

K3 curves with index $k > 1$

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*Dedicated to Fabrizio Catanese on the occasion of his 70th birthday,
with esteem, admiration and friendship.*

Abstract. Let \mathcal{KC}_g^k be the moduli stack of pairs (S, C) with S a *K3* surface and $C \subseteq S$ a genus g curve with divisibility k in $\text{Pic}(S)$. In this article we study the forgetful map $c_g^k : (S, C) \mapsto C$ from \mathcal{KC}_g^k to \mathcal{M}_g for $k > 1$. First we compute by geometric means the dimension of its general fibre. This turns out to be interesting only when S is a complete intersection or a section of a Mukai variety. In the former case we find the existence of interesting Fano varieties extending C in its canonical embedding. In the latter case this is related to delicate modular properties of the Mukai varieties. Next we investigate whether c_g^k dominates the locus in \mathcal{M}_g of k -spin curves with the appropriate number of independent sections. We are able to do this only when S is a complete intersection, and obtain in these cases some classification results for spin curves.

1 – Introduction

We are interested in the forgetful maps $c_g^k : \mathcal{KC}_g^k \rightarrow \mathcal{M}_g$ with $k > 1$ and $g = 1 + (g_1 - 1)k^2$ for some integer $g_1 > 1$, in the following notation (valid also in case $k = 1$):

- \mathcal{K}_g^k is the moduli stack of polarised *K3* surfaces of genus g and index k , i.e., pairs (S, L) such that S is a smooth *K3* surface and L is an ample, globally generated line bundle on S with $L^2 = 2g - 2$, such that $L = kL_1$ with L_1 a primitive line bundle on S ; note that (S, L_1) is a member of $\mathcal{K}_{g_1}^1$, which we usually denote by $\mathcal{K}_{g_1}^{\text{prim}}$;
- \mathcal{KC}_g^k is the moduli stack of pairs (S, C) with C a smooth curve on S and $(S, \mathcal{O}_S(C)) \in \mathcal{K}_g^k$;
- \mathcal{M}_g is the moduli space of curves of genus g .

Specifically our goal is to describe the general fibre and the image of the maps c_g^k .

In the primitive case $k = 1$ the situation is rather well understood and should be well-known by now, so we don't dwell on this here (the interested reader may consult, e.g., [8, §2] to find the relevant references), and concentrate on the case $k > 1$.

If C is a smooth curve of genus $g \geq 11$ and Clifford index $\text{Cliff}(C) > 2$ lying on a *K3* surface, it follows from work of Wahl [30], Arbarello–Bruno–Sernesi [2], and eventually the authors and Sernesi [8] that the fibre of the map

$$c_g : [S, C] \in \coprod_k \mathcal{KC}_g^k \mapsto [C] \in \mathcal{M}_g$$

(which is the aggregation of the maps c_g^k for all k such that there exists $g_1 > 1$ with $g = 1 + (g_1 - 1)k^2$) is essentially a linear space, of dimension $\nu = \text{cork}(\Phi_C) - 1$ (where Φ_C is the Gauss–Wahl map of C , see [8] for background and references), and there exists an arithmetically Gorenstein normal variety $X \subseteq \mathbf{P}^{g+\nu}$ of dimension $\nu + 2$ with $\omega_X = \mathcal{O}_X(-\nu)$, having as a linear section the curve C in its canonical embedding, as well as all *K3* surfaces S containing C , embedded by the linear system $|C|_S$; we call such an X the *universal extension* of C . (Recall that an *r-extension* (or simply extension, if $r = 1$) of a projectively embedded variety $V \subseteq \mathbf{P}^N$ is a variety $X \subseteq \mathbf{P}^{N+r}$ having V as a linear section; an extension is said to be trivial if it is a cone; again, see [8] for background and references).

We call *K3 curve of index k* (and genus g) a curve C which is in the image of the map c_g^k . For general such curves with $k > 1$ the corank of the Gauss–Wahl map has been computed by cohomological methods in [9], with some exceptions for $g_1 = 2$ documented in a corrigendum. In this article we compute by geometric considerations the dimension of the general fibre of the maps c_g^k , $k > 1$, thus concluding the task begun in [7] with the case $g_1 = 2$.

It turns out to be interesting only for low values of g_1 , for which the *K3* surfaces are either complete intersections or sections of more general homogeneous varieties found by Mukai and thence called *Mukai varieties* (see Section 4). Indeed, it follows from Prokhorov’s bound on the genus of Gorenstein Fano threefolds [29] that the maps c_g^k are generically injective if $g > 37$, see [8, Cor. 2.10].

(1.1) Proposition. *Let $k > 1$ and $g_1 > 1$ be integers, and set $g = 1 + (g_1 - 1)k^2$. Let $[C]$ be very general in the image of c_g^k .*

(1.1.1) *We have $\text{Cliff}(C) > 2$ and $g(C) \geq 11$ unless we are in one of the following cases:*

- $g_1 = 2$ and $k \leq 3$;
- $g_1 = 3$ and $k = 2$.

(1.1.2) *We have $g(C) > 37$ unless we are in one of the following cases:*

g_1	2	3	4	5	6	7	8	9	10
$k \leq$	6	4	3	3	2	2	2	2	2

Part (1.1.1) follows from Lemma (2.2) and it tells us that except for the exceptional cases $(g_1, k) \in \{(2, 2), (2, 3), (3, 2)\}$ the general theory of [30, 2, 8] applies. In the exceptional cases we have been able to analyze the situation by hand in [7]; we obtain a little more information here. Part (1.1.2) follows from a direct computation, and tells us the values of (g_1, k) for which it is possible that c_g^k has positive dimensional general fibre. We now give a brief summary of the results we obtain in these cases.

(1.2) Complete intersections (Section 3). For $g_1 \in \{3, 4, 5\}$ the general membre C of the image of c_g^k , with $g = 1 + (g_1 - 1)k^2$, has a model as a complete intersection in \mathbf{P}^{g_1} . This allows for a direct description of the fibre $(g_g^k)^{-1}(C)$; in particular we can compute its dimension, which gives $\text{cork}(\Phi_C)$, thus recovering the values obtained in [9].

In all cases, we provide a description of the universal extension X as a complete intersection in a weighted projective space, embedded by a suitable divisor in $\text{Pic}(X)$ of the anticanonical class. In fact this also works in the exceptional cases of (1.1.1), in which cases we obtain interesting examples showing the sharpness of Lvovski’s main theorem in [22]. This construction has been inspired by the corresponding one in case $g_1 = 2$, which has been communicated to us by Totaro (see [7, (4.8)]).

(1.3) Mukai models (Section 4). For $g_1 \in \{6, 7, 8, 9, 10\}$ the general membre C of the image of c_g^k , with $g = 1 + (g_1 - 1)k^2$, has a model as a complete intersection in a Mukai homogeneous variety M_{g_1} . For $g_1 \neq 6$ in this range, we show by geometric considerations that the maps c_g^k are generically injective for all $k > 1$. For $g_1 = 6$ and $k = 2$ the dimension of the general fibre is 1, see (4.5). On our way we obtain results about the related question whether all models of C are conjugate under the action of $G = \text{Aut}(M_{g_1})$, see Conjecture (4.3) and summary of results in (4.4).

We also show that in cases (a) $k = 1$ and $g_1 > 6$, and (b) $k = 2$ and $g_1 = 6$, the Mukai variety is the universal extension of the general membre of $\text{im}(c_g^k)$; this was certainly a natural thing to guess, but as far as we know this had not been proved yet.

A more detailed but still synthetic description of our results about Mukai varieties and their sections is given in subsection 4.1.

Our study (in Section 5) of the image of the maps c_g^k on the other hand is based on the following observation.

(1.4) K3 curves of index k and k -spin curves. For $g = 1 + k^2(g_1 - 1)$, the map c_g^k factorizes through

$$(S, C) \in \mathcal{KC}_g^k \mapsto (C, \mathcal{O}_S(\frac{1}{k}C)|_C) \in \mathcal{S}_g^{\frac{1}{k}, g_1},$$

where

— $\mathcal{S}_g^{\frac{1}{k}, g_1}$ is the moduli stack of k -spin curves of genus g having at least $g_1 + 1$ independent sections, i.e., pairs (C, θ) with C a smooth genus g curve and θ a line bundle on C such that $k\theta = K_C$ and $h^0(\theta) \geq g_1 + 1$.

A line bundle L such that $2L = K_C$ is called a theta-characteristic (hence the notation). Moreover we denote by

— $\mathcal{T}_g^{\frac{1}{k}, g_1}$ the image of $\mathcal{S}_g^{\frac{1}{k}, g_1}$ in \mathcal{M}_g by the (finite) forgetful map $(C, L) \mapsto C$, so that

$$(1.4.1) \quad \text{im}(c_g^k) \subseteq \mathcal{T}_g^{\frac{1}{k}, g_1}.$$

The question underlying our study in Section 5 is whether equality holds in (1.4.1).

For $k = 2$ there is an expected codimension for $\mathcal{T}_g^{\frac{1}{2}, g_1}$ in \mathcal{M}_g , viz. $\binom{g_1+1}{2}$; more precisely, the following holds.

(1.5) Theorem ([28] and [17, Cor. 1.11]). *Let $(C, \theta) \in \mathcal{S}_g^{\frac{1}{2}, g_1}$ be such that $h^0(\theta) = g_1 + 1$. Then for any deformation (C_t, θ_t) of (C, θ) one has $h^0(\theta_t) \equiv h^0(\theta) \pmod{2}$, and the codimension c of $\mathcal{S}_g^{\frac{1}{2}, g_1}$ in $\mathcal{S}_g^{\frac{1}{2}}$ at (C, θ) is $c \leq \binom{g_1+1}{2}$.*

However several examples are known of superabundant components of $\mathcal{S}_g^{\frac{1}{2}, g_1}$, see e.g., [13]. Here we observe that K3 curves of index 2 provide infinitely many such examples.

(1.6) Observation. We have the following remarkable formula:

$$(1.6.1) \quad \begin{aligned} \dim(\mathcal{KC}_g^2) - \text{expdim}(\mathcal{T}_g^{\frac{1}{2}, g_1}) &= (19 + g) - (3g - 3) + \binom{g_1+1}{2} \\ &= \frac{1}{2}(g_1 - 7)(g_1 - 8). \end{aligned}$$

Since for $g_1 \geq 7$ the map c_g^2 is generically injective (Corollary (4.16)), this formula implies that the image of c_g^2 lies in a superabundant component of $\mathcal{S}_g^{\frac{1}{2}, g_1}$ as soon as $g_1 \geq 9$.

Moreover this difference equals the dimension of the general fibre of c_g^2 for $3 \leq g_1 \leq 6$ (see Section 3 and paragraph (4.5)). For $3 \leq g_1 \leq 5$ the closure of $\text{im}(c_g^2)$ is an irreducible component of $\mathcal{T}_g^{\frac{1}{2}, g_1}$ (Proposition (5.3)) which thus has the expected dimension.

This makes the question whether the closure of $\text{im}(c_g^2)$ is an irreducible component of $\mathcal{T}_g^{\frac{1}{2}, g_1}$ for $g_1 = 6, 7, 8$ rather intriguing.

In the complete intersection cases, we can prove equality in (1.4.1) (see Proposition (5.3) below), and then it is possible to check that $\mathcal{T}_g^{\frac{1}{2}, g_1}$ has a component of the expected dimension.

(1.7) Classification of K3-like spin curves (Section 5). We make a thorough analysis of the components of $\mathcal{S}_g^{\frac{1}{2}, g_1}$ in the cases $g_1 = 3$ and $g_1 = 4$ (Theorems (5.10) and (5.12)), and give some partial information for $g_1 = 5$ (see Proposition (5.14)). In the absence of an estimate for the codimension of $\mathcal{T}_g^{\frac{1}{k}, g_1}$ in \mathcal{M}_g for $k \geq 3$, the cases $k \geq 3$ are more complicated, and we give only some partial answers for $k = 3$ and $3 \leq g_1 \leq 4$ (see Section 5.2).

This turns out to be unexpectedly involved, and is the occasion of discovering several interesting families of curves on rational normal scrolls in \mathbf{P}^{g_1} for $g_1 = 3, 4, 5$.

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2 – Preliminaries

(2.1) Notation. When the context is clear, we use the shorthand $h_n(k)$ for $h^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(k))$.

(2.2) Lemma. *Let $(S, L_1) \in \mathcal{K}_{g_1}^{\text{prim}}$ be very general, so that $\text{Pic}(S) = \mathbf{Z}\langle L_1 \rangle$. For smooth $C \in |kL_1|$, $k > 1$, we have $\text{Cliff}(C) = (2g_1 - 2)(k - 1) - 2$.*

Proof. Let C be a smooth membre of $|kL_1|$; it has genus $g = 1 + k^2(g_1 - 1)$. By [16], the Clifford index of C is either $\lfloor \frac{g-1}{2} \rfloor$ or it is computed by the restriction of a line bundle on S . Since $\text{Pic}(S) = \mathbf{Z}\langle L_1 \rangle$, we have to consider those $lL_1 \in \text{Pic}(S)$ such that $h^0(lL_1|_C) \geq 2$ and $h^0(K_C - lL_1|_C) \geq 2$, which amounts to the condition that $1 \leq l < k$. One computes

$$\text{Cliff}(lL_1|_C) = \deg(lL_1|_C) - 2(h^0(lL_1|_C) - 1) = (2g_1 - 2)l(k - l) - 2,$$

the minimal value of which for integral $l \in [1, k - 1]$ is $(2g_1 - 2)(k - 1) - 2$, indeed less than $\lfloor \frac{g-1}{2} \rfloor$. \square

In this article we use freely the notion of ribbon over a curve and its relation with the Gauss–Wahl map; we refer to [8, §4] and [7, §1.2] for the necessary background.

(2.3) Proposition. *Let C be a curve with $\text{Cliff}(C) > 2$. For any $v \in \ker(\text{T}\Phi_C)$ seen as a ribbon over $C \subseteq \mathbf{P}^{g-1}$, there exists at most one surface extension $S \subseteq \mathbf{P}^g$ up to projectivities (i.e., the integral of v , if it exists, is unique).*

Proof. It is given in [8, Remark 4.8]. There one finds the additional assumption that $g \geq 11$, but this is useless as far as unicity is concerned (it is needed only to ensure the existence of an integral).

What is really needed is that $C \subseteq \mathbf{P}^{g-1}$ is defined by quadratic equations with linear syzygies, which is ensured by the condition on the Clifford index. Under this assumption, one can indeed work out Wahl’s deformation construction, and then one sees [8, §4.9] that any two extensions differ by an element of $H^0(N_{C/\mathbf{P}^{g-1}}(-2))$. The conclusion stems from the fact that the latter cohomology group is 0 if $C \subseteq \mathbf{P}^{g-1}$ is defined by quadratic equations with linear syzygies (see [8, Lemma 3.6]). \square

(2.4) Proposition.

(2.4.1) Let (S, L) be a polarized K3 surface. Then the automorphisms group of (S, L) is finite.

(2.4.2) Let S be a projective K3 surface with $\text{Pic}(S) = \mathbf{Z}\langle L_1 \rangle$. Then $\text{Aut}(S)$ is trivial if $(L_1)^2 > 2$, and isomorphic to $\mathbf{Z}/2$ if $(L_1)^2 = 2$.

See [18, Cor. 15.2.12] for (2.4.2), and [18, Chap. 5 Prop. 3.3] for (2.4.1).

3 – Complete intersections

These are the cases $g_1 = 3, 4, 5$.

3.1 – Useful results

(3.1) Theorem. Let Z, Z' be two complete intersections in some projective space \mathbf{P}^n .

(3.1.1) The two polarized schemes $(Z, \mathcal{O}(1))$ and $(Z', \mathcal{O}(1))$ are isomorphic if and only if they are conjugate under the action of $\text{PGL}(n+1)$ on \mathbf{P}^n .

(3.1.2) If Z is non-degenerate and smooth (resp. general), then $\text{Aut}(Z, \mathcal{O}(1))$ is finite (resp. trivial) unless Z is a quadric hypersurface (resp. unless it is either a quadric hypersurface or the complete intersection of two quadrics).

To be noted that $\text{Aut}(Z, \mathcal{O}(1))$ is the stabilizer of Z in $\text{PGL}(n+1)$. For (3.1.1), see [4, Prop. 2.1]. Part (3.1.2) in the smooth case is [5, Thm. 1.6], and in the general case it is [6, Thm.1.3].

3.2 – Fibres and universal extensions

(3.2) $g_1 = 3$: dimension of the general fibre. Let $k \in \mathbf{N} \setminus \{0\}$, and consider the forgetful map $c_g^k : \mathcal{K}\mathcal{C}_{1+2k^2}^k \rightarrow \mathcal{M}_{1+2k^2}$, $g = 1 + 2k^2$. Its image contains $\mathcal{I}\mathcal{C}(4, k)$ as a dense subset (the locus of smooth complete intersection curves of type $(4, k)$ in \mathbf{P}^3).

Consider a general curve $[C] \in \mathcal{I}\mathcal{C}(4, k)$. Up to the action of $\text{PGL}(4)$, there is a unique¹ incarnation of C as a complete intersection in \mathbf{P}^3 , which we call C as well. The linear system of quartics containing C has dimension

$$h^0(\mathcal{I}\mathcal{C}(4)) - 1 = h_3(4 - k)$$

and dominates the fibre of c_g^k over C . In fact, the quotient of this linear system by the stabilizer of C in $\text{PGL}(4)$ is birational to this fibre. When $k > 1$ this stabilizer is trivial, and when $k = 1$ it is the affine group $\text{Aff}(3)$ which has dimension 4 (and indeed $h_3(3) - 4 = 16$, the dimension of the general fibre of c_3^1 which is dominant). We thus find the values indicated in the table below for the dimension of the general fibre of c_g^k .

$g_1 = 3$				
k	2	3	4	≥ 5
g	9	19	33	≥ 51
dim(fibre)	10	4	1	0

When $k \geq 3$ we have $g \geq 11$ and $\text{Cliff} > 2$ so we may apply the theory in [8], and this tells us the value of $\text{cork}(\Phi_C)$ for the general $[C] \in \mathcal{I}\mathcal{C}(4, k)$, as well as the number of times C may be extended in its canonical embedding. When $k \leq 2$ however there is little we can do but observe the coincidence in dimension with $\sum_{l \geq 1} h^0(N(-l))$, for C does not enjoy property N_2 (even in its canonical embedding).

¹unique indeed if we consider C as a curve with a k -spin structure, but in fact a finite number if we only take it as a naked curve

(3.3) $g_1 = 3$: universal extensions. We now describe the universal extensions of canonical curves in the image of c_g^k for $2 \leq k \leq 4$ (when $k > 4$ the universal extension is the unique quartic surface $S \subseteq \mathbf{P}^3$ containing C , re-immersed in \mathbf{P}^g by the linear system $|\mathcal{O}_S(k)|$). They are similar to the extensions found by Totaro (private communication) in the case $g_1 = 2$ (see [7, (4.8)] and (3.7) below). The case $k = 2$ is somewhat exceptional, as $h^0(N_{C/\mathbf{P}^9}(-2)) = 1$ hence there is no universal extension guaranteed by [8] (see also Remark (3.4)); still it works out as the others.

— $k = 4$: the universal extension is \mathbf{P}^3 , in its anticanonical embedding $v_4(\mathbf{P}^3) \subseteq \mathbf{P}^{34}$; it may also be presented as a quartic hypersurface $X_{4,4} \subseteq \mathbf{P}(1^4, 4)$ in its anticanonical embedding, in \mathbf{P}^{34} by $|\mathcal{O}(4)|$ ($|\mathcal{O}(4)|$ embeds $\mathbf{P}(1^4, 4)$ in \mathbf{P}^{35} as the cone over the 4-Veronese $v_4(\mathbf{P}^3) \subseteq \mathbf{P}^{34}$).

— $k = 3$: the universal extension has dimension 6, embedded in \mathbf{P}^{23} with index 4; it is again a quartic $X_{4,3}$, this time in $\mathbf{P}(1^4, 3^4)$; the anticanonical of $X_{4,3}$ is $\mathcal{O}(12)$, correspondingly the embedding with index 4 is given by (weighted) cubics (and indeed $h^0(\mathcal{O}_{X_{4,3}}(3)) = 24$). Specifically the equation is as follows. Assume $C \subseteq \mathbf{P}^4$ is the complete intersection of the quartic ($f = 0$) and the cubic ($g = 0$). Then $X_{4,3}$ is defined by

$$f(\mathbf{x}) + \mathbf{x} \cdot \mathbf{y} = f(x_0, \dots, x_3) + x_0y_0 + \dots + x_3y_3 = 0,$$

where the x_i 's and y_i 's are the homogeneous coordinates of weight 1 and 3 respectively. Indeed the general quartic surface in \mathbf{P}^3 containing C has an equation of the form $f + (a_0x_0 + \dots + a_3x_3)g = 0$, hence can be obtained by cutting $X_{4,3}$ by the 4 cubic equations

$$y_0 - a_0g(\mathbf{x}) = \dots = y_3 - a_3g(\mathbf{x}) = 0.$$

One gets C by cutting by the fifth cubic $g(\mathbf{x}) = 0$.

— $k = 2$: we can package together all the quartic surfaces containing a complete intersection curve of type $(2, 4)$ as a (weighted) quartic hypersurface $X_{4,2}$ in $\mathbf{P}(1^4, 2^{10})$. This has anticanonical $\mathcal{O}(20)$, hence may be embedded with index 10 by (weighted) quadrics; one has $h^0(\mathcal{O}_{X_{4,2}}(2)) = 20$, so one ends up with $X_{4,2} \subseteq \mathbf{P}^{19}$ as required. Take C the complete intersection of the quartic ($f = 0$) and the quadric ($g = 0$). Then the equation of $X_{4,2}$ is

$$f(\mathbf{x}) + (x_ix_j)_{0 \leq i \leq j \leq 3} \cdot \mathbf{y} = f(\mathbf{x}) + x_0^2y_0 + x_0x_1y_1 + \dots + x_3^2y_{10} = 0.$$

The general quartic homogeneous polynomial vanishing along C is of the form $f + qg$ for some quadratic form $q(\mathbf{x}) = a_0x_0^2 + \dots + a_{10}x_3^2$, and one gets the corresponding surface in \mathbf{P}^3 by cutting by the 10 degree 2 equations

$$y_0 - a_0g(\mathbf{x}) = \dots = y_{10} - a_{10}g(\mathbf{x}) = 0.$$

(3.4) Remark. The case $k = 2$ above provides interesting insight into Lvovski's main theorem in [22]. He proves that if $X \subseteq \mathbf{P}^n$ is a smooth, non-degenerate variety which is not a quadric, then X cannot be extended non-trivially more than $\alpha(X)$ times if $\alpha(X) < n$, with $\alpha(X) = h^0(N_{X/\mathbf{P}^n}(-1)) - n - 1$.

Consider a smooth complete intersection curve $C \subseteq \mathbf{P}^3$ of type $(2, 4)$, in its canonical embedding in \mathbf{P}^8 . The construction above proves that C is extendable 11 times, to the 12-fold $X_{4,2} \subseteq \mathbf{P}^{19}$, whereas

$$\alpha(C) = \text{cork}(\Phi_C) = 10.$$

Thus $C \subseteq \mathbf{P}^8$ is extendable strictly more than $\alpha(C)$ times; indeed the assumption " $\alpha(C) < 8$ " of Lvovski's theorem does not hold. Note moreover that $C \subseteq \mathbf{P}^8$ is not a complete intersection. This shows (a) that the condition " $\alpha(X) < n$ " is stronger than " $X \subseteq \mathbf{P}^n$ is not a complete

intersection”, and (b) that one may not replace the assumption “ $\alpha(X) < n$ ” by “ $X \subseteq \mathbf{P}^n$ is not a complete intersection” in Lvovski’s theorem.

On the other hand it follows from Lvovski’s theorem that the two Fano’s $X_{4,4} \subseteq \mathbf{P}^{34}$ and $X_{4,3} \subseteq \mathbf{P}^{23}$ constructed in (3.3) are not extendable.

The other complete intersection cases are similar, so we will give less details.

(3.5) $g_1 = 4$. Let $k \in \mathbf{N} \setminus \{0\}$, and consider the forgetful map $c_g^k : \mathcal{KC}_{1+3k^2}^k \rightarrow \mathcal{M}_{1+3k^2}$, $g = 1 + 3k^2$. Its image contains $\mathcal{IC}(2, 3, k)$ as a dense subset (the locus of smooth complete intersection curves of type $(2, 3, k)$ in \mathbf{P}^4).

$g_1 = 4$			
k	2	3	≥ 4
g	13	28	≥ 49
dim(fibre)	6	1	0
threefold	cubic	quadric	no

For $k = 2$ we consider complete intersection curves C of type $(2, 2, 3)$ in \mathbf{P}^4 . Then $h^0(\mathcal{I}_C(2)) = 2$ and $h^0(\mathcal{I}_C(3)) = 11$; there is a \mathbf{P}^1 of quadrics containing C , and for each of these the \mathbf{P}^{10} of cubics containing C cuts out a \mathbf{P}^5 of $K3$ ’s complete intersection of type $(2, 3)$. Eventually the fibre $(c_{13}^2)^{-1}(C)$ has dimension 6. The universal extension is a weighted complete intersection of type $(2, 3)$, $X_{(2,3),2} \subseteq \mathbf{P}(1^5, 2^6)$; it has anticanonical $\mathcal{O}(6 \cdot 2)$, hence it is embedded with index 6 in \mathbf{P}^{19} by the linear system of weighted quadrics. The equations of $X_{(2,3),2}$ are as follows: let $f = g = g' = 0$ be the equations of C , with $\deg(f) = 3$ and $\deg(g) = \deg(g') = 2$; then $X_{(2,3),2}$ may be defined by the equations

$$\begin{cases} g(\mathbf{x}) + y_5 = 0 \\ f(\mathbf{x}) + x_0 \cdot y_0 + \cdots + x_4 \cdot y_4 = 0. \end{cases}$$

with the x_i ’s and y_i ’s of weights 1 and 2 respectively. A complete intersection $K3$ in \mathbf{P}^4 containing C is the intersection of a quadric of the form $g + a_6 g' = 0$ and a cubic of the form $f + (a_0 x_0 + \cdots + a_4 x_4) g'$, which is obtained by cutting $X_{(2,3),2}$ by 6 quadric equations $y_5 - a_6 g' = 0$, etc.

For $k = 3$, a smooth complete intersection C of type $(2, 3, 3)$ is contained in a unique quadric, and there is a pencil of cubics containing C and independent of the quadric, hence a pencil of $K3$ surfaces containing C . The universal extension is a 3-fold, namely the quadric in \mathbf{P}^4 , in its anticanonical embedding $v_3(X_2) \subseteq \mathbf{P}^{29}$. This may also be presented in a uniform fashion with the other extensions: a quadric in \mathbf{P}^4 is also a complete intersection $X_{(2,3),3}$ of type $(2, 3)$ in $\mathbf{P}(1^5, 3)$.

It follows from Lvovski’s theorem that $X_{(2,3),2} \subseteq \mathbf{P}^{19}$ and $X_{(2,3),3} \subseteq \mathbf{P}^{29}$ are not extendable.

(3.6) $g_1 = 5$. In this case we consider complete intersection curves of type $(2^3, k)$ in \mathbf{P}^5 .

$g_1 = 5$		
k	2	≥ 3
g	17	≥ 37
dim(fibre)	3	0

For $k = 2$ the curve C is a complete intersection of type (2^4) in \mathbf{P}^5 . The $K3$ surfaces extending it correspond to the planes in the 3-space $\mathbf{P}(H^0(\mathcal{I}_C(2)))$, the family of which is

a \mathbf{P}^3 . The universal extension is a Fano 5-fold $X_{(2^3),2}$ complete intersection of type (2^3) in $\mathbf{P}(1^6, 2^3)$, with anticanonical $\mathcal{O}(3 \cdot 2)$ and embedded by quadrics in \mathbf{P}^{20} . For C with equations $g_0 = g_1 = g_2 = g_3 = 0$, where g_i are forms of degree 2 for $0 \leq i \leq 3$, we may take for $X_{(2^3),2}$ the equations

$$\begin{cases} g_0(\mathbf{x}) + y_0 = 0 \\ g_1(\mathbf{x}) + y_1 = 0 \\ g_2(\mathbf{x}) + y_2 = 0, \end{cases}$$

with the x_i 's and y_i 's of weights 1 and 2 respectively. Lvovski's theorem implies that $X_{(2^3),2} \subseteq \mathbf{P}^{20}$ is not extendable.

For $k \geq 3$ there is a unique net of quadrics containing C , hence C is extendable only once.

(3.7) Remark. The case of sextic double planes ($g_1 = 2$, which we studied in details in [7]) is closely analogous to that of quartics in \mathbf{P}^3 , as indeed a sextic double plane is a sextic hypersurface in $\mathbf{P}(1^3, 3)$ so in both cases we deal with hypersurfaces. For $g_1 = 2$, the dimension of the general fibre of c_g^k is given in the table below (see [7]).

$g_1 = 2$							
k	2	3	4	5	6	≥ 6	
g	5	10	17	26	37		
dim(fibre)	15	10	6	3	1	0	

For $k \leq 6$ we can package together the $K3$ surfaces extending a curve C in the image of c_g^k as a Fano $(2 + \nu)$ -fold of index ν in $\mathbf{P}^{g+\nu}$, where $\nu = \dim((c_g^k)^{-1}(C))$, which is the image of a sextic hypersurface in $X_{6,k} \subseteq \mathbf{P}(1^3, 3, k^\nu)$ (with equation similar to that given in (3.3)) by the linear system of weighted k -ics.

For $k \leq 2$ this linear system is hyperelliptic, i.e., it gives a $2 : 1$ map. For $k = 3$, we have $\text{cork}(\Phi_C) = 10$ and $h^0(N_{C/\mathbf{P}^3}(-2)) = 1$, and $X_{6,3}$ is an example similar to that in Remark (3.4), which shows the sharpness of Lvovski's theorem. For $k \leq 4$ the Fano $(2 + \nu)$ -fold $X_{6,k} \subseteq \mathbf{P}^{g+\nu}$ is not extendable, by Lvovski's theorem. For $k = 6$ in fact $X_{6,6}$ is merely $\mathbf{P}(1^3, 6)$ itself in its anticanonical embedding.

4 – Curves on Mukai varieties

In this section we consider the cases $g_1 = 6, 7, 8, 9, 10$.

4.1 – Introduction and a conjecture

(4.1) For $g_1 = 7, 8, 9, 10$, there exists a variety $M_{g_1} \subseteq \mathbf{P}(U_{g_1})$, homogeneous for some simple algebraic group G of which U_{g_1} is an irreducible representation, such that the general prime $K3$ surface of genus g_1 is a linear section of M_{g_1} [23, Cor. 0.3]. These groups, representations and homogeneous varieties are recalled in the table below.

g	G_g	dim(G)	U_g	dim(U_g)	M_g	dim(M_g)	k_g
7	Spin_{10}	45	Δ_+	16	S_{10}	10	4
8	SL_6	35	$\wedge^2 \mathbf{C}^6$	15	$\mathbf{G}(2, 6)$	8	3
9	Sp_6	21	$\wedge^{(3)} \mathbf{C}^6$	14	$\text{LG}(3, 6)$	6	2
10	G_2	14	\mathfrak{g}_2	14	G_2/P_2	5	2

Notation: we set $n(g) = \dim(U_g) - 1$. We will sometimes drop the subscript g in order to lighten the notation.

We will not use directly all the information in this table, so the reader unfamiliar with representation theory may safely read only those columns giving dimensions.

Two smooth surface linear sections of M_{g_1} are projectively equivalent if and only if one is image from the other by an automorphism coming from G [23, Thm. 0.2]. The proof of the latter statement may be adapted to prove the same result for smooth linear sections of higher dimension, but it breaks down in the case of curve linear sections of M_{g_1} .

In the present article we use ribbons to transport the results proved by Mukai for surface sections of M_{g_1} to curve sections, see (4.3) and (4.4) below.

A curve in the image of c_g^k is a complete intersection C of type $(1^{m-2}, k)$ in M_{g_1} ($m = \dim(M_{g_1})$), hence for $k > 1$ there is a unique surface linear section of M_{g_1} containing C . The description of the fibre $(c_g^k)^{-1}(C)$ depends on the truth of the Conjecture (4.3) below. Before we state it, let us describe the analogous results for $g_1 = 6$.

(4.2) For $g_1 = 6$ there is the Mukai–Fano threefold $V \subseteq \mathbf{P}^6$ which is a linear section of $\mathbf{G}(2, 5) \subseteq \mathbf{P}^9$ in its Plücker embedding; it has index 2 and degree 5. The generic prime $K3$ surface of genus 6 is a section of V by a quadric in \mathbf{P}^6 ; one has $\text{Aut}(V) \cong \text{PGL}(2)$, and two smooth quadric sections of V are projectively isomorphic if and only if one is image from the other by an automorphism of V [23, Thm. 4.1].

Smooth surface linear sections of V all are Del Pezzo surfaces of degree 5, in particular they are all isomorphic. Moreover, a general genus 6 curve C in its canonical embedding is a quadric section of the Del Pezzo surface of degree 5 in \mathbf{P}^5 , and therefore has a 6-dimensional family of models as a complete intersection of type $(1, 2)$ on V ; since $\text{Aut}(V)$ is only 3-dimensional, this shows that in general two isomorphic smooth complete intersections of type $(1, 2)$ are not conjugate under the action of $\text{Aut}(V)$.

(4.3) Conjecture. *Let $g_1 = 7, 8, 9, 10$, and $m = \dim(M_{g_1})$. For all $k \geq 1$, two smooth complete intersection curves of type $(1^{m-2}, k)$ in M_{g_1} are projectively isomorphic if and only if they are conjugate under the action of G_{g_1} . For all $k \geq 2$, two smooth complete intersection curves of type $(2, k)$ in $V \subseteq \mathbf{P}^6$ are projectively isomorphic if and only if they are conjugate under the action of $\text{Aut}(V)$.*

(4.4) Statement of results. We prove a weak form of Conjecture (4.3) for $g_1 = 7, 8, 9, 10$, see Corollary (4.13) and Proposition (4.14). If $k = 1$, a direct analysis of the representations U_{g_1} carried out in [12] shows that a general canonical curve C has only finitely many models as a linear section of M_{g_1} up to the action of G , see (4.7); we show (Corollary (4.13)) that there is in fact only one such model if C is general.²

If $k \geq 2$, we show (Proposition (4.14) and Remark (4.18)) that a general complete intersection curve of type $(1^{m-2}, k)$ has only finitely many models on M_{g_1} up to the action of G . From this we can deduce (Corollary (4.16) and Remark (4.18)) that the moduli map c_g^k is birational, and a general complete intersection curve C of type $(1^{m-2}, k)$ on M_g has $\text{cork}(\Phi_C) = 1$. Thus the canonical model of C can be extended only once, to a $K3$ surface which is the reembedding by k -ics of a linear section of M_{g_1} .

On our way, we show that the Mukai variety in $\mathbf{P}(U_{g_1})$ is the universal extension of its smooth curve linear sections, see Proposition (4.9). In particular it is not extendable, as was already observed in [9].

²After finishing this article, we have found out that for $g_1 = 7, 8, 9$ this had already been proven by Mukai himself [24].

(4.5) When $g_1 = 6$, we cannot apply the same arguments because canonical curves of genus 6 do not satisfy property N_2 . However for $k = 2$ the situation is pretty well understood.

A curve C corresponding to a general point in the image of c_{21}^2 is a complete intersection of type $(2, 2)$ in V . The canonical model of C in \mathbf{P}^{20} is a linear section of V anticanonically embedded in \mathbf{P}^{22} . This implies by [25, Prop. 6] and [3, Prop. 5.4] that the fibre $(c_{21}^2)^{-1}(C)$ has dimension at least 1. In fact, it is proven in [9] that $\text{cork}(\Phi_C) = 2$, hence the dimension of the fibre is exactly 1. Thus $V \subseteq \mathbf{P}^{22}$ is the universal extension of C ; in particular it is not extendable.

4.2 – Useful results

In the statements below, we use the notation of (4.1).

(4.6) **Theorem** [12]. *Let $g = 7, 8, 9, 10$, and let $k_g = 4, 3, 2, 2$ for $g = 7, 8, 9, 10$, respectively. Let $P \subseteq U_g$ be a generic vector subspace such that*

$$(\star) \quad \min(\text{codim}_{U_g}(P), \dim(P)) > k_g.$$

Then the stabilizer of P under the action of G is trivial.

(By the stabilizer of P under the action of G , we mean the subgroup of those $\gamma \in G$ such that $\gamma.P = P$; we do not ask that γ acts as the identity on P). The situation when (\star) is an equality is analyzed in detail in [12].

Following Mukai's arguments in [23], this implies that a general linear section of the Mukai variety M_g of positive dimension and codimension $k > k_g$ has only trivial automorphisms, see [12, Thm. 1].

(4.7) Theorem (4.6) implies that for a general canonical curve C of genus $g = 7, 8, 9$ (resp. a general canonical of genus 10 lying on a $K3$ surface), up to the action of G there are finitely many linear sections of M_g isomorphic to C . Consider the rational map

$$c_{M_g} : \mathbf{G}(g-1, \mathbf{P}(U_g))/G \dashrightarrow \mathcal{M}_g$$

mapping the orbit of a general $(g-1)$ -dimensional subspace $\Lambda \subseteq \mathbf{P}(U_g)$ to the modulus of the curve $\Lambda \cap M_g$. It is dominant if $g \neq 10$ by [23, Thm. 6.1], and dominant onto a divisor in \mathcal{M}_{10} if $g = 10$ by [11]. By Theorem (4.6), the source has dimension

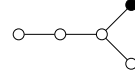
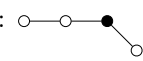
$$\dim(\mathbf{G}(g-1, \mathbf{P}(U_g))) - \dim(G),$$

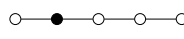
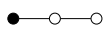
and a direct computation shows that this equals $\dim(\mathcal{M}_g)$ if $g \neq 10$, resp. $\dim(\mathcal{M}_{10}) - 1$ if $g = 10$. This implies that c_{M_g} is generically finite, which proves that for general C there are finitely many linear sections of M_g isomorphic to C , up to the action of G .

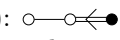

(4.8) **Lines through a point in a Mukai variety.** The results in this paragraph follow from [21, Thm. 4.8]. Let $g = 7, 8, 9, 10$, and M_g be a Mukai variety. For all $x \in M_g$, the variety of lines contained in M_g and passing through x is itself a homogeneous variety which we shall denote by M'_g . We give below a description of M'_g , but the only fact we shall use is that $\dim(M'_g) = \dim(M_g) - 4$.



g	M_g	$\dim(M_g)$	M'_g	$\dim(M'_g)$
7	S_{10}	10	$\mathbf{G}(2, 5)$	6
8	$\mathbf{G}(2, 6)$	8	$\mathbf{P}^1 \times \mathbf{P}^3$	4
9	$\text{LG}(3, 6)$	6	\mathbf{P}^2	2
10	G_2/P_2	5	\mathbf{P}^1	1

In [21, Thm. 4.8] there is a recipe in terms of Dynkin diagrams to find M'_g . The homogeneous varieties we consider are of the form G/P , with G a complex Lie group with semisimple Lie algebra, and P a maximal parabolic subgroup; the diagrams below encode the group G , and the coloured vertex encodes the subgroup P . For these matters we refer to [14, §23.3]. The recipe for finding M'_g is that one should cancel the coloured vertex, and colour those vertices that were adjacent to the cancelled vertex.

$g = 7, S_{10}$:  gives $\mathbf{G}(2, 5)$:  in its Plücker embedding.

$g = 8, \mathbf{G}(2, 6)$:  gives $\mathbf{P}^1 \times \mathbf{P}^3$:  as a Segre variety.

$g = 9, \mathbf{LG}(3, 6)$:  gives \mathbf{P}^2 :  embedded as a Veronese surface in \mathbf{P}^5 , as the double edge indicates.

$g = 10, G_2/P_2$:  gives \mathbf{P}^1 :  embedded as a rational normal cubic, as the triple edge indicates, contained in a hyperplane in \mathbf{P}^4 given by the contact structure on the adjoint variety G_2/P_2 .

There exist projective constructions of the Mukai varieties explaining this correspondence in another way, which are instances of the general construction given in [20].

4.3 – Results and proofs

Let $g_1 \in \{7, 8, 9, 10\}$.

(4.9) Proposition. *Let $[C] \in \mathcal{M}_{g_1}$ be a general point if $g_1 \neq 10$, (resp. a general point in $\text{im}(c_{10}^1)$ if $g_1 = 10$) and consider C in its canonical embedding.*

(4.9.1) For all $e \in \ker(\mathbb{T}\Phi_C)$, the ribbon $C_e \subseteq \mathbf{P}^{g_1}$ is integrable to a unique surface (which is a K3 for general e).

(4.9.2) The Mukai variety M_{g_1} is its universal extension.

The generality condition on $[C]$ may be explicated as the requirement that C is a smooth linear section of the Mukai variety M_{g_1} , see Corollary (4.11).

Proof of (4.9.1). In this paragraph, we show the first part of the proposition. If $g_1 \leq 9$, by [23, Thm. 6.1] the general curve of genus g_1 lies on a K3, so the map $c_{g_1}^1 : \mathcal{KC}_{g_1}^1 \rightarrow \mathcal{M}_{g_1}$ is dominant, and its generic fibre has dimension $22 - 2g_1$; on the other hand, we know that $\text{cork}(\Phi_C) = 23 - 2g_1$ by [10, Cor. 4.4]. If $g_1 = 10$, by [11] $\text{im}(c_{10}^1)$ is a divisor in \mathcal{M}_{10} , and the general fibre of c_{10}^1 has dimension $3 = \text{cork}(\Phi_C) - 1$. The upshot is that in both cases there is a family of surface extensions of C of the same dimension as the moduli space $\mathbf{P}(\ker(\mathbb{T}\Phi_C))$ of ribbons on C .

Moreover C has Clifford index $\lfloor \frac{g_1-1}{2} \rfloor > 2$, so it satisfies property N_2 . It follows that for every $e \in \ker(\mathbb{T}\Phi_C)$ the corresponding ribbon is integrable to at most one surface, see Proposition (2.3). By the dimension count above, this implies that the general such ribbon must indeed be integrable; we shall see in the proof of (4.9.2) that they are in fact all integrable, which will end the proof of the first part of the proposition. \square

(4.10) Lemma. *Let Λ be a (g_1-1) -dimensional linear subspace of $\mathbf{P}(U_{g_1})$ such that $C = \Lambda \cap M_{g_1}$ is a smooth curve. Then there does not exist any g_1 -dimensional linear space $\tilde{\Lambda} \subseteq \mathbf{P}(U_{g_1})$ containing Λ , and such that $\tilde{\Lambda} \cap M_{g_1}$ is the cone over C (with vertex a point).*

Proof. Let $v \in M = M_{g_1}$ be a point such that the cone $J(v, C)$ is contained in M . For all $x \in C$ the line $\langle v, x \rangle$ is contained in M hence in the projective tangent space $\mathbf{T}_{M,v}$, a \mathbf{P}^m , $m = \dim(M)$. Since C spans Λ , it follows that $\tilde{\Lambda} = \langle v, C \rangle$ is a \mathbf{P}^{g_1} contained in $\mathbf{T}_{M,v}$. On the other hand the lines contained in M and passing through v form a cone K_v over $M' \subseteq \mathbf{P}(T_{M,v})$ (the homogeneous variety indicated in (4.8)) with vertex the point v . Since M' has dimension $m - 4$, the cone K_v is $(m - 3)$ -dimensional contained in $\mathbf{T}_{M,v}$. Since also $\tilde{\Lambda}$ is contained in the m -dimensional $\mathbf{T}_{M,v}$, the intersection $K_v \cap \tilde{\Lambda}$ must be at least $(g_1 - 3)$ -dimensional, which is in contradiction with the fact that $\Lambda \cap M$ is a curve. \square

Proof of (4.9.2). Let Λ be a general $\mathbf{P}^{g_1-1} \subseteq \mathbf{P}(U_{g_1})$, and $C = \Lambda \cap M_{g_1}$. We denote by $\mathbf{P}(U_{g_1})/\Lambda$ the closed subset of $\mathbf{G}(g_1, \mathbf{P}(U_{g_1}))$ of all linear subspaces of $\mathbf{P}(U_{g_1})$ of dimension g_1 containing Λ . We want to prove that the rational map

$$r : \tilde{\Lambda} \in \mathbf{P}(U_{g_1})/\Lambda \dashrightarrow 2C_{\tilde{\Lambda} \cap M_{g_1}} \in \mathbf{P}(\ker({}^T\Phi_C))$$

is an isomorphism (this is the map which sends $\tilde{\Lambda}$ to the ribbon of C in the surface $\tilde{\Lambda} \cap M_{g_1}$).

This is a projectivity (i.e., it is induced by a linear map at vector space level) between projective spaces of the same dimension $n(g_1) - g_1$ (remember that $n(g_1) = \dim(\mathbf{P}(U_{g_1}))$, and the values of $\text{cork}(\Phi_C)$ are given in the proof of (4.9.1)). Therefore it suffices to prove that the associated linear map between vector spaces has trivial kernel. Since $\text{Cliff}(C) > 2$ the trivial ribbon over C may only be integrated to the cone over C (Proposition (2.3)). By Lemma (4.10) there is no $\tilde{\Lambda}$ cutting out the cone over C in M_{g_1} , so our map r is indeed injective, hence an isomorphism. \square

(4.11) Corollary. *Let C be a smooth curve linear section of M_{g_1} . Then (4.9.1) and (4.9.2) hold for C . In particular, the corank of the Gauss–Wahl map Φ_C is the same for all smooth curve linear sections of M_{g_1} .*

Proof. By Lemma (4.12) below, one has $\text{Cliff}(C) > 2$. Therefore, Lemma (4.10) and the proof of (4.9.2) show that the map

$$r : \tilde{\Lambda} \in \mathbf{P}(U_{g_1})/\Lambda \rightarrow 2C_{\tilde{\Lambda} \cap M_{g_1}} \in \mathbf{P}(\ker({}^T\Phi_C))$$

(in the same notation as in the proof of (4.9.2)) is an injective linear map of projective spaces. This implies that the universal extension of C is an extension of M_{g_1} . But M_{g_1} is not extendable (because for a general C' curve linear section of M_{g_1} , M_{g_1} is the universal extension of C' by what we have already proved), therefore the Mukai variety is also the universal extension of C , and the corank of the Gauss–Wahl map of C equals that of a general curve linear section of M_{g_1} . \square

(4.12) Lemma. *Let $g_1 = 7, 8, 9, 10$ and C be a smooth linear section of the Mukai variety M_{g_1} . Then one has $\text{Cliff}(C) > 2$.*

Proof. For $g_1 = 7, 8, 9$, Mukai [24, 26, 27] has proven that a curve of genus g_1 is a linear section of M_{g_1} if and only if it has no g_5^1 , resp. no g_7^2 , resp. no g_6^1 . On the other hand a curve of genus $g_1 < 10$ has Clifford index strictly larger than 2 if and only if it has no g_4^1 . So for $g_1 = 7, 9$, any smooth curve linear section of M_{g_1} has Clifford index strictly larger than 2.

For $g_1 = 8$ we use a different argument. By [19] the Mukai variety $M_8 = \mathbf{G}(2, 6)$ satisfies the property N_2 . Then we find by applying repeatedly Green’s hyperplane sections theorem [15, Thm.3.b.7] that C enjoys property N_2 as well. In turn this implies by the Green–Lazarsfeld theorem [15, Appendix] that $\text{Cliff}(C) > 2$.

For $g_1 = 10$, it is proven in [11, Rmk. 2.7] that any smooth curve linear section of M_{10} has Clifford index strictly larger than 2. \square

(4.13) Corollary. *Two smooth curve linear sections of M_{g_1} are isomorphic if and only if they are conjugate under the action of G .*

Proof. Let C and C' be two distinct general curve linear sections of M_{g_1} , and assume that they are isomorphic (being canonical curves, they are isomorphic as abstract curves if and only if they are isomorphic as polarized varieties).

By (4.9.2) (Corollary (4.11)) we may choose two $K3$ surfaces S and S' sections of M_{g_1} containing C and C' respectively, and such that the two ribbons $2C_S$ and $2C'_{S'}$ are isomorphic. By unicity of the integral of this ribbon, there exists an isomorphism of polarized surfaces $\phi : (S, C) \cong (S', C')$ taking C to C' . By [23, Thm. 0.2] there exists $\gamma \in G$ inducing ϕ , and in particular $\gamma.C = C'$. \square

(4.14) Proposition. *Let C be a general complete intersection of type $(1^\nu, 2)$ in M_{g_1} , $\nu = n(g_1) - g_1$. There are at most finitely many curves C' complete intersection of the same kind, projectively isomorphic to C but not conjugate to it modulo G .*

Proof. We argue by contradiction and assume that for general C as above there exists a positive dimensional family of curves C' projectively isomorphic to it but not G -conjugate.

We claim that a fortiori the same holds for all (even singular) complete intersection curves of type $(1^\nu, 2)$ in $M = M_{g_1}$. Indeed, as C moves in the family of such complete intersections, the dimension of the family of curves in M projectively isomorphic to C (resp. of the stabilizer of C in G) is upper semi-continuous, so that the family of curves isomorphic to C gets bigger whereas that of curves conjugate to C gets smaller. The second of these semi-continuity statements follows from the fact that there is a universal family of stabilizers over the family of complete intersection curves in M , which we shall denote by $\mathcal{IC}(1^\nu, 2; M)$. For the former, we have to consider the rational map from complete intersections in M to the moduli space \mathcal{M}_g , $g = 1 + 4(g_1 - 1)$; one may resolve its indeterminacy locus by suitably blowing-up $\mathcal{IC}(1^\nu, 2; M)$; the obtained morphism gives us the semi-continuity we want.

In particular the assumption we made by contradiction holds for $C = 2C_1$, a very general ribbon over a curve linear section C_1 of M , which is indeed a complete intersection of type $(1^\nu, 2)$ in M for which the quadratic equation is a square. Our contradiction assumption implies that there exists another ribbon C' , projectively isomorphic but not G -conjugate to C (we need to assume that the general complete intersection C has infinitely many projectively isomorphic but not conjugate copies to reach this conclusion, to avoid that finitely many such copies all degenerate to the same one with multiplicity, as C degenerates to a ribbon).

The ribbon C' is incarnated on a copy C'_1 of C_1 , which is a curve linear section of M as well. Both C and C' are integrable to surfaces S and S' respectively, both linear sections of M . By the unicity of the extension (see (2.3)), there is a projectivity ω such that $\omega(S) = S'$ and $\omega(C) = C'$. By [23, Thm. 0.2], S and S' are conjugate under the action of G , so there exists $\gamma \in G$ such that $\gamma.S = S'$. The very generality of C implies that of S , so we may assume that S has no non-trivial projective automorphisms by Proposition (2.4.2). This implies that $\omega = \gamma$, hence $\gamma.C = C'$, a contradiction. \square

(4.15) Remark. The rational map from $\mathcal{IC}(1^\nu, 2; M)$ to \mathcal{M}_g considered in the proof is indeterminate on the locus corresponding to ribbons $2C_1$, but this indeterminacy may be generically resolved at a general point of the ribbons locus by a single blow-up along this locus: a general

point in the exceptional divisor lying over a ribbon $2C_1$ is mapped to a double cover of C_1 branched over a bicanonical divisor, as one sees by stable reduction.

(4.16) Corollary. *Let $k = 2$ and $g = 4g_1 - 3$. For general $[C] \in \text{im}(c_g^2)$, one has*

$$\dim((c_g^2)^{-1}(C)) = \text{cork}(\Phi_C) - 1 = 0.$$

Proof. Being a general member of the image of c_g^2 , the curve C has a model as a complete intersection of type $(1^\nu, 2)$ in M_{g_1} . Such a curve spans a \mathbf{P}^{g_1} in $\mathbf{P}(U_{g_1})$, so there is a unique surface S linear section of M_{g_1} containing it. For $\gamma \in G$, the surface spanned by $\gamma.C$ is $\gamma.S$, and the two pairs (S, C) and $(\gamma.S, \gamma.C)$ give the same point in the fibre of c_g^2 over C .

By Proposition (4.14) C has finitely many models as a complete intersection in M_{g_1} , up to the action of G . Therefore the fibre $(c_g^2)^{-1}(C)$ has dimension 0. By Proposition (1.1), the results of [8] apply to C , hence $\dim((c_g^2)^{-1}(C)) = \text{cork}(\Phi_C) - 1$ (and in fact the fibre consists of a single point). \square

(4.17) Corollary. *The stabilizer of a general complete intersection curve of type $(1^\nu, 2)$ in M_{g_1} under the action of G is finite.*

Proof. There is a dominant rational map

$$\mathcal{IC}(1^\nu, 2; M)/G \dashrightarrow \text{im}(c_g^2),$$

and by Proposition (4.14) it is generically finite. Therefore its source and target have the same dimension. By Corollary (4.16) the dimension of the target equals that of \mathcal{KC}_g^2 . A direct computation shows that

$$\dim(\mathcal{KC}_g^2) = \dim(\mathcal{IC}(1^\nu, 2; M)) - \dim(G),$$

and the assertion follows. \square

(4.18) Remark. The statements (4.14), (4.16), and (4.17) generalize mutatis mutandis to general complete intersection curves of type $(1^\nu, k)$ for all $k \geq 2$. To do so, one ought to replace in the proofs the ribbons $2C_1$ by “higher order ribbons” kC_1 , cut out by ν linear equations and one k -th power of a linear equation, and note that $2C_1 \subseteq kC_1$.

We conclude by noting that we cannot directly reproduce the argument of Corollary (4.13) to prove that « two very general complete intersection curves of type $(1^\nu, 2)$ in M_{g_1} are isomorphic if and only if they are conjugate under the action of G », because of the possibility that two non isomorphic curves C and C' in \mathbf{P}^{g_1} may give the same canonical curve after reimmersion in \mathbf{P}^g ; in other words the problem is that the canonical class may have distinct square roots.

4.4 – Maximal variation

(4.19) Proposition. *Let $g_1 = 7, 8, 9$ (resp. $g_1 = 10$), and C be a general genus g_1 curve (resp. a curve of genus 10 general among those that lie on a K3 surface). Let S be a polarized K3 surface having C as a hyperplane section. There are only finitely many members $C' \in |\mathcal{O}_S(C)|$ that are isomorphic to C .*

As in Proposition (4.9), the generality condition on C may be replaced by the condition that C is a smooth linear section of the Mukai variety M_{g_1} , see Corollary (4.11).

Proof. By Proposition (4.9) there exists a universal family of surface extensions of C defined over $\mathbf{P}(\ker({}^T\Phi_C))$ and a rational map $s : \mathbf{P}(\ker({}^T\Phi_C)) \dashrightarrow \mathcal{K}_{g_1}^{\text{can}}$, which sends a (non-trivial) ribbon over C to the modulus of its unique $K3$ integral. We may thus apply [8, Prop. 8.4] and conclude that $s|_U$ is finite, with U the dense open subset on which s is well-defined. In particular the ribbons $2C'_S$ of the various copies C' of C in $|\mathcal{O}_S(C)|$ are only finitely many (in other words: the first infinitesimal neighbourhoods of C' in S fall into finitely many isomorphism classes).

On the other hand, arguing exactly as in [8, Cor. 8.6] we conclude that for all ribbon C_e over C the copies C' of C in $|\mathcal{O}_S(C)|$ such that $2C'_S = C_e$ are finitely many, and this ends the proof.

Before we close this proof, we emphasize that we have all the necessary assumptions for the arguments of [8, Prop. 8.4] and [8, Cor. 8.6] to apply without any single change. There the assumption that “ $g \geq 11$ and $\text{Cliff}(C) > 2$ ” is made only to ensure that every ribbon over C is integrable to a unique surface, and this in the present situation is granted by Proposition (4.9). \square

(4.20) Remark. In fact, if S is general then C and C' have the same ribbon in S only if they are the same curve. Indeed if C and C' have the same ribbon, then by Proposition (2.3) there exists a projective automorphism of S mapping C to C' . By [12, Thm. 1] the automorphism group of S is trivial, hence $C = C'$.

(4.21) Corollary. *Let (S, L) be a general primitively polarized $K3$ surface of genus $g_1 = 7, 8, 9, 10$. Then for general $C \in |L|$, there are only finitely many members $C' \in |L|$ such that C and C' are isomorphic.*

Proof. The generality assumptions ensure that C is liable to Proposition (4.19). \square

(4.22) Problem. If C is a general curve of genus $g_1 \leq 6$, then it is not true that the integral of a ribbon over C is unique, so both the arguments given in [8] to prove [8, Prop. 8.4] and [8, Cor. 8.6] break down, and it is not clear to us whether Proposition (4.19) holds in this case.

5 – Theta-characteristics, spin curves, etc.

In this section we will discuss some properties of $\mathcal{S}_g^{\frac{1}{k}, g_1}$, of $\mathcal{T}_g^{\frac{1}{k}, g_1}$ and of $\text{im}(c_g^k)$.

(5.1) Sextic double planes. First we consider $K3$ surfaces which are double cover of the projective plane branched along a smooth sextic curve. In relation with these surfaces we can consider \mathcal{K}_5^2 . The image of c_5^2 is the hyperelliptic locus in \mathcal{M}_5 , and this equals $\mathcal{T}_5^{\frac{1}{2}, 2}$. Indeed if $(C, L) \in \mathcal{S}_5^{\frac{1}{2}, 2}$, $|L|$ gives a special g_4^2 on C and by Clifford’s Theorem C is hyperelliptic.

The image of c_{10}^3 is the locus of plane sextics in \mathcal{M}_{10} , which coincides with $\mathcal{T}_{10}^{\frac{1}{3}, 2}$. In fact a cubic root θ of the canonical bundle K with $h^0(\theta) = 3$ is such that $|\theta| = g_6^2$.

For $k \geq 4$ it is more complicated to understand $\mathcal{S}_{1+k^2}^{\frac{1}{k}, 2}$, $\mathcal{T}_{1+k^2}^{\frac{1}{k}, 2}$ and whether the equality holds in (1.4.1).

(5.2) Complete intersections. Next we consider the cases $g_1 = 3, 4, 5$, in which the general polarised $K3$ surface (S, L_1) with L_1 a primitive line bundle with $(L_1)^2 = 2g_1 - 2$ is a complete intersection. In this case we face the following question.

(5.2.1) *Question.* Let $g_1 = 3, 4, 5$. Is it true that for any $k \geq 2$ the moduli space $\mathcal{S}_g^{\frac{1}{k}, g_1}$ is irreducible and that the general member of $\mathcal{S}_g^{\frac{1}{k}, g_1}$ is a complete intersection of type $(4, k)$ in \mathbf{P}^3 if $g_1 = 3$, of type $(2, 3, k)$ in \mathbf{P}^4 if $g_1 = 4$, and of type $(2, 2, 2, k)$ in \mathbf{P}^5 if $g_1 = 5$?

It is worth noticing that for $k = 2$ we have

	g_1	3	4	5
$\dim(\mathcal{K}\mathcal{C}_g^2) - \text{expdim}(\mathcal{T}_g^{\frac{1}{2}, g_1})$		10	6	3
$\dim(\text{fibre of } c_g^2)$		10	6	3

where the values on the first line are computed with (1.6.1), and those on the second line are obtained as in Section 3 for $g_1 = 3$.

In the rest of this section we discuss Question (5.2.1). First we give a definition. We say that an irreducible component \mathcal{S} of $\mathcal{S}_g^{\frac{1}{k}, g_1}$ is *birational* if for $(C, \theta) \in \mathcal{S}$ general the linear series $|\theta|$ determines a birational map. There are three levels of problems related to Question (5.2.1):

- (a) prove that $\mathcal{S}_g^{\frac{1}{k}, g_1}$ is irreducible, and its general element is a complete intersection of type $(4, k)$ in \mathbf{P}^3 if $g_1 = 3$, of type $(2, 3, k)$ in \mathbf{P}^4 if $g_1 = 4$, and of type $(2, 2, 2, k)$ in \mathbf{P}^5 if $g_1 = 5$;
- (b) prove that $\mathcal{S}_g^{\frac{1}{k}, g_1}$ has only one birational irreducible component and its general element is a complete intersection of type $(4, k)$ in \mathbf{P}^3 if $g_1 = 3$, of type $(2, 3, k)$ in \mathbf{P}^4 if $g_1 = 4$, and of type $(2, 2, 2, k)$ in \mathbf{P}^5 if $g_1 = 5$;
- (c) prove that the closure of the family of complete intersections of type $(4, k)$ in \mathbf{P}^3 if $g_1 = 3$, of type $(2, 3, k)$ in \mathbf{P}^4 if $g_1 = 4$, and of type $(2, 2, 2, k)$ in \mathbf{P}^5 if $g_1 = 5$, is an irreducible component of $\mathcal{S}_g^{\frac{1}{k}, g_1}$.

A justification for restricting our attention to the birational components is that there are in general several non-birational components, and we do not care entering the corresponding botanic. We provide examples in (5.5), (5.6) and (5.7) below.

Also, one could try to characterize the birational components by some Brill–Noether theoretic property. We will see examples of this in Propositions (5.14), (5.16), (5.17) and (5.20).

We now answer affirmatively to problem (c) above:

(5.3) Proposition. *For $3 \leq g_1 \leq 5$ and for any integer $k \geq 2$ the closure of the family of complete intersections of type $(4, k)$ in \mathbf{P}^3 if $g_1 = 3$, of type $(2, 3, k)$ in \mathbf{P}^4 if $g_1 = 4$, and of type $(2, 2, 2, k)$ in \mathbf{P}^5 if $g_1 = 5$, is an irreducible component of $\mathcal{S}_g^{\frac{1}{k}, g_1}$.*

Proof. Let \mathcal{S} be an irreducible component of $\mathcal{S}_g^{\frac{1}{k}, g_1}$ containing the family of complete intersections of type $(4, k)$ in \mathbf{P}^3 if $g_1 = 3$, of type $(2, 3, k)$ in \mathbf{P}^4 if $g_1 = 4$, and of type $(2, 2, 2, k)$ in \mathbf{P}^5 if $g_1 = 5$. We have to prove that if $(C, \theta) \in \mathcal{S}$ is general, it corresponds to a complete intersection of the same type in \mathbf{P}^{g_1} . Note that the linear series $|\theta|$ has dimension g_1 , it is very ample, and maps C to a smooth curve in \mathbf{P}^{g_1} . Since a complete intersection is projectively normal, then also C is projectively normal in \mathbf{P}^{g_1} . Moreover $k\theta = K_C$. By Gherardelli's Theorem (see [1, p. 147]), this settles the case $g_1 = 3$.

Next we treat the case $g_1 = 4$. Let C' be a smooth complete intersection of type $(2, 3, k)$ in \mathbf{P}^4 . For any positive integer h we have, by semicontinuity and by projective normality,

$$h^0(\mathcal{O}_{\mathbf{P}^4}(h)) - h^0(\mathcal{I}_C(h)) = h^0(\mathcal{O}_C(h)) \leq h^0(\mathcal{O}_{C'}(h)) = h^0(\mathcal{O}_{\mathbf{P}^4}(h)) - h^0(\mathcal{I}_{C'}(h))$$

hence $h^0(\mathcal{I}_C(h)) \geq h^0(\mathcal{I}_{C'}(h))$. On the other hand, by semicontinuity, we have $h^0(\mathcal{I}_C(h)) \leq h^0(\mathcal{I}_{C'}(h))$ so that equality holds. This immediately implies that C , as well as C' , is a complete intersection of type $(2, 3, k)$.

The case $g_1 = 5$ is similar and we mostly leave it to the reader. In this case we find a net of quadrics containing C . The base locus of this net is a complete intersection surface of type (2^3) because it so happens for complete intersection curves, of which C is a deformation. \square

It is useful, in order to compute the number of moduli of non-birational components of $\mathcal{S}_g^{\frac{1}{k}, g_1}$, to record the following well known lemma.

(5.4) Lemma. *The following loci in \mathcal{M}_g have the indicated dimensions:*

- k -gonal locus $\mathcal{M}_{g,k}^1$: $2g + 2k - 5$;
- locus of k -elliptic curves, i.e., $k : 1$ covers of elliptic curves: $2g - 2$;
- $k : 1$ covers of a genus $h > 1$ curve: $2g - (2k - 3)h + 2k - 5$.

Proof. A $k : 1$ cover of genus g of \mathbf{P}^1 amounts to the datum of $2g - 2 + 2k$ branch points, and these have $2g - 2 + 2k - 3$ moduli.

A $k : 1$ cover of genus g of an elliptic curve E amounts to the datum of $2g - 2$ branch points, and these have $2g - 2 - 1$ moduli. There is one additional modulus for the choice of E .

A $k : 1$ cover of genus g of a genus h curve C amounts to the datum of $2g - 2 - k(2h - 2)$ branch points, and each point gives a modulus. To this we have to add the dimension $3h - 3$ of \mathcal{M}_h . \square

(5.5) Hyperelliptic curves. Let (C, ι) be a genus g hyperelliptic curve. By this we mean that ι is the g_2^1 on C . Then $K_C = (g - 1)\iota$. If $g \equiv 1 \pmod{r}$, then $\theta = \frac{g-1}{r}\iota$ is such that $r\theta = K_C$, and one has $h^0(\theta) = \frac{g-1}{r} + 1$. We thus get a locus

$$\mathcal{H}_{g, \frac{g-1}{r}} \subseteq \mathcal{S}_g^{\frac{1}{r}, \frac{g-1}{r}}$$

of dimension $2g - 1$. By Clifford's Theorem, all points in $\mathcal{H}_{g, \frac{g-1}{r}}$ correspond to hyperelliptic curves.

More generally, assume there exist non-negative integers a, b, h such that $b(2a + h) = g - 1$, and $h \leq 2g + 2$. Then we can choose h distinct Weierstrass points p_1, \dots, p_h of C , and consider the linear series $\theta = a\iota + p_1 + \dots + p_h$. Then $2b\theta = K_C$ and $h^0(\theta) = a + 1$. In this way we obtain a locus

$$\mathcal{H}_{g,a}^h \subseteq \mathcal{S}_g^{\frac{1}{2b}, a} = \mathcal{S}_g^{\frac{2a+h}{2g-2}, a}$$

of dimension $2g - 1$.

(5.6) Bielliptic curves. Let us fix an integer $k \geq 2$ and an integer $g_1 \geq 1 + \frac{2}{k-1}$. Set, as usual, $g = 1 + k^2(g_1 - 1)$. A bielliptic canonical curve C of genus g in \mathbf{P}^{g-1} sits on a cone S with vertex a point p over an elliptic normal curve E of degree $g - 1$ spanning a hyperplane of \mathbf{P}^{g-1} , and it is cut out on S by a quadric hypersurface not passing through p . The bielliptic involution γ_2^1 is cut out on C by the rulings of S . The linear series g_{2g-2}^{g-2} cut out on C by the hyperplanes through p is composed with the bielliptic involution γ_2^1 , being the pull-back on C of the complete hyperplane series g_{g-1}^{g-2} on E . The latter series is certainly divisible by k , any of its k -tuple divisors is a $g_{k(g_1-1)}^{k(g_1-1)-1}$ on E , its pull-back θ on C is such that $(C, \theta) \in \mathcal{S}_g^{\frac{1}{k}, k(g_1-1)-1}$, and $k(g_1 - 1) - 1 \geq g_1$ by the hypothesis $g_1 \geq 1 + \frac{2}{k-1}$. In this way we get a locus

$$\mathcal{E}_{g,k} \subseteq \mathcal{S}_g^{\frac{1}{k}, k(g_1-1)-1}$$

of dimension $2g - 2$. As we will see later in the case $k = 2$, $\mathcal{E}_{g,k}$ sometimes is a component of $\mathcal{S}_g^{\frac{1}{k}, k(g_1-1)-1}$, and sometimes not.

(5.7) Double covers of genus 2 curves. Again we fix integers $k \geq 2$ and $g_1 \geq 1 + \frac{3}{k-1}$ and set $g = 1 + k^2(g_1 - 1)$. Fix in \mathbf{P}^g a line L and a subspace Π of dimension $g - 3$ skew with L . Consider in Π a linearly normal, smooth genus two curve Γ of degree $g - 1$. Consider the g_2^1 on Γ as a $2 : 1$ morphism $f : \Gamma \rightarrow L$. For each point $p \in \Gamma$ consider the line $\langle p, f(p) \rangle$, and take the union S of all such lines when p varies in Γ . Then S is a scroll of genus 2 and degree $g + 1$, having L as a double directrix line. Take the intersection curve C of S with a quadric intersecting L in two general points q_1, q_2 , off the four rulings issuing from q_1, q_2 . The curve C is smooth, does not intersect L , and it is easy to see that it has genus g , hence it is a canonical curve with a $2 : 1$ morphism onto the genus 2 curve Γ , and the genus 2 involution γ_2^1 is cut out on C by the rulings of S . The base point free linear series g_{2g-2}^{g-3} cut out on C by the hyperplanes containing L is composed with the genus 2 involution γ_2^1 , being the pull-back on C of the complete hyperplane series g_{g-1}^{g-3} on Γ . This series is certainly divisible by k , any of its k -tuple divisors is a $g_{k(g_1-1)}^{k(g_1-1)-2}$ on Γ , its pull back θ on C is such that $(C, \theta) \in \mathcal{S}_g^{\frac{1}{k}, k(g_1-1)-2}$, and $k(g_1 - 1) - 2 \geq g_1$. In this way we get a locus

$$\mathcal{D}_{g,k} \subseteq \mathcal{S}_g^{\frac{1}{k}, k(g_1-1)-2}$$

of dimension $2g - 3$.

(5.8) Curves on quadrics in \mathbf{P}^3 . Fix an integer $k \geq 3$. We shall now consider irreducible curves of degree $4k$ and geometric genus $1 + 2k^2$ that lie on a quadric in \mathbf{P}^3 , and study the possibility that the pull-back of the hyperplane bundle is a k -th root of the canonical bundle on the normalization.

We look at the case $k = 3$ and we consider irreducible curves of type $(6, 6)$ on a smooth quadric $S = \mathbf{P}^1 \times \mathbf{P}^1$. These curves must have exactly 6 nodes, or equivalent singularities, to have genus 19. Let us assume they have only nodes. For such a curve Γ the adjoint system $K_S + \Gamma$ has bidegree $(4, 4)$, so we find a 3-rd root of the canonical bundle on the normalization C when the 6 nodes lie on a plane conic section D of S , and are the complete intersection of Γ and D . Let us check that the curves in question exist indeed.

Fix an irreducible conic D on S and 6 general points p_1, \dots, p_6 on it. Consider the linear system \mathcal{L} of curves of type $(6, 6)$ singular at p_1, \dots, p_6 . One has

$$\dim(\mathcal{L}) \geq 48 - 3 \cdot 6 = 30.$$

Note that \mathcal{L} contains the subsystem \mathcal{L}' consisting of the curves containing D as a fixed component, with variable part consisting of curves of type $(5, 5)$ containing p_1, \dots, p_6 . One has

$$\dim(\mathcal{L}') = 35 - 6 = 29$$

(the points p_1, \dots, p_6 clearly impose independent conditions to curves of type $(5, 5)$) and the general curve in \mathcal{L}' has nodes at p_1, \dots, p_6 . Since \mathcal{L}' is strictly contained in \mathcal{L} , we see that the general curve in \mathcal{L} is irreducible, has nodes at p_1, \dots, p_6 and no other singularity, so it is of the required type. Note that in fact $\dim(\mathcal{L}) = 30$. Indeed, with only one condition D splits off the curves of \mathcal{L} and the residual system is just \mathcal{L}' , which therefore has codimension 1 in \mathcal{L} .

This construction gives us a locus \mathcal{S} inside $\mathcal{S}_{19}^{\frac{1}{3}, 3}$. Let us compute its dimension. The choice of the plane section D of S depends on 3 parameters. The choice of p_1, \dots, p_6 on D depends on 6 parameters. The linear system \mathcal{L} has dimension 30. The automorphisms of $\mathbf{P}^1 \times \mathbf{P}^1$ have 6 dimensions. In conclusion we find $\dim(\mathcal{S}) = 33$. Note that one can make the same construction on a quadric cone, thus getting curves in the closure of \mathcal{S} .

It is interesting to compare the dimension of \mathcal{S} with the dimension of the image of c_{19}^3 , which is 34 (see Section 3). Since $\dim(\mathcal{S}) < \dim(\text{im}(c_{19}^3))$, this suggests the following conjecture.

(5.8.1) *Conjecture.* The locus \mathcal{S} is contained in the closure of $\text{im}(c_{19}^3)$.

It would be tempting to see the above example as a particular case of a more general situation. Namely, for every integer $k \geq 3$ one would like to consider irreducible curves of type $(2k, 2k)$ on $\mathbf{P}^1 \times \mathbf{P}^1$, with exactly $2k(k-2)$ nodes, so that their genus is $1 + 2k^2$. For such a curve Γ , in order to have a k -th root of the canonical bundle on the normalization C , the $2k(k-2)$ nodes should lie on a curve D of type $(k-2, k-2)$, and should be the complete of intersection Γ and D . However, as soon as $k \geq 4$, it is not at all clear that irreducible curves of the required type exist indeed. It is also possible to consider similar examples on rational normal scrolls in \mathbf{P}^4 and \mathbf{P}^5 respectively; these pose the same kind of questions, which we don't answer in this text.

5.1 – Theta-characteristics (the case $k = 2$)

The case of theta-characteristics is special in that we have an expected dimension for $\mathcal{S}_g^{\frac{1}{2}, g_1}$ or equivalently for $\mathcal{T}_g^{\frac{1}{2}, g_1}$.

(5.9) $g_1 = 3$ and $k = 2$. Consider the locus $\mathcal{H}_{9,4} \subseteq \mathcal{S}_9^{\frac{1}{2}, 4} \subseteq \mathcal{S}_9^{\frac{1}{2}, 3}$ introduced in paragraph (5.5). The elements in $\mathcal{H}_{9,4}$ are of the type $(C, 4\mathfrak{l})$, where (C, \mathfrak{l}) is a genus 9 hyperelliptic curve. One has $h^0(4\mathfrak{l}) = 5$ (odd), hence $(C, 4\mathfrak{l})$ cannot be a specialization of some $(C_1, \theta) \in \mathcal{S}_9^{\frac{1}{2}, 3}$ with $h^0(\theta) = 4$ (even) by Theorem (1.5) (the invariance of the parity is due to Mumford [28]). Note that $\dim(\mathcal{H}_{9,4}) = 17$, and this agrees with Theorem (1.5) because

$$17 = \dim(\mathcal{H}_{9,4}) \geq \dim(\mathcal{M}_9) - \frac{4(4+1)}{2} = 14.$$

Let M be any irreducible component of $\mathcal{S}_9^{\frac{1}{2}, 3}$ whose general element (C, θ) is such that $h^0(\theta)$ is even. By Clifford's Theorem, if (C, θ) is general in M , then $h^0(\theta) = 4$. Moreover, by Theorem (1.5) one has

$$(5.9.1) \quad \dim(M) \geq \dim(\mathcal{M}_9) - \frac{3(3+1)}{2} = 18.$$

Note that the locus $\mathcal{H}_{9,3}^2$ consisting of hyperelliptic curves (see paragraph (5.5)), does not fill up an irreducible component of $\mathcal{S}_9^{\frac{1}{2}, 3}$ because it has only dimension 17 and it is not contained in $\mathcal{H}_{9,4}$ by the constancy of $h^0(\theta)$.

(5.10) Theorem. $\mathcal{S}_9^{\frac{1}{2}, 3}$ has two irreducible components, one equal to $\mathcal{H}_{9,4}$ and the other whose general element is a complete intersection curve of type $(2, 4)$ in \mathbf{P}^3 (see Proposition (5.3)).

Proof. Let M be an irreducible component of $\mathcal{S}_9^{\frac{1}{2}, 3}$ containing $\mathcal{H}_{9,4}$. If (C, θ) is general in M , by constancy of the parity one has $h^0(\theta) = 5$, so M coincides with $\mathcal{H}_{9,4}$ by Clifford's Theorem.

Let now M be an irreducible component of $\mathcal{S}_9^{\frac{1}{2}, 3}$ whose general element (C, θ) has $h^0(\theta) = 4$, so $|\theta|$ is a g_8^3 . By (5.9.1), C is not hyperelliptic, since $\dim(\mathcal{M}_{9,2}^1) = 17$. Let us show that θ determines an embedding of C in \mathbf{P}^3 .

Let b be the number of base points of $|\theta|$. Then $|\theta|$ induces a special g_{8-b}^3 hence $b < 2$ by Clifford's Theorem. If $b = 1$, then the g_7^3 must give a birational map since 7 is prime. This is impossible because by Castelnuovo's bound a curve of genus 7 in \mathbf{P}^3 has at most genus 6. Hence $|\theta|$ is base-point-free.

Now assume that the morphism $C \rightarrow \mathbf{P}^3$ determined by $|\theta|$ is not birational. Then it is $2 : 1$ onto an elliptic normal quartic curve and C is bielliptic. Then C depends on at most 16 moduli (see Lemma (5.4)), in contradiction with (5.9.1).

We have thus proved that $|\theta|$ determines a birational morphism $C \rightarrow \mathbf{P}^3$ onto a degree 8 curve Γ in \mathbf{P}^3 . Since the maximal geometric genus of a curve of degree 8 in \mathbf{P}^3 is 9, then Γ is a Castelnuovo curve, so it is a complete intersection of type $(2, 4)$, proving the assertion. \square

(5.11) $g_1 = 4$ and $k = 2$. Next we consider $\mathcal{S}_{13}^{\frac{1}{2},4}$. If M is an irreducible component of $\mathcal{S}_{13}^{\frac{1}{2},4}$ whose general element (C, θ) has $h^0(\theta)$ odd, then by Theorem (1.5) one has

$$(5.11.1) \quad \dim(M) \geq \dim(\mathcal{M}_{13}) - \frac{4 \cdot 5}{2} = 26.$$

The locus $\mathcal{H}_{13,6}$ introduced in paragraph (5.5) has dimension 25 and it is contained in $\mathcal{S}_{13}^{\frac{1}{2},4}$; its general element (C, θ) has $h^0(\theta) = 7$ odd, so we conclude by Theorem (1.5) that $\mathcal{H}_{13,6}$ cannot fill up an irreducible component of $\mathcal{S}_{13}^{\frac{1}{2},4}$.

There are two more hyperelliptic loci of dimension 25 contained in $\mathcal{S}_{13}^{\frac{1}{2},4}$, namely $\mathcal{H}_{13,5}^2$ and $\mathcal{H}_{13,4}^4$, see (5.5). For the same reasons as above, $\mathcal{H}_{13,4}^4$ cannot fill up an irreducible component of $\mathcal{S}_{13}^{\frac{1}{2},4}$.

(5.12) Theorem. $\mathcal{S}_{13}^{\frac{1}{2},4}$ has three irreducible components, namely:

(a) one whose general elements correspond to complete intersections of type $(2, 2, 3)$ in \mathbf{P}^4 (see Proposition (5.3)), which contains the loci $\mathcal{H}_{13,6}$ and $\mathcal{H}_{13,4}^4$;

(b) $\mathcal{H}_{13,5}^2$;

(c) $\mathcal{E}_{13,2}$ (see paragraph (5.6)).

Proof. Let (C, θ) be general in some irreducible component M of $\mathcal{S}_{13}^{\frac{1}{2},4}$. By Clifford's Theorem, we have $h^0(\theta) \leq 7$, and, as we saw, the case $h^0(\theta) = 7$ cannot occur.

Suppose first that $h^0(\theta) > 5$; then the only possibility is $h^0(\theta) = 6$. The linear system $|\theta|$ cannot be birational by Castelnuovo's bound, so it determines a $2 : 1$ morphism of C to a non-degenerate, linearly normal curve Γ of degree $d \leq 6$ in \mathbf{P}^5 . If $d = 6$ then Γ has genus 1 and C is bi-elliptic. In this way we find the locus $\mathcal{E}_{13,2} \subseteq \mathcal{S}_{13}^{\frac{1}{2},5} \subseteq \mathcal{S}_{13}^{\frac{1}{2},4}$. This is an irreducible component of $\mathcal{S}_{13}^{\frac{1}{2},4}$, because it cannot be contained in an irreducible component of $\mathcal{S}_{13}^{\frac{1}{2},4}$ whose general element (C, θ) has $h^0(\theta) = 5$, since the parity of $h^0(\theta)$ has to be preserved in a component. If $d = 5$, then Γ is a rational normal curve, C is hyperelliptic and we find the locus $\mathcal{H}_{13,5}^2$. This is a component of $\mathcal{S}_{13}^{\frac{1}{2},4}$ because it cannot be contained in an irreducible component of $\mathcal{S}_{13}^{\frac{1}{2},4}$ whose general element (C, θ) has $h^0(\theta) = 5$, and neither can it be contained in $\mathcal{E}_{13,2}$ which has smaller dimension.

Next we turn to the case in which the general element (C, θ) in M is such that C is not hyperelliptic and $|\theta|$ is a g_{12}^4 . Let b be the number of its base points, so that $|\theta|$ induces a g_{12-b}^4 . By Clifford's Theorem, we have $b < 4$. If $b > 0$ then $|\theta|$ cannot be birational by Castelnuovo's bound. If $b = 3$, the g_9^4 must give a birational map, and this is impossible. If $b = 2$, we have a g_{10}^4 , which is $2 : 1$ onto an elliptic normal quintic. Then C is bi-elliptic. However bielliptic curves depend on 24 moduli (see Lemma (5.4)), whereas $\dim(M) \geq 26$ by (5.11.1), so this is against the generality of (C, θ) in M . Finally, if $b = 1$, $|\theta|$ is a g_{11}^4 hence it gives a birational map, which is impossible.

So $|\theta|$ is a base point free g_{12}^4 . First we discuss the case in which this series is not birational. In this case we have two possibilities:

- (a) either the g_{12}^4 determines a $3 : 1$ morphism onto a rational normal quartic,
(b) or the g_{12}^4 determines a $2 : 1$ morphism onto a degree 6 curve.

We first show by contradiction that case (a) cannot happen. Indeed, in this case C is trigonal, we denote by \mathfrak{g} the g_3^1 on C , and we have $K_C = 8\mathfrak{g}$. The canonical model of C in \mathbf{P}^{12} sits on a rational normal scroll S of degree 11, which is easily seen to be smooth. On S we have $C \sim 3H - 9F$ where H is the hyperplane class on S , and F the class of a ruling. Now $F|_C = \mathfrak{g}$ and $H|_C = K_C = 8\mathfrak{g}$. Hence S should have a hyperplane section consisting of 8 rulings plus a curve E of degree 3 such that $E \cdot C = 0$. The curve E is irreducible (because $E \cdot F = 1$ and it cannot contain fibres), and one has $E^2 = -5$. It follows that S is an \mathbf{F}_5 , and $H \sim L + 3F$, with $L \sim E + 5F$. Thus $C \sim 3L$. The linear system $|3L|$ on \mathbf{F}_5 has dimension 33, and $\text{Aut}(\mathbf{F}_5)$ has dimension 10, so the curve C has at most 23 moduli, and therefore cannot be a general element of M by (5.11.1). This proves that (a) cannot happen.

Next we show that case (b) also cannot happen. In that case in fact the image curve of C via the g_{12}^4 is linearly normal of degree 6 in \mathbf{P}^4 , so it has genus 2. By Lemma (5.4), genus 13 double covers of genus 2 curves have 23 moduli, so also this possibility is in contradiction with the generality of C by (5.11.1).

Therefore the only remaining possibility is that $|\theta|$ is a birational g_{12}^4 . Let Γ be the image of C via the morphism determined by $|\theta|$. One has

$$h^0(\mathcal{I}_\Gamma(2)) \geq h_4(2) - h^0(\omega_C) = 15 - 13 = 2.$$

Assume for a moment that $h^0(\mathcal{I}_\Gamma(2)) = 2$. Then, since $3\theta = K_C + \theta$ is non-special, one has $h^0(3\theta) = 24$ by Riemann–Roch Theorem, hence

$$h^0(\mathcal{I}_\Gamma(3)) \geq h_4(3) - h^0(3\theta) = 35 - 24 = 11,$$

and therefore there exists at least one cubic containing Γ , not a combination of the quadrics containing Γ . For degree reasons we may conclude that Γ is a complete intersection of type $(2, 2, 3)$ as asserted.

We are thus left to prove that $h^0(\mathcal{I}_\Gamma(2)) = 2$. Assume by contradiction that $h^0(\mathcal{I}_\Gamma(2)) > 2$. Since $\deg(\Gamma) > 8$, the quadrics containing Γ must have base locus a non-degenerate surface, and the only possibility is that this is a cubic scroll S and then $h^0(\mathcal{I}_\Gamma(2)) = h^0(\mathcal{I}_S(2)) = 3$. Now, with the same computations as above, we see that $h^0(\mathcal{I}_\Gamma(5)) \geq 78$, whereas $h^0(\mathcal{I}_S(5)) = 75$, so there are quintic hypersurfaces containing Γ but not S . Let H be the hyperplane section of S and set $D = 5H - \Gamma$. We have $\dim(|D|) \geq 2$ and $\deg(D) = 3$. Given this, the possibilities for D are the following:

- (i) $D \sim H$;
- (ii) S is not a cone and D consists of 3 rulings;
- (iii) S is not a cone and D consists of a conic plus the line directrix E of S .

We will see that neither one of these cases is possible.

In case (i), we have $\Gamma \sim 4H$. Then the arithmetic genus of Γ is 15, hence Γ has two nodes or equivalent singularities because the geometric genus is 13. The adjoint system is $4H + K_S \sim 2H + F$, where F is a ruling of S , hence it is possible that the pull-back on C of a hyperplane section is semi-canonical if the two nodes of Γ lie on a ruling of S . Let us count the number of moduli for such curves Γ . We have $h^0(4H) = 35$. For each ruling we have 2 parameters for the choice of the nodes of Γ on that ruling, hence 3 parameters in total for the position of the nodes. Given the nodes we have a linear system of dimension $34 - 6 = 28$ (and not larger) of curves in $|4H|$ with the chosen nodes. Finally the automorphism group of S has at least dimension 6, so the number of moduli is not larger than $3 + 28 - 6 = 25$. By (5.11.1) this contradicts the generality of (C, θ) in M .

In case (ii) we have

$$\Gamma \sim 5H - 3F = 5E + 7F \quad \text{and} \quad K_S + \Gamma \sim 3E + 4F,$$

with E the negative curve on S . Then Γ has arithmetic genus 14, so it has only one node (or one cusp). It is possible that the pull-back on C of a hyperplane section is semi-canonical if the double point of Γ lies on E . Again, let us count the number of moduli for such curves Γ . We have $h^0(5E + 7F) = 33$. The choice of the double point of Γ depends on only 1 parameter (a point on E). Given the double point we have a linear system of dimension $32 - 3 = 29$ (and not larger) of curves in $|5E + 7F|$ with the chosen double point. Finally the automorphism group of S has dimension 6, so the number of moduli is $1 + 29 - 6 = 24$. By (5.11.1) this again contradicts the generality of (C, θ) in M .

Finally in case (iii) we have $\Gamma \sim 3(E + 3F)$, whose arithmetic genus is 13. So $\Gamma = C$ is smooth. We have

$$K_S + C \sim E + 6F.$$

Assume that the hyperplane series on C is semi-canonical. This means that

$$E + 6F|_C \sim 2E + 4F|_C \quad \text{hence} \quad E|_C \sim 2F|_C,$$

which is impossible: indeed $|2F|$ cuts out on C a complete g_6^2 composed with the g_3^1 cut out by $|F|$. If $E|_C \sim 2F|_C$, then the six points cut out by E on C would also belong to the union of two rulings, which is not possible. \square

(5.13) $g_1 = 5$ and $k = 2$. This case is more complicated than the previous ones and we do not have complete results. We have here $g = 17$. If M is an irreducible component of $\mathcal{S}_{17}^{\frac{1}{2},5}$ with general element (C, θ) such that $h^0(\theta)$ is even then, as usual, we have

$$(5.13.1) \quad \dim(M) \geq 48 - \frac{5 \cdot 6}{2} = 33.$$

There is the *main component* $(\mathcal{S}_{17}^{\frac{1}{2},5})^{\text{main}}$, that of complete intersections $(2, 2, 2, 2)$, that has dimension

$$\dim((\mathcal{S}_{17}^{\frac{1}{2},5})^{\text{main}}) = \dim(\mathbf{G}(4, H^0(\mathcal{O}_{\mathbf{P}^5}(2)))) - \dim(\text{PGL}(6)) = 4 \cdot 17 - 35 = 33$$

equal to the expected one since the general element (C, θ) has $h^0(\theta) = 6$.

Let us look for non-birational loci. The interesting hyperelliptic loci we found in Section (5.5) are $\mathcal{H}_{17,8}$, $\mathcal{H}_{17,7}^2$, $\mathcal{H}_{17,6}^4$, $\mathcal{H}_{17,5}^6$, all of dimension 33. If (C, θ) is in $\mathcal{H}_{17,8}$, one has $h^0(\theta) = 9$ which has different parity than 6. So if M is a component of $\mathcal{S}_{17}^{\frac{1}{2},5}$ containing $\mathcal{H}_{17,8}$ we have the estimate

$$(5.13.2) \quad \dim(M) \geq 48 - \frac{6 \cdot 7}{2} = 27.$$

The same estimate holds for a component M of $\mathcal{S}_{17}^{\frac{1}{2},5}$ containing $\mathcal{H}_{17,6}^4$. By contrast, if (C, θ) is in $\mathcal{H}_{17,7}^2$ or in $\mathcal{H}_{17,5}^6$ then $h^0(\theta)$ is even, and for a component of $\mathcal{S}_{17}^{\frac{1}{2},5}$ containing these loci the estimate (5.13.1) holds.

Look next at the bielliptic locus $\mathcal{E}_{17,2} \subseteq \mathcal{S}_{17}^{\frac{1}{2},7} \subseteq \mathcal{S}_{17}^{\frac{1}{2},5}$ (see Section (5.6)). We have $\dim(\mathcal{E}_{17,2}) = 32$. If M is an irreducible component of $\mathcal{S}_{17}^{\frac{1}{2},5}$ containing $\mathcal{E}_{17,2}$, we have the estimate (5.13.1). So $\mathcal{E}_{17,2}$ has at least codimension 1 in an irreducible component of $\mathcal{S}_{17}^{\frac{1}{2},5}$ containing it.

Finally, look at the locus $\mathcal{D}_{17,2} \subseteq \mathcal{S}_{17}^{\frac{1}{2},6} \subseteq \mathcal{S}_{17}^{\frac{1}{2},5}$ (see Section (5.7)). We have $\dim(\mathcal{D}_{17,2}) = 33$. If M is an irreducible component of $\mathcal{S}_{17}^{\frac{1}{2},5}$ containing $\mathcal{D}_{17,2}$, we have the estimate (5.13.2).

(5.14) Proposition. *Let M be an irreducible component of $\mathcal{S}_{17}^{\frac{1}{2},5}$ and let (C, θ) be its general element. Suppose that C has no g_4^1 . Then $h^0(\theta) = 6$, and $|\theta|$ is base point free and birational.*

Proof. Let M be an irreducible component of $\mathcal{S}_{17}^{\frac{1}{2},5}$ and let (C, θ) be its general element. By the hypothesis, C is not hyperelliptic. Hence $h^0(\theta) < 9$ by Clifford's Theorem. Moreover, if $h^0(\theta) > 6$, then $|\theta|$ is not birational, by Castelnuovo's bound.

If $h^0(\theta) = 7$, since $|\theta|$ is not birational, it must determine a $2 : 1$ morphism of C onto a non-degenerate linearly normal curve Γ of degree $d \leq 8$ in \mathbf{P}^6 . Then Γ has genus at most 2, hence C has a g_4^1 , a contradiction.

Similarly, if $h^0(\theta) = 8$, $|\theta|$ determines a $2 : 1$ morphism of C onto a non-degenerate linearly normal curve Γ of degree $d \leq 8$ in \mathbf{P}^7 . Then Γ has genus at most 1, so that C has a g_4^1 a contradiction again.

So we may assume $h^0(\theta) = 6$ and the estimate (5.13.1) holds. We claim that $|\theta|$ is birational. Suppose this is not the case. Let $h \geq 2$ be the degree of the map determined by $|\theta|$ onto a non-degenerate curve Γ of degree d in \mathbf{P}^5 . We have

$$5 \leq d \leq \frac{16}{h} \quad \text{hence} \quad 5h \leq 16.$$

Then either $h = 3$, $d = 5$, or $h = 2$ and one has $d > 5$ since C is not hyperelliptic. In the former case C would be trigonal, a contradiction. In the latter cases, by Lemma (5.4), C would depend on at most 32 moduli, contradicting the estimate (5.13.1). This proves that $|\theta|$ is birational.

Finally we prove that $|\theta|$ is base point free. Let b be the number of base points of $|\theta|$, and consider the complete, base-point-free, special g_{16-b}^5 induced by $|\theta|$. We have $b \leq 1$ because, by Castelnuovo's bound, a curve in \mathbf{P}^5 of degree $d \leq 14$ has genus at most 15. So let us discuss the case $b = 1$.

If $b = 1$ we have a birational, base point free g_{15}^5 , which maps C to a curve Γ of degree 15 in \mathbf{P}^5 . We claim that Γ is contained in at least 6 linearly independent quadrics. Indeed we have

$$h^0(\mathcal{O}_\Gamma(2)) \leq h^0(\omega_C(-2p)) = 15$$

where p is the base point of $|\theta|$ (the last equality is because C is not hyperelliptic), hence

$$h^0(\mathcal{I}_\Gamma(2)) \geq h_5(2) - h^0(\mathcal{O}_\Gamma(2)) \geq 21 - 15 = 6.$$

If the quadrics containing Γ have base locus a curve, then there exists a line ℓ such that $\Gamma + \ell$ is a complete intersection of type $(2, 2, 2, 2)$. Then Γ , and its normalization C , have strictly less number of moduli than 33, the number of moduli of complete intersections of type $(2, 2, 2, 2)$, in contradiction with the estimate (5.13.1).

Otherwise the base locus of the quadrics containing Γ is a rational normal quartic scroll. In fact this base locus cannot be a threefold because the maximum number of quadrics containing an irreducible, non-degenerate threefold in \mathbf{P}^5 is 3, and it is achieved by the Segre variety $\mathbf{P}^1 \times \mathbf{P}^2$ of degree 3. So the base locus in question is a surface, and it is a minimal degree surface S , which in fact has the property that $h^0(\mathcal{I}_S(2)) = 6$. On the other hand S cannot be the Veronese surface because there are no curves of odd degree on it.

Finally we discuss the case in which Γ is contained in a rational normal quartic scroll S . If S is a cone with vertex p , let m be the number of points in which the rulings intersect Γ off p , so that C has a g_m^1 . By intersecting Γ with a general hyperplane through p we see that $4m \leq 15$, which implies $m \leq 3$, a contradiction.

Next we assume S to be smooth. By Riemann–Roch we have $h^0(\mathcal{O}_\Gamma(5)) \leq 59$ and $h^0(\mathcal{O}_S(5)) = 66$ (whether S be an \mathbf{F}_0 or an \mathbf{F}_2), so

$$h^0(\mathcal{I}_\Gamma(5)) \geq 252 - 59 > 252 - 66 = h^0(\mathcal{I}_S(5)).$$

Hence there are quintic hypersurfaces containing Γ that do not contain S . They cut out on S a linear system \mathcal{L} of dimension at least 6 of quintic curves D . The curves in \mathcal{L} cut out on a general hyperplane section H of S a linear series g_5^r with $r \leq 5$. Hence there are curves in \mathcal{L} containing H . So $D \sim H + R$, where R is a line on S , and we have $\Gamma \sim 5H - D \sim 4H - R$. If F is a ruling of S , then $F \cdot \Gamma \leq 4$, hence C has a g_4^1 , a contradiction. \square

Unfortunately the discussion whether there is only one component as in the statement of Proposition (5.14), coinciding with $(\mathcal{S}_{17}^5)^{\text{main}}$, is quite intricate and we will not dwell on this here.

5.2 – The case $k = 3$

(5.15) $g_1 = 3$ and $k = 3$. Here we analyze the possibilities for curves in $\mathcal{S}_{19}^{\frac{1}{3},3}$, i.e., pairs (C, θ) with C of genus 19, $h^0(\theta) \geq 4$, and $3\theta \sim K_C$, hence $\deg(\theta) = 12$. We do not have an expected dimension for $\mathcal{S}_{19}^{\frac{1}{3},3}$, but we know that

$$\dim(\text{im}(c_{19}^3)) = \dim(\mathcal{K}\mathcal{C}_{19}^3) - 4 = 34$$

(see (3.2)).

There are certainly irreducible loci in $\mathcal{S}_{19}^{\frac{1}{3},3}$, whose general element (C, θ) is such that $|\theta|$ is not birational. For example, consider $\mathcal{H}_{19,6}$ which has dimension 37 and its general element (C, θ) is such that $h^0(\theta) = 7$. However we can prove the following:

(5.16) Proposition. *If $(C, \theta) \in \mathcal{S}_{19}^{\frac{1}{3},3}$ and C has no g_6^1 , then $|\theta|$ is birational.*

(By way of comparison, a general curve complete intersection of type (3, 4) has Clifford index 6 by Lemma (2.2), hence has no g_7^1).

Proof. Suppose (C, θ) is such that $|\theta|$ is not birational. Since C is not hyperelliptic, by Clifford's Theorem we have $h^0(\theta) < 7$, which leaves the possibilities $4 \leq h^0(\theta) \leq 6$.

Suppose that $h^0(\theta) = 4$. Let b be the number of base points of $|\theta|$. Since C is non-hyperelliptic, again by Clifford's Theorem we have $b \leq 5$. If $b = 5$, $|\theta|$ determines a g_7^3 , which should be birational, a contradiction. If $b = 4$ we obtain a g_8^3 . It may give a double cover of a curve of degree 4, which has genus at most 1. Then C has a g_4^1 , a contradiction. If $b = 3$ we have a g_9^3 . This gives a $3 : 1$ map to a rational normal cubic curve, hence C is trigonal, a contradiction. If $b = 2$ we have a g_{10}^3 , which should give a double cover of a quintic; a quintic in \mathbf{P}^3 has genus at most 2, so C has a g_4^1 , a contradiction. If $b = 1$ we have a g_{11}^3 , necessarily birational, a contradiction. If $|\theta|$ is a non-birational, base point free g_{12}^3 , it gives a double cover of a sextic curve, or a triple cover of a quartic curve. In both cases we see that C has a g_6^1 , a contradiction again. So the case $h^0(\theta) = 4$ is ruled out.

If $h^0(\theta) = 5$, with the same notation as above, by Clifford's Theorem we have $b \leq 3$. If $b = 3$, $|\theta|$ determines a g_9^4 which should be birational, a contradiction. If $b = 2$, we have a g_{10}^4 which gives a $2 : 1$ map to a quintic, which has genus 1, so C has a g_4^1 , a contradiction. If $b = 1$, we have a g_{11}^4 which should be birational, a contradiction. If $b = 0$, then $|\theta|$ is a g_{12}^4 which either determines a $3 : 1$ morphism of C to a rational normal quartic, or a $2 : 1$ morphism of C to a genus 2 sextic curve, both cases leading to a contradiction. So also the case $h^0(\theta) = 5$ is ruled out.

The case $h^0(\theta) = 6$ can be ruled out with similar arguments, we leave the details to the reader. \square

Actually we can be more precise:

(5.17) Proposition. *Let $(C, \theta) \in \mathcal{S}_{19}^{\frac{1}{3}, 3}$ be such that C has no g_6^1 . Then C is a complete intersection of type $(3, 4)$ in \mathbf{P}^3 .*

Proof. By Proposition (5.16), we have that $|\theta|$ is birational. By Castelnuovo's bound we must have $h^0(\theta) = 4$. Let $\Gamma \subseteq \mathbf{P}^3$ be the image of C via the morphism determined by $|\theta|$. One has $\deg(\Gamma) \leq 12$. We claim that Γ does not lie on a quadric. In fact, suppose that C lies on a smooth quadric $Q = \mathbf{P}^1 \times \mathbf{P}^1$. Then it is a curve of type (a, b) with $a + b = \deg(\Gamma) \leq 12$. This implies that $\min\{a, b\} \leq 6$, so that C has a g_6^1 a contradiction. A similar argument shows that Γ does not lie on a quadric cone.

Suppose next that $|\theta|$ has $b > 0$ base points. By Castelnuovo's bound, one has $b = 1$, so there is only one base point p . We have $h^0(\mathcal{I}_\Gamma(3)) \geq h_3(3) - h^0(\mathcal{O}_\Gamma(3))$ and

$$h^0(\mathcal{O}_\Gamma(3)) \leq h^0(\mathcal{O}_C(3)) = h^0(3\theta - 3p) < h^0(K_C) = 19.$$

Then $h^0(\mathcal{I}_\Gamma(3)) \geq 2$. Hence Γ is contained in two distinct cubics X and Y , with no common component. This is a contradiction because we would have $11 = \deg(\Gamma) \leq 9$. This proves that $|\theta|$ is base point free.

Then we have

$$h^0(\mathcal{O}_\Gamma(3)) \leq h^0(\mathcal{O}_C(3\theta)) = h^0(K_C) = 19.$$

Hence

$$h^0(\mathcal{I}_\Gamma(3)) \geq h_3(3) - 19 = 1,$$

so there is an irreducible cubic X containing Γ . Similarly, we see that $h^0(\mathcal{I}_\Gamma(4)) \geq 5$, so there is some quartic surface Y containing C which does not contain the cubic X . Then Γ is the complete intersection of X and Y , as desired. \square

(5.18) Remark. The hypothesis that C has no g_6^1 in Proposition (5.17) is to exclude that $|\theta|$ is non-birational and that, if $|\theta|$ is birational, the curve Γ lies on a quadric. On the other hand we do know that there are points $(C, \theta) \in \mathcal{S}_{19}^{\frac{1}{3}, 3}$ corresponding to curves on a quadric. One example is the locus \mathcal{S} considered in paragraph (5.8) although, if we believe in Conjecture (5.8.1), it should not give a new irreducible component of $\mathcal{S}_{19}^{\frac{1}{3}, 3}$.

Another example of points $(C, \theta) \in \mathcal{S}_{19}^{\frac{1}{3}, 3}$, with $|\theta|$ birational, corresponding to curves on a quadric, is the following. Consider a smooth quadric $S = \mathbf{P}^1 \times \mathbf{P}^1$ in \mathbf{P}^3 . Consider on S curves Γ of type $(5, 6)$ with one single double point q (a node or a simple cusp), and such that the 5-secant ruling through q intersects Γ , off q , in a divisor of the type $3p$. Let $\nu : C \rightarrow \Gamma$ be the normalization. Since the arithmetic genus of Γ is 20, then C has genus 19. Consider the line bundle $\theta = \nu^*(\mathcal{O}_\Gamma(1)) \otimes \mathcal{O}_C(p)$, where we abuse notation and see p as a point of C . One checks that $3\theta = K_C$, so $(C, \theta) \in \mathcal{S}_{19}^{\frac{1}{3}, 3}$. Here $|\theta|$ has the base point p . One sees that the number of moduli on which the construction of these curves depend is 32. As in (5.8.1), one may conjecture that these curves are limits of complete intersections.

(5.19) $g_1 = 4$ and $k = 3$. Here we analyze the possibilities for elements (C, θ) in $\mathcal{S}_{28}^{\frac{1}{3}, 4}$, with C of geometric genus 28, $h^0(\theta) \geq 5$, and $3\theta \sim K_C$, hence $\deg(\theta) = 18$. We focus on the case when $|\theta|$ defines a birational map. Then we must have $5 \leq h^0(\theta) \leq 6$ by Castelnuovo's bound.

(5.20) Proposition. *Assume (C, θ) in $\mathcal{S}_{28}^{\frac{1}{3}, 4}$ is such that $|\theta|$ is base point free, defines a birational map and C has no g_8^1 . Then C is a complete intersection of type $(2, 3, 3)$ in \mathbf{P}^4 .*

(By way of comparison, a general curve complete intersection of type $(3, 4)$ has Clifford index 10 by Lemma (2.2), hence has no g_{11}^1).

Proof. First we assume $h^0(\theta) = 6$. The maximum genus of non-degenerate curves of degree 18 in \mathbf{P}^5 is 28, so $|\theta|$ embeds C in \mathbf{P}^5 as a smooth Castelnuovo curve. These curves are classified (see [1, p. 122–123]), and they are of two types, namely:

- (a) $C \subseteq \mathbf{P}^5$ is the 2-Veronese image of a smooth plane curve of degree 9;
- (b) C is a smooth element of $|5H - 2F|$ on a rational normal scroll S , with H the hyperplane section and F a ruling of S .

In either case C has a g_d^1 with $d \leq 8$, so we can exclude these cases.

Assume next that $h^0(\theta) = 5$ and let Γ be the image of C via $|\theta|$. We have $h^0(\mathcal{O}_\Gamma(2)) \leq h^0(2\theta) = h^1(\theta) = 14$ by Riemann–Roch, hence $h^0(\mathcal{I}_\Gamma(2)) \geq 1$, thus C sits on an irreducible quadric Q . Next $h^0(\mathcal{O}_\Gamma(3)) \leq h^0(K_C) = 28$, hence $h^0(\mathcal{I}_\Gamma(3)) \geq 7$, thus C sits on at least two cubics X and X' that do not contain Q . If Q, X and X' intersect in a curve, by degree reasons this curve is Γ , which is a complete intersection of type $(2, 3, 3)$ as desired.

So we have to discuss the case in which Q, X and X' intersect along an irreducible surface S containing Γ . Since X' is independent of Q and X , we see that $\deg(S) \leq 5$.

Assume first $\deg(S) = 5$. In this case S is the complete intersection of a quadric and a cubic off a plane Π . This is an arithmetically Cohen–Macaulay surface whose Hilbert–Burch matrix is of the type

$$(5.20.1) \quad \begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \end{pmatrix}$$

(the numbers at the entries stand for the degrees of the corresponding forms) and its general hyperplane section D is a curve with arithmetic genus 2. By adjunction, the hyperplanes containing Π cut out on S , off the intersection scheme of S with Π , a pencil \mathcal{P} which is adjoint to the hyperplane section of S , so the curves in \mathcal{P} have degree 2. If the general curve in \mathcal{P} is reducible, then S is a scroll. Otherwise the general curve in \mathcal{P} is an irreducible conic and S is rational.

We compute $h^0(\mathcal{O}_\Gamma(4)) \leq h^0(K_C + \theta) = 45$, hence $h^0(\mathcal{I}_\Gamma(4)) \geq 25$. From the matrix (5.20.1) we deduce that $h^0(\mathcal{I}_S(4)) = 23$, so there are quartics containing Γ and not S . If S is a scroll, its genus is the geometric genus of D , which is at most 2. This implies that C has a g_8^1 , so this case is not possible. If the general conic of \mathcal{P} is irreducible, the pull-back on C of the linear series cut out by these conics on Γ is a g_d^1 , with $d \leq 8$, so that we can rule out this case too.

Assume next $\deg(S) = 4$. Since Γ is linearly normal, it cannot sit on a non-linearly normal surface, so S is the complete intersection of two quadrics and it has only isolated singularities. So S can be either a cone over an elliptic normal quartic curve in \mathbf{P}^3 or a del Pezzo surface.

If S is a cone, with vertex p , let m be the number of intersection points of Γ with a ruling, off p . By taking a general hyperplane section of Γ through p , we see that $4m \leq 18$ so that $m \leq 4$, which implies that C has a g_8^1 , and we can rule out this case.

Suppose S is a del Pezzo surface. Thus S is Gorenstein and $K_S = \mathcal{O}_S(-1)$. Moreover S is represented on \mathbf{P}^2 by a linear system of cubics passing simply through five points p_1, \dots, p_5 , which can be proper or infinitely near. With computations usual by now, we find that $h^0(\mathcal{I}_\Gamma(6)) \geq 129$, whereas $h^0(\mathcal{I}_S(6)) = 125$, so there are sextic hypersurfaces containing Γ and not S . We let $\nu : \tilde{S} \rightarrow S$ be the minimal desingularization of S and, abusing notation, we denote by Γ its proper transform on \tilde{S} . We set $H = \nu^*(\mathcal{O}_S(1))$, and notice that $K_{\tilde{S}} = -H$. We have $6H \sim \Gamma + D$, with $\dim(|D|) \geq 3$. Moreover, $\Gamma + K_{\tilde{S}} \sim 5H - D$. Since the pull-back of H on

C is θ and $3\theta = K_C$, we have that the pull-back to C of the divisors in $|2H - D|$ verifying the adjoint conditions with respect to Γ is the 0 divisor (off the pull-back of singular points). Thus we have $2H \sim D + E$, where E is effective and $H \cdot E = 2$, so the curves in $|E|$ correspond to curves of degree 2 on S . There are two possibilities for the linear system $|E|$:

- (a) $|E|$ has dimension 1 and it corresponds to a linear system of conics on S plus possibly a fixed part consisting of (-2) -curves: we may then suppose that the movable part of $|E|$ is represented in \mathbf{P}^2 by the pencil of lines through one of the points p_1, \dots, p_5 ;
- (b) $h^0(E) = 1$ and $E = E_1 + E_2 + Z$, where E_1, E_2 are two (-1) -curves, either distinct or not, and Z is a (possibly empty) union of (-2) -curves.

In case (a), $|D|$ is represented on \mathbf{P}^2 by a linear system of curves of degree 5 with multiplicity 1 at one of the points p_1, \dots, p_5 and 2 at the others. Since $\Gamma \sim 6H - D$, then Γ is represented on \mathbf{P}^2 by a curve of degree 13 with multiplicity at least 5 at one of the points p_1, \dots, p_5 , and this implies that C has a g_8^1 , a contradiction.

In case (b) we can either have $E_1 = E_2$ or $E_1 \neq E_2$. Assume that $E_1 = E_2$. Then we may suppose that E_1 is the (-1) -curve corresponding to one of the points p_1, \dots, p_5 . Then $|D|$ is represented on \mathbf{P}^2 by a linear system of curves of degree 6 with a point of multiplicity 4 and four points of multiplicity 2. Then Γ is represented on \mathbf{P}^2 by a curve of degree 12 with a point of multiplicity 2 and 4 points of multiplicity 4. Again this implies that C has a g_8^1 , a contradiction. The case in which E_1 and E_2 are distinct is similar, and also leads to an impossibility, we leave the details to the reader.

Finally we assume $\deg(S) = 3$, so that S is a rational normal scroll. If S is smooth, then S is isomorphic to \mathbf{P}^2 blown-up at a point p . Suppose that Γ is represented on \mathbf{P}^2 by a curve of degree d with a point of multiplicity m at p . We have

$$18 = \deg(\Gamma) = 2d - m$$

hence $d - m \leq 18 - d$. Since C has no g_8^1 we must have $d \geq 10$. But then $d - m \leq 8$. Since C has a g_{d-m}^1 we can rule out this case.

If S is a cone with vertex p , we let m be the number of intersection points of the rulings with Γ off p , so that C has a g_m^1 . By intersecting Γ with a general hyperplane through p , we see that $3m \leq 18$, so $m \leq 6$, and we can rule out this case too. \square

(5.21) Remark. The hypothesis that $|\theta|$ is base point free and C has no g_8^1 in Proposition (5.20) helps to rule out several cases which in fact can occur. We point out here some of these possibilities.

(a) As seen in the proof of Proposition (5.20), the 2-Veronese image C in \mathbf{P}^5 of a smooth plane curve of degree 6, is such that $(C, \mathcal{O}_C(1)) \in \mathcal{S}_{28}^{\frac{1}{3}, 4}$. This gives a locus in $\mathcal{S}_{28}^{\frac{1}{3}, 4}$ of dimension 46.

(b) Again, we saw in the proof of Proposition (5.20) that the smooth Castelnuovo curves C in $|5H - 2F|$ on a rational normal scroll S in \mathbf{P}^5 , with H the hyperplane section and F a ruling of S are such that $(C, \mathcal{O}_C(1)) \in \mathcal{S}_{28}^{\frac{1}{3}, 4}$. This gives a locus in $\mathcal{S}_{28}^{\frac{1}{3}, 4}$ of dimension 47.

(c) This is an example of elements $(C, \theta) \in \mathcal{S}_{28}^{\frac{1}{3}, 4}$, with $h^0(\theta) = 5$ and $|\theta|$ with a base point. Consider a smooth cubic rational normal scroll S in \mathbf{P}^4 . We denote by E the (-1) -curve on S and F a ruling. Consider a curve Γ in $|10F + 7E|$ which has an ordinary triple point in a point p of E , two distinct nodes p_1, p_2 along a ruling F_0 of S not passing through p , and such that F_0 cuts out on Γ , off p_1 and p_2 , a divisor of type $3q$. The arithmetic genus of Γ is 33, so its geometric genus is 28. If $\nu : C \rightarrow \Gamma$ is the normalization, one sees that $\theta = \nu^*(\mathcal{O}_\Gamma(1)) \otimes \mathcal{O}_C(q)$ (where we abuse notation and denote by q its pre-image on C) is such that $3\theta = K_C$.

(d) This is an example of elements $(C, \theta) \in \mathcal{S}_{28}^{\frac{1}{3}, 4}$, with $h^0(\theta) = 5$ and $|\theta|$ with no base point. Consider a del Pezzo quartic surface S in \mathbf{P}^4 which is the image of the linear system of cubics in

\mathbf{P}^2 with simple base points at general points p_1, \dots, p_5 . Consider the linear system \mathcal{L} of plane curves of degree 13 with multiplicity 5 at p_1 , with multiplicity 4 at p_2, \dots, p_5 and with 4 nodes located along a line L passing through p_1 . The image of a general curve of \mathcal{L} is a 4-nodal curve of geometric genus 28 and degree 18, with the 4 nodes located along a conic E on S . The adjoint linear system of \mathcal{L} consists of curves of degree 10, having multiplicity 4 at p_1 , 3 at p_2, \dots, p_5 and passing through the 4 nodes. The line L cuts out the zero divisor off the singular points of the general curve $\Gamma \in \mathcal{L}$. So the canonical system is cut out by curves of degree 9 with points of multiplicity 3 at p_1, \dots, p_5 , which maps to the the triple of the hyperplane system of S .

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