ON THE IRREDUCIBILITY OF SEVERI VARIETIES ON $K3$ SURFACES

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Abstract. Let (S, L) be a polarized $K3$ surface of genus $p \geqslant 11$ such that $\mathrm{Pic}(S) = \mathbf{Z}[L]$, and δ a non-negative integer. We prove that if $p \geqslant 4\delta - 3$, then the Severi variety of δ-nodal curves in $|L|$ is irreducible.

1. Introduction

Given a polarized surface (S, L) and an integer $\delta \geqslant 0$, the *Severi variety* $V^{L,\delta}$ is the parameter space for irreducible, δ-nodal curves in the linear system $|L|$ (see § 2.1). This text is dedicated to the proof of the following result:

Theorem 1. *Let (S, L) be a primitively polarized $K3$ surface of genus $p \geqslant 11$ such that $\mathrm{Pic}(S) = \mathbf{Z}[L]$, and δ a non-negative integer such that $4\delta - 3 \leqslant p$. The Severi variety $V^{L,\delta}$ is irreducible.*

It had already been proven by Keilen [14] that in the situation of Theorem 1, for all integer $k \geqslant 1$ the Severi variety $V^{kL,\delta}$ is irreducible if

$$\delta < \frac{6(2p-2)+8}{\big(11(2p-2)+12\big)^2} \cdot k^2 \cdot (2p-2)^2 \qquad \Big(\sim_{p\to\infty} \frac{12}{121} \cdot k^2 \cdot p\Big),$$

and later by Kemeny [15] that the same holds if $\delta \leqslant \frac{1}{6}\big(2 + k(p-1)\big)$. Our result is valid only in the case $k = 1$, i.e., for curves in the *primitive class*, but in this case our condition is better. In a slightly different direction, we have proven some time ago in [6] that the *universal families* of the $V^{L,\delta}$'s are irreducible for all δ ($\delta = p$ included) if $3 \leqslant p \leqslant 11$ and $p \neq 10$.

Kemeny's result is based on the observation that for any smooth polarized surface (S, L), the Severi variety $V^{L,\delta}$ is somehow trivially irreducible if L is $(3\delta - 1)$-very ample: Indeed, in this case the curves in $|L|$ with nodes at $p_1, \ldots, p_\delta$ form a dense subset of a projective space of constant dimension for *any* set of pairwise distinct points $p_1, \ldots, p_\delta$. Kemeny then applies a numerical criterion for n-very ampleness on $K3$ surfaces due to Knutsen [16].

The central idea of the present article is close in spirit to Kemeny's observation, to the effect that provided $\dim |L| \geqslant 3\delta$, the curves in $|L|$ with nodes at $p_1, \ldots, p_\delta$ should form in nice circumstances a dense subset of a projective space of constant dimension for a *general* choice of δ pairwise disjoint points. It is indeed so for curves in the primitive class of a $K3$ surface, thanks to a result of Chiantini and the first-named author, see Proposition 14. One thus gets a distinguished irreducible component of the Severi variety $V^{L,\delta}$ which we call its *standard component*. For any other irreducible component V, the nodes of the members of V sweep out a locus of positive codimension h_V in the Hilbert scheme $S^{[\delta]}$, see Section 3; we call h_V the excess of V.

Our applications then rely on the observation that, in the $K3$ situation of Theorem 1, for all $C \in V$ the preimage of the nodes defines a linear series of type $g^h_{2\delta}$ on the normalisation of C (see Lemma 20), together with some recent results in [7] and [17] (Theorems 9 and 8 respectively) which give some control on the families of linear series that may exist on the normalisations of primitive curves on $K3$ surfaces. The latter results hold only for curves in the primitive class, and this is the main obstruction to carry out our approach in the non-primitive situation.

One may for instance give a two-lines proof of irreducibility in the range $p \geqslant 5\delta - 3$, as follows. Assume by contradiction that there is a non-standard irreducible component V of the Severi variety $V^{L,\delta}$. Then for all $C \in V$ the normalisation of C has a $g^1_{2\delta}$. By [17] this implies $\dim(V) = p - \delta \leqslant 4\delta - 2$, which is impossible in the range under consideration.

We obtain the better bound in Theorem 1 by proving the estimate $h_V > 2$ for all non-standard components of $V^{L,\delta}$. This is done in Section 4 by a careful study of the singularities of curves in the intersection of the standard component with a hypothetical non-standard component, which we are again able to control thanks to Brill–Noether theoretic results for singular curves on $K3$ surfaces.

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2. Preliminaries

2.1. **Severi varieties.** We work over $\mathbf{C}$ throughout the text. We denote by $\mathcal{K}_p$ the irreducible, 19-dimensional stack of primitively polarized $K3$ surfaces (S, L) of genus $p \geqslant 2$, i.e., S is a compact, complex surface with $h^1(S, \mathcal{O}_S) = 0$ and $\omega_S \cong \mathcal{O}_S$, and L a big and nef, primitive line bundle on S with $L^2 = 2p - 2$, hence $\dim(|L|) = p$. The *arithmetic genus* of the curves $C \in |L|$ is $p_a(C) = p$.

In this paper we will often assume that $\mathrm{Pic}(S) = \mathbf{Z}[L]$, which is the case if $(S, L) \in \mathcal{K}_p$ is very general, so that L is globally generated and ample, and very ample if $p \geqslant 3$.

For any non-negative integer $g \leqslant p$, we consider the locally closed subset V_g^L of $|L|$ consisting of curves $C \in |L|$ of *geometric genus* $p_g(C) = g$, i.e., curves C whose normalization has genus g (see [8, § 1.2]). We will set $\delta = p - g$, which is usually called the *δ-invariant* of the curve.

Proposition 2 (see [8, Proposition 4.5])**.** *Every irreducible component of V_g^L has dimension g.*

For every non-negative integer $\delta \leqslant p$, we will denote by $V^{L,\delta}$ the *Severi variety*, i.e., the locally closed subset of $|L|$ consisting of curves with δ nodes and no other singularities, whose geometric genus is $g = p - \delta$. The following is classical:

Proposition 3 (see [8, §3–4])**.** *The Severi variety $V^{L,\delta}$, if not empty, is smooth and pure of dimension g. More precisely, if $C \in V^{L,\delta}$, and Δ is the set of nodes of C, then the projective tangent space to $V^{L,\delta}$ at C in $|L|$ is the g-dimensional linear system $|L(-\Delta)| := \mathbf{P}(\mathrm{H}^0(S, L \otimes \mathcal{I}_{\Delta,S}))$ of curves in $|L|$ containing Δ.*

It is indeed true that the Severi varieties of a general primitively polarized $K3$ surface are non-empty.

Proposition 4 (see [3])**.** *If $(S, L) \in \mathcal{K}_p$ is general, then $V^{L,\delta}$ is not empty for every non-negative integer $\delta \leqslant p$.*

By Propositions 2 and 3, each irreducible component of $V^{L,\delta}$ is dense in a component of V_g^L. Xi Chen [4] has shown that moreover if $g > 0$, then $V^{L,\delta}$ is dense in V_g^L for general $(S, L) \in \mathcal{K}_p$. We shall need the following weaker result, in which however the generality assumption is explicit.[1]

Proposition 5 ([8, Proposition 4.8])**.** *Let $(S, L) \in \mathcal{K}_p$ be such that $\mathrm{Pic}(S) = \mathbf{Z}[L]$. If $2\delta < p$, then $V^{L,\delta}$ is dense in V_g^L.*

2.2. **Local structure of Severi varieties.** The following is a restatement of the well-known fact that the nodes of a nodal curve on a $K3$ surface may be smoothed independently. It is a consequence of Proposition 3.

Proposition 6. *Let $(S, L) \in \mathcal{K}_p$, $\delta < \varepsilon$ be two non-negative integers, and V be an irreducible component of $V^{L,\varepsilon}$. Consider a curve $C \in V$, and let $\{p_1, \ldots, p_\varepsilon\}$ be the set of its nodes. Then:*
(i) the Zariski closure $\overline{V}^{L,\delta}$ of $V^{L,\delta}$ contains V;
(ii) locally around C, $\overline{V}^{L,\delta}$ consists of $\binom{\varepsilon}{\delta}$ analytic sheets $\mathcal{V}_{\mathfrak{d}}$, which are in $1:1$ correspondence with the subsets $\mathfrak{d} \subset \{p_1, \ldots, p_\varepsilon\}$ of order δ, and such that when the general point C' of $\mathcal{V}_{\mathfrak{d}}$ specializes at C, the set of δ nodes of C' specializes at $\mathfrak{d}$;
(iii) for each such $\mathfrak{d}$, the sheet $\mathcal{V}_{\mathfrak{d}}$ is smooth at C of dimension $p - \delta$, relatively transverse to all other similar sheets.[2]

[1] Actually, the assumption in [8, Proposition 4.8] is that (S, L) be very general; it is straightforward to check that the condition $\mathrm{Pic}(S) = L$ is indeed sufficient for the proof in [8].

[2] in the sense that for all $\mathfrak{d}'$ of cardinality δ, the sheets $\mathcal{V}_{\mathfrak{d}}$ and $\mathcal{V}_{\mathfrak{d}'}$ intersect exactly along the local sheet $V_{\mathfrak{d} \cup \mathfrak{d}'}$ of $\overline{V}^{L,|\mathfrak{d} \cup \mathfrak{d}'|}$ at C, and their respective tangent spaces at C intersect exactly along the tangent space of $V_{\mathfrak{d} \cup \mathfrak{d}'}$ at C.

As an immediate consequence, we have:

Corollary 7. *Let $(S,L)\in\mathcal{K}_p$ and let V, V' be irreducible components of $V^{L,\delta}$ and $V^{L,\delta'}$, with $\delta\leqslant\delta'$. If V' intersects the Zariski closure $\overline{V}$ of V, then $V'\subset\overline{V}$.*

2.3. **Brill–Noether theory of curves on K3 surfaces.** We will use the following results.

Theorem 8 ([17, Theorem 5.3 and Remark 5.6])**.** *Let (S,L) be such that $\mathrm{Pic}(S)=\mathbf{Z}[L]$, and $V\subset V^L_g$ a non-empty reduced scheme. Let k be a positive integer. Assume that for all $C\in V$, there exists a g^1_k on the normalisation $\tilde{C}$ of C. Then one has*

$$\dim(V)+\dim(G^1_k(\tilde{C}))\leqslant 2k-2$$

for general $C\in V$.

Theorem 9 ([7, Theorem 3.1])**.** *Let $(S,L)\in\mathcal{K}_p$ be such that $\mathrm{Pic}(S)=\mathbf{Z}[L]$, and $C\in V^L_g$; let $\delta=p-g$. Let r,d be nonnegative integers. If there exists a g^r_d on the normalization of C, then*

$$\delta\geqslant\alpha\big(rg-(d-r)(\alpha r+1)\big),\quad \textit{where}\quad \alpha=\left\lfloor\frac{gr+(d-r)(r-1)}{2r(d-r)}\right\rfloor.$$

Theorem 10 ([18, 11, 1, 10])**.** *Let $(S,L)\in\mathcal{K}_p$ be such that $\mathrm{Pic}(S)=\mathbf{Z}[L]$, and $C\in|L|$. The Clifford index of C, computed with sections of rank one torsion free sheaves on C (see [8, p. 202] or [1]), equals $\lfloor\frac{p-1}{2}\rfloor$.*

3. Standard components

3.1. **The nodal map.** Let $(S,L)\in\mathcal{K}_p$. For any positive integer n, we denote by $S^{[n]}$ the Hilbert scheme of 0-dimensional subschemes of S of length n. Recall that $S^{[n]}$ is smooth of dimension $2n$ (see [9]).

Consider the morphism

$$\varphi_{L,\delta}:V^{L,\delta}\longrightarrow S^{[\delta]},$$

called the *nodal map*, which maps a curve $C\in V^{L,\delta}$ to the scheme Δ of its nodes, indeed 0-dimensional of length δ. We set $\Phi_{L,\delta}:=\mathrm{Im}(\varphi_{L,\delta})$. If V is an irreducible component of $V^{L,\delta}$, we set

$$\varphi_V:=\varphi_{L,\delta}|_V\quad\text{and}\quad\Phi_V:=\mathrm{Im}(\varphi_V).$$

Let Δ be a general point in Φ_V. Then $\varphi_V^{-1}(\Delta)$ is an open subset of the linear system $|L(-2\Delta)|:=\mathbf{P}(\mathrm{H}^0(S,L\otimes\mathcal{I}^2_{\Delta,S}))$ of curves in $|L|$ singular at Δ. We set

$$\dim(|L(-2\Delta)|)=p-3\delta+h_V,$$

which defines the non-negative integer h_V, called the *excess* of V. By Proposition 3, one has

$$\dim(\Phi_V)=2\delta-h_V. \tag{1}$$

The following is immediate:

Lemma 11. *Let $(S,L)\in\mathcal{K}_p$, and let V_1,V_2 be two distinct irreducible components of $V^{L,\delta}$. Then Φ_{V_1} and Φ_{V_2} have distinct Zariski closures in $S^{[\delta]}$.*

3.2. **A useful lemma.** Let $C\in|L|$ be a reduced curve, and consider the conductor ideal $A\subset\mathcal{O}_C$ of the normalization $\nu:\tilde{C}\to C$. There exists a divisor $\tilde{\Delta}$ on $\tilde{C}$ such that $A=\nu_*\mathcal{O}_{\tilde{C}}(-\tilde{\Delta})$, and one has $\omega_{\tilde{C}}=\nu^*\omega_C\otimes\mathcal{O}_{\tilde{C}}(-\tilde{\Delta})$. It is a classical result that $\nu^*|L\otimes A|=|\omega_{\tilde{C}}|$, see [8, Lemma 3.1]. The same argument proves that $\nu^*|L\otimes A^{\otimes 2}|=|\omega_{\tilde{C}}(-\tilde{\Delta})|$.

Consider the particular case when C has ordinary cusps $p_1,\dots,p_k$ and nodes $p_{k+1},\dots,p_\delta$ as its only singularities. Denote by $p_1,\dots,p_k\in\tilde{C}$ the respective preimages of $p_1,\dots,p_k\in C$ by the normalisation ν, abusing notations, and by p'_i and p''_i the two preimages of p_i for $i=k+1,\dots,\delta$. Then A is the product of the maximal ideals of $p_1,\dots,p_\delta$, i.e., $A=\mathcal{I}_{\Delta,S}\otimes\mathcal{O}_C$ with $\Delta=\{p_1,\dots,p_\delta\}$, and

$$\tilde{\Delta}=2\sum\nolimits_{i=1}^k p_i+\sum\nolimits_{i=k+1}^{\delta}(p'_i+p''_i).$$

The previous identity $\nu^*|L\otimes A^{\otimes 2}|=|\omega_{\tilde{C}}(-\tilde{\Delta})|$ readily implies the following.

Lemma 12. *Let j be the closed immersion $C \hookrightarrow S$. One has*
$$(j \circ \nu)^*\big(\,|L(-2\Delta)|\,\big) = |\omega_{\tilde{C}}(-\tilde{\Delta})|,$$
and therefore $\dim\big(\,|L(-2\Delta)|\,\big) = h^0(\omega_{\tilde{C}}(-\tilde{\Delta}))$.

3.3. **Standard components.** Let V be an irreducible component of $V^{L,\delta}$. We call V *standard* if $h_V = 0$. If V is standard and $\Delta \in \Phi_V$ is general, then
$$0 \leqslant \dim(\varphi_V^{-1}(\Delta)) = \dim(|L(-2\Delta)|) = p - 3\delta,$$
hence $p \geqslant 3\delta$. Moreover if V is standard, then $\dim(\Phi_V) = 2\delta$, hence Φ_V is dense in $S^{[\delta]}$. We will prove in Proposition 16 below that if $p \geqslant 3\delta$ and if $\mathrm{Pic}(S) = \mathbf{Z}[L]$, then there is a unique standard component of $V^{L,\delta}$. To do this, we need to recall some basic fact from [5].

Let $Y \subset \mathbf{P}^N$ be an irreducible, n-dimensional, non-degenerate, projective variety. Let $\mathcal{H}$ be the linear system cut out on Y by the hyperplanes of $\mathbf{P}^N$, i.e.,
$$\mathcal{H} = \mathbf{P}(\mathrm{Im}(r)) \quad \text{where} \quad r : \mathrm{H}^0(\mathbf{P}^N, \mathcal{O}_{\mathbf{P}^N}(1)) \to \mathrm{H}^0(Y, \mathcal{O}_Y \otimes \mathcal{O}_{\mathbf{P}^N}(1))$$
is the restriction map. Let k be a non-negative integer. The variety Y is said to be *k-weakly defective* if given $p_0, \dots, p_k \in Y$ general points, the general element of $\mathcal{H}(-2p_0 - \dots - 2p_k)$ has a positive dimensional singular locus, where $\mathcal{H}(-2p_0 - \dots - 2p_k)$ denotes the linear system of divisors in $\mathcal{H}$ singular at $p_0, \dots, p_k$.

Proposition 13 ([5, Theorem 1.4])**.** *Let $Y \subset \mathbf{P}^N$ be an irreducible, n-dimensional, non-degenerate, projective variety. Let k be a non-negative integer such that $N \geqslant (n+1)(k+1)$. If Y is not k-weakly defective, then given $p_0, ..., p_k$ general points on Y, one has:*
(i) $\dim(\mathcal{H}(-2p_0 - ... - 2p_k)) = N - (n+1)(k+1)$*;*
(ii) the general divisor $H \in \mathcal{H}(-2p_0 - ... - 2p_k)$ has ordinary *double points at $p_0, ..., p_k$, i.e., double points with tangent cone of maximal rank n, and no other singularity.*

In [5, Theorem 1.3] one finds the classification of k-weakly defective surfaces. After an inspection which we leave to the reader, one sees that:

Proposition 14. *Let $(S, L) \in \mathcal{K}_p$ be such that* $\mathrm{Pic}(S) = \mathbf{Z}[L]$*, and assume $p \geqslant 3$. Consider S embedded in $\mathbf{P}^p$ via the morphism determined by $|L|$. Then S is not k-weakly defective for any non-negative integer k.*

We can therefore apply Proposition 13 and conclude that:

Proposition 15. *Maintain the assumptions of Proposition 14, and let δ be a non-negative integer such that $3\delta \leqslant p$. Then given $\Delta \in S^{[\delta]}$ general, one has* $\dim(|L(-2\Delta)|) = p - 3\delta$ *and the general curve in $|L(-2\Delta)|$ has nodes at Δ and no other singularities.*

As a consequence we have:

Proposition 16. *Under the assumptions of Proposition 15, there is a unique standard component $V_{\mathrm{st}}^{L,\delta}$ of $V^{L,\delta}$, which is the unique irreducible component V of $V^{L,\delta}$ such that $\varphi_V : V \to S^{[\delta]}$ is dominant.*

Proof. Proposition 15 implies that there is a standard component V of $V^{L,\delta}$ such that $\varphi_V : V \to S^{[\delta]}$ is dominant. By Lemma 11, it is the unique standard component. □

4. A lower bound on the excess

This section is entirely devoted to the proof of the following:

Proposition 17. *Let $p \geqslant 11$ and $\delta > 1$, $(p, \delta) \neq (12, 4)$, be integers such that $3\delta \leqslant p$. We consider $(S, L) \in \mathcal{K}_p$ such that* $\mathrm{Pic}(S) = \mathbf{Z}[L]$*. For all non-standard component V of $V^{L,\delta}$, one has $h_V \geqslant 3$.*

Let V be a non-standard component of $V^{L,\delta}$ as above. One has $h_V > 0$ by definition, and we shall proceed by contradiction to show that h_V may neither equal 1 nor 2.

4.1. **Proof that** $h_V \neq 1$**.** In the setup of Proposition 17, we assume by contradiction that $h_V = 1$. Then the closure of Φ_V is an irreducible divisor in $S^{[\delta]}$. Let $\Delta \in \Phi_V$ be a general point. It can be seen as the limit of a general 1-dimensional family $\{\Delta_t\}_{t\in\mathbf{D}}$, where $\mathbf{D}$ is a complex disk, and Δ_t is general in $S^{[\delta]}$ for $t \neq 0$. In particular, we may assume $\dim(\varphi_{L,\delta}^{-1}(\Delta_t)) = p - 3\delta$ for $t \in \mathbf{D} - \{0\}$. We define the limit $\mathcal{L}_\Delta$ of $\varphi_{L,\delta}^{-1}(\Delta_t)$ as $t \to 0$ as the fibre over $0 \in \mathbf{D}$ of the closure of $\bigcup_{t\neq 0}(\varphi_{L,\delta}^{-1}(\Delta_t))$ inside $|L| \times \mathbf{D}$. Then:
(i) $\mathcal{L}_\Delta$ is a $(p-3\delta)$-dimensional sublinear system of $|L(-2\Delta)|$;
(ii) $\mathcal{L}_\Delta$ is contained in $\overline{V} \cap \overline{V}^{L,\delta}_{\mathrm{st}}$;
(iii) since $V^{L,\delta}$ is smooth, by (ii) the general curve in $\mathcal{L}_\Delta$ does not belong to $V^{L,\delta}$, i.e., it has singularities worse than only nodes at the points of Δ;
(iv) as Δ moves in a suitable dense open subset U of Φ_V, the union $\bigcup_{\Delta\in U}\mathcal{L}_\Delta$ describes a locally closed subset of dimension

$$\dim(\Phi_V) + (p-3\delta) = (2\delta - 1) + (p - 3\delta) = g - 1,$$

which is dense in an irreducible component W of $\overline{V} \cap \overline{V}^{L,\delta}_{\mathrm{st}}$, where $g = p - \delta$ as usual.

Let C be the general curve in W, which belongs to $\mathcal{L}_\Delta$ for some general $\Delta \in \Phi_V$. By (i) and (iii) above, C is singular at Δ but it is not δ-nodal. By Proposition 2 one has $p_g(C) \geqslant g-1$, hence $g - 1 \leqslant p_g(C) \leqslant g$. We will show that each of these two possible values leads to a contradiction, thus proving that $h_V \neq 1$.

4.1.1. *Case* $p_g(C) = g - 1$. Since $\dim(W) = g - 1$, it follows from Proposition 5 that W is dense in the closure of a component of $V^{L,\delta+1}$, i.e., C is a $(\delta+1)$-nodal curve, with only one extra node $p_{\delta+1} \notin \Delta$. By Proposition 6, locally around C there is only one smooth branch $\mathcal{V}$ of $\overline{V}^{L,\delta}$ containing W and such that when the general point of $\tilde{C}$ of $\mathcal{V}$ specializes at C, then set of δ nodes of $\tilde{C}$ specializes at Δ. This is a contradiction, because both $\overline{V}$ and $\overline{V}^{L,\delta}_{\mathrm{st}}$ contain W. Therefore, it is impossible that $p_g(C) = g - 1$.

4.1.2. *Case* $p_g(C) = g$. Since C is singular at $\Delta = p_1 + \ldots + p_\delta$, it is singular only there, and has only nodes and (simple) cusps (with local equation $x^2 = y^3$); it must have at least one cusp by (iii).

Claim 18. C has only one cusp.

Proof of the Claim. Suppose that C has cusps at $p_1, \ldots, p_k$ and nodes at $p_{k+1}, \ldots, p_\delta$, with $k \geqslant 1$. The tangent space to the equisingular deformations of C in S is $\mathrm{H}^0(C, L \otimes \mathcal{I} \otimes \mathcal{O}_C)$, where $\mathcal{I}$ is the ideal sheaf associated to the *equisingular ideal* (see [8, § 3]) $I = \prod_{i=1}^{\delta} I_{p_i}$, where:

- $I_{p_i} = (x, y^2)$, if the local equation of C around p_i is $x^2 = y^3$, for $i = 1, \ldots, k$;
- I_{p_i} is the maximal ideal at p_i, for $i = k+1, \ldots, \delta$.

Let $\nu : \tilde{C} \to C$ be the normalization. We abuse notation and denote by $p_1, \ldots, p_k$ their counterimages by ν, whereas we denote by p_i' and p_i'' the two points of $\tilde{C}$ in the preimage of p_i by ν, for $i = k+1, \ldots, \delta$. By pulling back by ν the sections of $\mathrm{H}^0(C, L \otimes \mathcal{I} \otimes \mathcal{O}_C)$ and dividing by sections vanishing at the fixed divisor $2\sum_{i=1}^{k} p_i + \sum_{i=k+1}^{\delta}(p_i' + p_i'')$ (see [8, §3.3]), we find an isomorphism

$$\nu^* : \mathrm{H}^0(C, L \otimes \mathcal{I} \otimes \mathcal{O}_C) \cong \mathrm{H}^0(\tilde{C}, \omega_{\tilde{C}}(-p_1 - \ldots - p_k)),$$

hence

$$h^0(\tilde{C}, \omega_{\tilde{C}}(-p_1 - \ldots - p_k)) = h^0(C, L \otimes \mathcal{I} \otimes \mathcal{O}_C) \geqslant \dim(W) = g - 1. \tag{2}$$

This implies that the points $p_1, \ldots, p_k$ are all identified by the canonical map of $\tilde{C}$, which is possible only if either $k = 1$, or $k = 2$ and $\dim(|p_1 + p_2|) = 1$. We now prove that $\tilde{C}$ may not be hyperelliptic, hence the latter case does not occur.

By Theorem 8, if $\tilde{C}$ is hyperelliptic then $\dim(W) = g - 1 \leqslant 2$. This contradicts our assumptions that $3\delta \leqslant p$ and $p \geqslant 11$: indeed, as $g = p - \delta$ they imply that $g > 3$. Hence the only possibility left is that $k = 1$, which proves the claim. □

Note moreover that since $k = 1$, equality holds in (2).

Let $N_{C/S} \cong L|_C$ be the normal bundle of C in S. We have the exact sequence

$$0 \to N'_{C/S} \to N_{C/S} \to T^1_C \cong \mathcal{O}^2_{p_1} \oplus \bigoplus_{i=2}^{\delta} \mathcal{O}_{p_i} \to 0$$

where $N'_{C/S}$ is the *equisingular normal sheaf* of C in S, and one has $N'_{C/S} \cong N_{C/S} \otimes \mathcal{I}$. So $\mathrm{H}^0(C, N'_{C/S}) = \mathrm{H}^0(C, L \otimes \mathcal{I} \otimes \mathcal{O}_C)$ is the tangent space to the equisingular deformations of C in S.

We have $h^0(C, N_{C/S}) = p$ and, as we saw, $h^0(C, N'_{C/S}) = g - 1 = p - \delta - 1$. Thus the map

$$\mathrm{H}^0(C, N_{C/S}) \to T^1_C \tag{3}$$

is surjective, and $\mathrm{H}^1(C, N'_{C/S}) \cong \mathrm{H}^1(C, N_{C/S}) \cong \mathbf{C}$. Moreover the obstruction space to deformations of C in S, contained in $\mathrm{H}^1(C, N_{C/S})$, is zero as is well-known (see, e.g., [8, § 4.2]). This implies that, locally around C, $\overline{V}^{L,\delta}$ is the product of the equigeneric deformation spaces inside the versal deformation spaces of the singularities of C. By looking at the versal deformation space of a cusp (see, e.g., [12, p. 98]), we deduce that $\overline{V}^{L,\delta}$ has a double point at C with a single cuspidal sheet. This is a contradiction, because we assumed that both $\overline{V}$ and $\overline{V}^{L,\delta}_{\mathrm{st}}$ contain C. This contradiction proves that $p_g(C) = g$ cannot occur.

In conclusion we have proved that if $h_V = 1$ then $p_g(C)$ equals either $g-1$ or g, but both these possibilities lead to contradictions, hence $h_V \neq 1$.

4.2. **Proof that $h_V \neq 2$.** Still in the setup of Proposition 17, we now assume by contradiction that $h_V = 2$. Then $\dim(\Phi_V) = 2\delta - 2$. Let $\Delta \in \Phi_V$ be a general point. Again Δ can be seen as the limit of general 1-dimensional families $\{\Delta_t\}_{t\in\mathbf{D}}$, where $\mathbf{D}$ is a disk, and Δ_t is general in $S^{[\delta]}$ for $t \neq 0$. We consider the closure $\mathcal{L}_\Delta$ of the union of all $(p-3\delta)$-dimensional sublinear systems $\lim_{t\to 0}(\varphi^{-1}_{L,\delta}(\Delta_t)) \subset |L(-2\Delta)|$ as $\{\Delta_t\}_{t\in\mathbf{D}}$ varies among all families as above. Similarly to the case $h_V = 1$, we have:
(i) $\mathcal{L}_\Delta$ is contained in $\overline{V} \cap \overline{V}^{L,\delta}_{\mathrm{st}}$ and $\dim(\mathcal{L}_\Delta) = p - 3\delta + \varepsilon$, with $0 \leqslant \varepsilon \leqslant 1$;
(ii) the general curve in $\mathcal{L}_\Delta$ is singular at Δ but has singularities worse than only nodes at the points of Δ;
(iii) as Δ moves in a suitable dense open subset U of Φ_V, the union $\bigcup_{\Delta\in U} \mathcal{L}_\Delta$ describes a locally closed subset of dimension

$$\dim(\Phi_V) + \dim(\mathcal{L}_\Delta) = g - 2 + \varepsilon,$$

which is dense in an irreducible component W of $\overline{V} \cap \overline{V}^{L,\delta}_{\mathrm{st}}$.

If $\varepsilon = 1$, then $\dim(W) = g-1$ and the discussion goes as in the case $h_V = 1$. So we assume $\varepsilon = 0$, hence $\dim(W) = g-2$. Let C be the general curve in W. By Proposition 2, we have $g - 2 \leqslant p_g(C) \leqslant g$. We will prove that this cannot happen, thus proving that $h_V \neq 2$. The proof parallels the one for $h_V \neq 1$.

4.2.1. *Case $p_g(C) = g-2$.* By Proposition 5, C is a $(\delta+2)$-nodal curve, with two extra nodes $p_{\delta+1}, p_{\delta+2} \notin \Delta$ and W is dense in the closure of a component of $V^{L,\delta+2}$. By Proposition 6, locally around C there is only one smooth branch $\mathcal{V}$ of $\overline{V}^{L,\delta}$ containing W and such that when the general point of C' of $\mathcal{V}$ specializes at C, then set of δ nodes of C' specializes at Δ. This is a contradiction, because both $\overline{V}$ and $\overline{V}^{L,\delta}_{\mathrm{st}}$ contain W. Hence $p_g(C) = g-2$ cannot happen.

4.2.2. *Case $p_g(C) = g-1$.* In this case we have the two following disjoint possibilities for C:
(a) C has precisely one more singularity p_0 besides the ones in Δ;
(b) C has no singularities besides the ones in Δ, either an ordinary tacnode or a ramphoid cusp (with local equation $x^2 = y^{4+\varepsilon}$, $\varepsilon = 0$ or 1 respectively) at one of the points of Δ, and nodes or ordinary cusps at the other points of Δ.

Subcase (a). The points $p_0, \ldots, p_\delta$ are either nodes or cusps. Arguing as for Claim 18, we see that at most one of these points can be a cusp.

If C is $(\delta+1)$-nodal, then W sits in an irreducible component of $\overline{V}^{L,\delta+1}$, and we get a contradiction as in the proof of case $p_g(C) = g-1$ for $h_V = 1$.

If C is δ-nodal and 1-cuspidal, then again the map (3) is surjective and the deformation space of C is locally the product of the versal deformation spaces at $p_0, \ldots, p_\delta$. We then have the two following possibilities.

If p_0 is a node, then W sits in a $(g-1)$-dimensional irreducible variety W' parametrizing curves which are $(\delta-1)$-nodal and 1-cuspidal, such that when the general member of W' tends to C, its singularities tend to Δ. Moreover the map (3) is surjective for the general member of W'. Then W' should be contained in both $\overline{V}$ and $\overline{V}^{L,\delta}_{\mathrm{st}}$. On the other hand, as usual by now, $\overline{V}^{L,\delta}$ should be unibranched along W', a contradiction.

If p_0 is the cusp, then W sits in a $(g-1)$-dimensional irreducible component W' of $\overline{V}^{L,\delta+1}$, such that when the general member of W' tends to C, its singularities tend to $p_0, \dots, p_\delta$. By Corollary 7, W' should be contained in both $\overline{V}$ and $\overline{V}^{L,\delta}_{\mathrm{st}}$, leading again to a contradiction.

Subcase (b). Suppose the tacnode or ramphoid cusp is located at p_1, that $p_2, \dots, p_k$ are cusps, and $p_{k+1}, \dots, p_\delta$ are nodes: one has $1 \leqslant k \leqslant \delta$, and $k = 1$ (resp. δ) means that there is no cusp (resp. no node). If C has local equation $x^2 = y^{4+\varepsilon}$ around p_1, then the equisingular ideal I_{p_1} at p_1 is $(x, y^{3+\varepsilon})$ (see [8, §3]). As usual set $I = \prod_{i=1}^{\delta} I_{p_i}$ and let $\mathcal{I}$ be the corresponding ideal sheaf.

We have

$$h^0(C, N'_{C/S}) = h^0(C, N_{C/S} \otimes \mathcal{I}) \geqslant \dim(W) = g - 2. \tag{4}$$

Now we can look at $\mathrm{H}^0(C, N'_{C/S})$ as defining a linear series of *generalized divisors* on the singular curve C (see [13] and [8, §3.4]). Then $N'_{C/S} = N_{C/S} \otimes \mathcal{I} \cong \omega_C(-E)$ where E is the effective generalized divisor on C defined by the ideal sheaf $\mathcal{I}$ and (4) reads

$$h^0(C, \omega_C(-E)) \geqslant g - 2. \tag{5}$$

The subscheme of C defined by $\mathcal{I}$ has length $3 + \varepsilon$ at the tacnode, length 2 at each cusp and length 1 at the nodes, so that

$$\deg(E) = 3 + \varepsilon + 2(k-1) + \delta - k = \delta + k + 1 + \varepsilon.$$

By Riemann–Roch and Serre duality [13, Theorems 1.3 and 1.4], one has

$$h^0(C, \omega_C(-E)) = h^1(C, \mathcal{O}_C(E)) = h^0(C, \mathcal{O}_C(E)) - \deg(E) + p - 1 = h^0(C, \mathcal{O}_C(E)) + g - k - 2 - \varepsilon. \tag{6}$$

Next we argue as in the proof of [8, Prop. 4.8]. If $h^1(C, \mathcal{O}_C(E)) < 2$, then by (5) we have $g \leqslant 3$, which contradicts our assumptions that $3\delta \leqslant p$ and $\delta > 1$. If on the other hand $h^0(C, \mathcal{O}_C(E)) < 2$, then by (5) and (6) we have

$$g - 2 \leqslant h^1(C, \mathcal{O}_C(E)) \leqslant g - k - 1 - \varepsilon,$$

hence $\varepsilon = 0$ and $k = 1$, i.e., the singularities of C are precisely one ordinary tacnode and $\delta - 1$ nodes. There is then equality in both (4) and (5), hence once more (3) is surjective and the deformation space of C is locally the product of the versal deformation spaces at $p_1, \dots, p_\delta$. By looking at the versal deformation space of a tacnode (see [2, p. 181]) we see that W is contained in $\overline{V}^{L,\delta}$ which should be unibranched along W, a contradiction.

So one has necessarily that $h^i(C, \mathcal{O}_C(E)) \geqslant 2$, for $i = 1, 2$. Then, since $\mathrm{Cliff}(C) = \lfloor \frac{p-1}{2} \rfloor$ by Theorem 10, one has

$$p + 1 - h^0(C, \mathcal{O}_C(E)) - h^1(C, \mathcal{O}_C(E)) = \deg(E) - 2h^0(C, \mathcal{O}_C(E)) + 2 \geqslant \lfloor \tfrac{p-1}{2} \rfloor$$

hence

$$g - 2 \leqslant h^1(C, \mathcal{O}_C(E)) \leqslant p + 1 - \lfloor \tfrac{p-1}{2} \rfloor - h^0(C, \mathcal{O}_C(E)) \leqslant p - 1 - \lfloor \tfrac{p-1}{2} \rfloor = \lceil \tfrac{p-1}{2} \rceil.$$

Plugging in the inequality $3\delta \leqslant p$, one finds

$$\frac{2}{3}p - 2 \leqslant p - \delta - 2 = g - 2 \leqslant \lceil \frac{p-1}{2} \rceil \leqslant \frac{p}{2} \tag{7}$$

which implies $p \leqslant 12$, hence $p = 11$ or 12. Case $p = 11$ is impossible by (7), since there is no integer between the two extremes in (7). If $p = 12$, then (7) implies $g = 8$, hence $\delta = 4$, which is excluded by assumption. Hence subcase (b) is impossible. This concludes the proof that $p_g(C) \neq g - 1$.

4.2.3. *Case $p_g(C) = g$.* As in the case $h_V = 1$, C is singular only at $\Delta = p_1 + \ldots + p_\delta$, having only nodes and simple cusps, and it must have at least one cusp.

Claim 19. *C has at most two cusps.*

Proof of the Claim. The proof goes as the one of Claim 18, from which we keep the notation. If C has cusps at $p_1, \ldots, p_k$, we have

$$h^0(\tilde{C}, \omega_{\tilde{C}}(-p_1 - \ldots - p_k)) \geqslant \dim(W) = g - 2. \tag{8}$$

We argue by contradiction and assume $k \geqslant 3$. As in the proof of Claim 18, we see that $\tilde{C}$ is not hyperelliptic: this would imply by Theorem 8 that $g - 2 = \dim(W) \leqslant 2$, hence $p = 6$ and $g = 4$; but in this case $\delta = 2$ and since $k \leqslant \delta$ we are out of the range $k \geqslant 3$.

The only other possibility is that $\tilde{C}$ is trigonal, $k = 3$, and $\dim(|p_1+p_2+p_3|) = 1$. In this case, one would have $g-2 = \dim(W) \leqslant 4$ by Theorem 8, which together with the inequality $p \geqslant 3\delta$ implies that $p \leqslant 9$: This is in contradiction with our assumptions. It is thus impossible that $k \geqslant 3$, and the Claim is proved. ☐

By Claim 19, we have only the following two mutually disjoint possibilities:
(a) C has precisely one cusp at p_1, and $h^0(\tilde{C}, \omega_{\tilde{C}}(-p_1)) = g - 1 > g - 2 = \dim(W)$;
(b) C has precisely two cusps at p_1 and p_2, and $h^0(\tilde{C}, \omega_{\tilde{C}}(-p_1 - p_2)) = g - 2 = \dim(W)$.

Subcase (a). We have $h^0(C, N'_{C/S}) = h^0(\tilde{C}, \omega_{\tilde{C}}(-p_1)) = g - 1$, hence the map (3) is surjective. This implies as in the case $h_V = 1$ and $p_g = g$ that W is contained in a subvariety W' of dimension $g - 1$ contained in $\overline{V}^{L,\delta}$, whose general point corresponds to a curve which has $\delta - 1$ nodes and one cusp, and, as in the proof of case $h_V = 1$, $\overline{V}^{L,\delta}$ is unibranched locally at any point of W' corresponding to such a curve for which the map (3) is surjective. This contradicts the fact that W is an irreducible component of $\overline{V} \cap \overline{V}^{L,\delta}_{\mathrm{st}}$.

Subcase (b). In this case W is dense in the equisingular deformation locus of C and again the map (3) is surjective. This again implies that $\overline{V}^{L,\delta}$ is unibranched locally around C, which leads to a contradiction.

This concludes the proof that $h_V \neq 2$, hence also the proof of Proposition 17.

5. Proof of Irreducibility if $p > 4\delta - 4$

In this section we conclude the proof of Theorem 1. So let (S, L) be a primitively polarized $K3$ surface of genus $p \geqslant 11$ such that $\mathrm{Pic}(S) = \mathbf{Z}[L]$, and δ be a non-negative integer such that $4\delta - 3 \leqslant p$.

These assumptions imply that $p \geqslant 3\delta$, so that the notion of standard component makes sense, and the Severi variety $V^{L,\delta}$ has a unique standard component by Proposition 16. We assume by contradiction that $V^{L,\delta}$ is not irreducible: this means that there exists a non-standard component V of the Severi variety $V^{L,\delta}$, and we shall see this contradicts the inequality $p > 4\delta - 4$.

Let $h = h_V$. If $\delta \leqslant 1$, then Theorem 1 is trivial; else we're in the range of application of Proposition 17 (note that the case $(p, \delta) = (12, 4)$ is excluded by the hypothesis $p \geqslant 4\delta - 3$), hence $h \geqslant 3$.

Consider a general member $C \in V$, and let $\Delta = \{p_1, \ldots, p_\delta\}$ be the set of its nodes. Let $\nu : \tilde{C} \to C$ be the normalization map, and for all $i = 1, \ldots, \delta$, p'_i and p''_i the two antecedents of p_i by ν. We consider the divisor $\tilde{\Delta} = \sum_{i=1}^{\delta}(p'_i + p''_i)$ on $\tilde{C}$.

Lemma 20. *The complete linear series $|\tilde{\Delta}|$ is a $g^h_{2\delta}$.*

Proof. One has $h^1(\tilde{\Delta}) = p - 3\delta + h$ by Lemma 12, and then the result follows from the Riemann–Roch formula. ☐

Conclusion of the proof of Theorem 1. We maintain the above setup. We first apply Theorem 9: Let $g = p - \delta$ denote the geometric genus of C, and set

$$\alpha = \left\lfloor \frac{gh + (2\delta - h)(h - 1)}{2h(2\delta - h)} \right\rfloor = \left\lfloor \frac{g}{2(2\delta - h)} + \frac{h - 1}{2h} \right\rfloor;$$

the existence of a $g^h_{2\delta}$ on $\tilde{C}$ implies the inequality

$$\alpha hg + \alpha h(\alpha h + 1) \leqslant \delta(2\alpha^2 h + 2\alpha + 1). \tag{9}$$

Let us also apply Theorem 8: The existence of a $g^h_{2\delta}$ on $\tilde{C}$ induces the existence of a family of dimension $2(h-1)$ of $g^1_{2\delta}$'s on $\tilde{C}$, parametrizing the lines in the $g^h_{2\delta}$, so it holds that

$$\dim(V) + \dim(G^1_{2\delta}(\tilde{C})) \geqslant g + 2(h-1),$$

which implies by Theorem 8 that

$$g \leqslant 2(2\delta - h). \tag{10}$$

Inequality (10) implies that

$$\alpha = \left\lfloor \frac{g}{2(2\delta - h)} + \frac{h-1}{2h} \right\rfloor \leqslant \left\lfloor 1 + \frac{1}{2} \right\rfloor = 1.$$

Let us now show by contradiction that $\alpha = 1$. If $\alpha \leqslant 0$, then

$$\frac{gh + (2\delta - h)(h-1)}{2h(2\delta - h)} < 1 \iff g < (2\delta - h)(1 + \frac{1}{h}) \iff p < \delta(3 + \frac{2}{h}) - h - 1;$$

plugging in the inequality $h \geqslant 3$, we get that $\alpha \leqslant 0$ implies $p < \frac{11}{3}\delta - 4$, in contradiction with our assumption that $p > 4\delta - 4$. Hence $\alpha = 1$.

Therefore, (9) gives the inequalities

$$hg + h(h+1) \leqslant \delta(2h+3) \iff p \leqslant \delta(3 + \frac{3}{h}) - h - 1.$$

Taking into account the fact that $h \geqslant 3$, this implies that $p \leqslant 4\delta - 4$. In conclusion, the existence of a non-standard component of $V^{L,\delta}$ is in contradiction with the inequality $p > 4\delta - 4$. □

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