathcal{L}_2 &\simeq \mathcal{O}_C(p + q + x' + y') \simeq \mathcal{O}_C(x + y + p' + q'), \end{aligned}$$

in particular, $\eta := \mathcal{L}_1 - \mathcal{L}_2$ is 2-torsion. Moreover, one can easily verify that $\omega_C(\eta) \simeq \mathcal{L}_1 + \mathcal{L}_2$. Thus, the normalization $\nu : C \rightarrow X \subset \mathbb{P}^1 \times \mathbb{P}^1$ followed by the embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ as a quadric in \mathbb{P}^3 induces the Prym-canonical map associated to $\omega_C(\eta)$, so that (C, η) has a 4-nodal Prym-canonical image. One readily checks that this map is the inverse of the map f defined above. Thus, we have proved that \mathcal{D}_5^0 is irreducible of dimension $\dim \mathcal{V} / (\text{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)) = 16 - 6 = 10$.

We have left to prove that the closure of \mathcal{D}_5^0 in \mathcal{R}_5 coincides with the closure of T_5 . We proved above that \mathcal{D}_5^0 is contained in T_5 and that the latter maps, via the finite forgetful map $\mathcal{R}_5 \rightarrow \mathcal{M}_5$, to the locus \mathcal{B}_5 of curves with two autoresidual g_4^1 s, which is irreducible of dimension 10 by [11, Thm. 2.10]. Below we give a direct proof of the latter fact, which also proves that the general member of \mathcal{B}_5 carries exactly two g_4^1 s, equivalently two theta characteristics θ and θ' such that $h^0(\theta) = h^0(\theta') = 2$. It will follow that there is an inverse rational map $\mathcal{B}_5 \dashrightarrow T_5$ mapping C to $(C, \theta - \theta')$, proving that also T_5 is irreducible of dimension 10. Its closure must therefore coincide with the closure of \mathcal{D}_5^0 , finishing the proof of the proposition.

So let C be a smooth, irreducible curve of genus 5 and consider its canonical embedding $C \subset \mathbb{P}^4$. Given $\xi = |D|$ a (complete) g_4^1 on C , the divisors in ξ span planes which sweep out a quadric Q_ξ of rank $r < 5$. If ξ is not autoresidual, then Q_ξ has rank $r = 4$ and it has another 1-dimensional system of planes which cut out on C the divisors of $\xi' = |K_C - D|$. In this case $Q_\xi = Q_{\xi'}$. Hence ξ is autoresidual if and only if Q_ξ has rank 3, and therefore it possesses only one 1-dimensional family of planes. This means that the homogeneous ideal of a curve in \mathcal{B}_5 in its canonical embedding contains two distinct rank 3 quadrics. Hence the general curve C in \mathcal{B}_5 is obtained by intersecting two general rank 3 quadrics in \mathbb{P}^4 with another general quadric. Note that the two rank 3 quadrics cut out a Del Pezzo surface S with 4 nodes, hence C is a general quadric section on S . The two autoresidual g_4^1 on C are cut out on C by the conics of the two pencils on S with base points two of the nodes.

From this description it follows that \mathcal{B}_5 is irreducible, 10-dimensional and that its general member contains precisely two autoresidual g_4^1 s. Indeed, consider the \mathbb{P}^{14} parametrizing all quadrics in \mathbb{P}^4 . The locus \mathcal{X} of quadrics of rank $r \leq 3$ is non-degenerate and has dimension 11. The net of quadrics defining a general curve C in \mathcal{B}_5 corresponds to a plane in \mathbb{P}^{14} containing a general secant line to \mathcal{X} (which, by its generality, contains only two points in \mathcal{X}), and an easy count of parameters shows that these planes clearly fill up a variety of dimension 34. Modding out by the 24-dimensional group of projective transformations of \mathbb{P}^4 , we get dimension 10 for \mathcal{B}_5 . \square

Remark 5.4. Denote, as in the last proof, by \mathcal{D}_0^5 the locus of Prym curves (C, η) carrying a theta-characteristic θ with $h^0(\theta) = h^0(\theta + \eta) = 2$. By [4, Prop. 7.3 and Thm. 7.4] the locus \mathcal{D}_0^5 maps, via the Prym map $\mathcal{P}_5 : \mathcal{R}_5 \rightarrow \mathcal{A}_4$, to the irreducible divisor θ_{null} of principally polarized abelian varieties whose theta-divisor has a singular point at a 2-torsion point, and moreover the general member of $\mathcal{P}_5(\mathcal{D}_0^5)$ has precisely one ordinary double point, cf. [4, Pf. of Prop. 7.5]. It would be interesting to know if \mathcal{D}_0^5 dominates θ_{null} .

By [4, Prop. 7.3] one knows that the closure of $\mathcal{P}_5^{-1}(\theta_{\text{null}})$ is the closure of the locus of Prym curves (C, η) carrying a theta-characteristic θ such that $h^0(\theta) + h^0(\theta + \eta)$ is even, which *properly* contains \mathcal{D}_0^5 .

Remark 5.5. By contrast, if we consider the adjoint line bundle of the one in Example 5.2, that is, $H' := \mathcal{O}_S(2E_1 + 2E_2)$, then by [8, Prop. 4.1.2, Thm. 4.7.1, (F) p. 277] the morphism $\varphi_{H'}$ defined by $|H'|$ is of degree 2 onto a quartic Del Pezzo surface. In particular, $\varphi_{H'}$ maps any smooth $C \in |H|$ doubly onto an elliptic quartic curve in \mathbb{P}^3 . Hence, the Prym curve $(C, \mathcal{O}_C(K_S))$ belongs to the locus $\mathcal{R}_5^{0, \text{nb}}$ described in Corollary 2.2.

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A. ADDENDUM

ABSTRACT. We give some additional details on the proof of [1, Prop. 3.1], which states the irreducibility and unirationality of the locus \mathcal{R}_g^0 of Prym curves such that the Prym-canonical linear system is base point free but does not define an embedding, for $g \geq 5$.

We use freely the notation and setup introduced in [1]. At the very end of the proof of Prop. 3.1, p. 78 (see footnote 1), we claim that “since $\mathcal{H} \dashrightarrow \mathcal{M}_{g,4}^1$ is dominant, then $\alpha : \mathcal{G} \dashrightarrow \mathcal{R}_g^0$ is also dominant”. The aim of this addendum is to provide a complete proof of this claim.

Let us denote by β the map $\mathcal{H} \dashrightarrow \mathcal{M}_{g,4}^1$ and by U its domain of definition. Note that by [1, Lemma 2.1], \mathcal{R}_g^0 is a sublocus of $\mathcal{R}_{g,4}^1$. In order to prove the above mentioned claim, it is necessary and sufficient to check that there are no irreducible components of \mathcal{R}_g^0 dominating a sublocus of $\mathcal{M}_{g,4}^1 \setminus \beta(U)$. This had not been carried out in [1], and we provide such a verification in the present addendum.

We denote by $\overline{\mathcal{M}}_g$ the moduli space of Deligne and Mumford stable curves of genus g . The map β extends to a rational map $\overline{\beta} : \overline{\mathcal{H}} \dashrightarrow \overline{\mathcal{M}}_{g,4}^1$, where $\overline{\mathcal{H}}$ and $\overline{\mathcal{M}}_{g,4}^1$ are the closures of \mathcal{H} and $\mathcal{M}_{g,4}^1$ in $|\mathcal{O}_{\mathbb{P}^2}(n)|$ and $\overline{\mathcal{M}}_g$ respectively. We denote by $\overline{\mathcal{G}}$ the closure of \mathcal{G} in $\overline{\mathcal{H}}$.

Claim A.1. *Let \mathcal{G}' be an irreducible family of irreducible plane curves of degree n and genus g , having multiplicity $n - 4$ at p , whose normalization, endowed with the g_4^1 cut out by the lines through p , belongs to $\mathcal{R}_g^0 \subset \mathcal{R}_{g,4}^1$. Then \mathcal{G}' is contained in $\overline{\mathcal{G}}$.*

Proof. We may assume that \mathcal{G}' is maximal for the above properties with respect to inclusion. Let C be a general member of \mathcal{G}' . By [1, Lemma 2.1] and the deformation theory for logarithmic Severi varieties, see [2, Thm. 1.4], C has an ordinary $(n - 4)$ -tuple point at p and its other singularities are all nodal. Therefore C is a member of some linear system $\mathcal{L}_n(p_1, \dots, p_\delta, [q_1, q_2, q'_1, q'_2])$, with the possibility that the points $p_1, \dots, p_\delta, q_1, q_2, q'_1, q'_2$ be in a special position.

However the dimension of these linear systems is always the same, no matter the position of the points $p_1, \dots, p_\delta, q_1, q_2, q'_1, q'_2$: indeed the corresponding characteristic series on C is always non-special, as follows from the computation of its degree, which does not depend on the position of the points $p_1, \dots, p_\delta, q_1, q_2, q'_1, q'_2$. The upshot is that the union of the linear systems $\mathcal{L}_n(p_1, \dots, p_\delta, [q_1, q_2, q'_1, q'_2])$ is irreducible, even if one includes special configurations of points $p_1, \dots, p_\delta, q_1, q_2, q'_1, q'_2$ in taking the union. This union therefore has \mathcal{G} as a dense subset, hence C belongs to the closure of \mathcal{G} , and the closures of \mathcal{G} and \mathcal{G}' are equal, which proves our claim. \square

Claim A.2. *There is no irreducible component of \mathcal{R}_g^0 dominating a sublocus of $\mathcal{M}_{g,4}^1 \setminus \beta(U)$.*

Proof. We consider $\widetilde{\mathcal{H}}$ a blow-up of $\overline{\mathcal{H}}$ such that the map $\beta : \mathcal{H} \dashrightarrow \mathcal{M}_{g,4}^1$ induces a regular map $\widetilde{\beta} : \widetilde{\mathcal{H}} \rightarrow \overline{\mathcal{M}}_{g,4}^1$ (in particular $\widetilde{\mathcal{H}} \rightarrow \overline{\mathcal{H}}$ factorizes through the normalization

of $\overline{\mathcal{H}}$). A point \tilde{C} of the exceptional locus of $\tilde{\mathcal{H}} \rightarrow \overline{\mathcal{H}}$ is the datum of a point $[C] \in \overline{\mathcal{H}}$ and an infinitesimal curvilinear arc in $\overline{\mathcal{H}}$ centered at $[C]$, and the image of \tilde{C} in $\overline{\mathcal{M}}_{g,4}^1$ is determined by the stable reduction of the deformation of the partial normalization of genus g of C corresponding to this curvilinear arc. Since $\tilde{\mathcal{H}}$ and $\overline{\mathcal{M}}_{g,4}^1$ are projective and β is dominant, $\tilde{\beta}$ is surjective; in particular the closure of the image of \mathcal{R}_g^0 in $\overline{\mathcal{M}}_g$ is contained in the image of $\tilde{\beta}$. To prove our claim we may thus consider a point \tilde{C} in $\tilde{\mathcal{H}}$ sitting over $[C] \in \overline{\mathcal{H}}$, such that $\tilde{\beta}(\tilde{C})$ is a general point in some irreducible component of the closure of the image of \mathcal{R}_g^0 in $\overline{\mathcal{M}}_g$, and show that then $\tilde{\beta}(\tilde{C})$ necessarily belongs to the closure of $\beta(\mathcal{G}) = \alpha(\mathcal{G})$.

First consider the case that C is irreducible and reduced. If C has geometric genus g , then $[C]$ sits in the domain of definition of β in $\overline{\mathcal{H}}$ (or rather of the rational map $\tilde{\beta}$ induced by β on the normalization of $\overline{\mathcal{H}}$), and, by Claim A.1, the curve C is in the closure of \mathcal{G} . Then $\tilde{\beta}(\tilde{C}) = \tilde{\beta}([C])$ sits in the closure of $\beta(\mathcal{G}) = \alpha(\mathcal{G})$, and we are done.

If the geometric genus of C is smaller than g , then for all \tilde{C} in $\tilde{\mathcal{H}}$ over $[C]$, the curve $\tilde{\beta}(\tilde{C})$ has an irreducible component of genus smaller than g , so it sits in the boundary of $\overline{\mathcal{M}}_g$, hence it cannot be a general member of a component of the closure of the image of \mathcal{R}_g^0 . This ends the proof in the case C integral.

Assume next that C is not integral. Let us first consider the case in which C has several irreducible components. If none of these is contracted in the stable reduction, then the stable model of \tilde{C} is in the boundary of $\overline{\mathcal{M}}_g$, hence it cannot be a general member of a component of the closure of the image of \mathcal{R}_g^0 .

Next we examine the case in which some component C' is contracted in the stable reduction. If the restriction to C' of the g_4^1 cut out by lines through p is non-trivial, then the g_4^1 on the stable model of \tilde{C} has base points. If there is only one such base point, then the stable model cannot lie in \mathcal{R}_g^0 by [1, Lemma 2.1]. If there are more base points, then the stable model is hyperelliptic, hence the number of moduli of such curves is at most $2g - 1$. On the other hand the irreducible components of \mathcal{R}_g^0 have dimension at least $2g + 1$ as a direct application of [1, Lemma 2.1] shows. Therefore in this case, $\tilde{\beta}(\tilde{C})$ cannot be general in a component of the closure of the image of \mathcal{R}_g^0 .

If the restriction to C' of the g_4^1 cut out by lines through p is trivial, then C' is a line passing through p . In this case a direct computation shows that the number of moduli for $C - C'$ is at most $2g$, hence we conclude as above.

So the only remaining case is the one in which $C = mC_0$ with $m > 1$ and C_0 irreducible. We want to check that these curves cannot give rise to an irreducible component of \mathcal{R}_g^0 . First, since the pencil of lines through p cuts out a g_4^1 on the general member of \mathcal{H} , one sees that m may only take the values 2 and 4. By explicit computation of the stable reduction, as for instance in [3, 3.C], one sees that in the former case it gives a curve C' of genus g which is a double cover of a hyperelliptic curve, and in the latter case it gives a tetragonal curve for which all ramification points of the g_4^1 are triple.

Suppose first C' is a double cover of a hyperelliptic curve Γ of genus γ . An easy count of parameters shows that the number of moduli on which C' depends is at most $2g + 3 - 2\gamma$ if $\gamma \geq 2$, is at most $2g - 2$ if $\gamma = 1$, and is at most $2g - 1$ if $\gamma = 0$. Suppose next that C' has a g_4^1 for which all ramification points are triple. Then the number of moduli of C' is at most $\frac{2}{3}g - 1$.

In any case the curves C' have too few moduli to fill up a component of \mathcal{R}_g^0 . This proves our claim. \square

Claim A.2 above implies that indeed $\alpha : \mathcal{G} \dashrightarrow \mathcal{R}_g^0$ is dominant. The proof of [1, Prop. 3.1] is thus complete, and \mathcal{R}_g^0 is indeed irreducible and unirational.

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CIRO CILIBERTO, DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DI ROMA TOR VERGATA, VIA DELLA RICERCA SCIENTIFICA, 00173 ROMA, ITALY

Email address: `cilibert@mat.uniroma2.it`

THOMAS DEDIEU, INSTITUT DE MATHÉMATIQUES DE TOULOUSE — UMR5219, UNIVERSITÉ DE TOULOUSE — CNRS, UPS IMT, F-31062 TOULOUSE CEDEX 9, FRANCE

Email address: `thomas.dedieu@math.univ-toulouse.fr`

CONCETTINA GALATI, DIPARTIMENTO DI MATEMATICA E INFORMATICA, UNIVERSITÀ DELLA CALABRIA, VIA P. BUCCI, CUBO 31B, 87036 ARCAVACATA DI RENDE (CS), ITALY

Email address: `concettina.galati@unical.it`

ANDREAS LEOPOLD KNUTSEN, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF BERGEN, POSTBOKS 7800, 5020 BERGEN, NORWAY

Email address: `andreas.knutsen@math.uib.no`