Comment on: On the irreducibility of the Severi variety of nodal curves in a smooth surface, by E. Ballico

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Abstract. In this short note, I point out that results of Ballico and Kool–Shende–Thomas together imply that on $K3$, Enriques, and Abelian surfaces, if L is a very ample and $(2p_a(L)-2g-1)$-spanned line bundle, then the equigeneric Severi variety $V_g(L)$ of all curves in $|L|$ having genus g is non-empty, irreducible, of the expected dimension, and its general member is a $(p_a(L)-g)$-nodal curve.

Let S be a smooth, complex, projective surface, and L an effective line bundle on S. We denote by $p(L)$ the common arithmetic genus of all members of the linear system $|L|$. For nonnegative integers g and δ, we consider the *equigeneric Severi variety* $V_g(L)$ (resp. *nodal Severi variety* $V^\delta(L)$), namely the locally closed subset in $|L|$ corresponding to reduced curves of geometric genus g (resp. with δ ordinary double points and no further singularity). In particular, $V^\delta(L)$ is an open subset of $V_{p(L)-\delta}(L)$.

In the recent paper [1] Ballico has proven that if L is very ample and $(2\delta-1)$-spanned, then the nodal Severi variety $V^\delta(L)$, if non-empty, is irreducible of codimension δ in $|L|$. Here I show how this result can be enhanced by taking in consideration a former result due to Kool, Shende and Thomas. This text is merely intended as a complement to [1], and I thank Edoardo Ballico for giving me the opportunity to write this up.

Theorem 1. *Let S be a $K3$ (resp. Enriques, resp. Abelian) surface, and L a line bundle on it. Consider an integer $g \leqslant p(L)$. If L is very ample and $(2p(L)-2g-1)$-spanned, then the equigeneric Severi variety $V_g(L)$ is non-empty and irreducible of dimension g (resp. $g-1$, resp. $g-2$), and the general member of $V_g(L)$ is a nodal curve.*

On $K3$, Enriques, and Abelian surfaces, there are explicit necessary and sufficient conditions for a line bundle to be k-spanned, resp. k-very ample, [2, 3, 13, 9]. In particular, they say that being k-spanned and k-very ample are two equivalent conditions.

Remark 2. The arguments given here don't ensure that the general member of $V_g(L)$ is irreducible. In practice, this may be obtained by studying the various possible splittings of L and a dimension argument.

It is now common knowledge that if (S, L) is a polarized $K3$ or Abelian surface, then the equigeneric Severi variety $V_g(L)$ is pure of the expected dimension, see [8] and the references therein (this is stated here in Proposition 7). For a general such surface, it is also known that the nodal Severi variety is non-empty by [4] for $K3$'s and [10] for Abelian surfaces. The density of the nodal Severi variety in the equigeneric one was so far only known if in addition L is primitive (and $g \geqslant 5$ in the Abelian case), see [5, 6] (as well as [8, 11]) for the $K3$ case, and [11] for the Abelian case.

Remark 3. On Enriques surfaces, it is proved in [7] that the irreducible components of the nodal Severi variety $V^{p(L)-g}(L)$ have dimension either $g-1$ or g. In the range of application of Theorem 1, there is only one component of dimension $g-1$, and the condition given in [7] to distinguish between the two cases tells us that for a general $[C] \in V^{p(L)-g}(L)$, the pull-back of K_S to the normalization of C is non-trivial.

As the main step in his proof, Ballico establishes the following statement.

Proposition 4. *Let L be a line bundle on a smooth complex projective surface. If L is very ample and $(2\delta - 1)$-spanned, then the family $\Sigma_\delta(L)$ of all members of $|L|$ which are singular in (at least) δ points is irreducible of codimension δ in $|L|$.*

The result of Kool, Shende and Thomas that we use is the following, see [12, Prop. 2.1].

Proposition 5. *Let L be a line bundle on a smooth complex projective surface. If L is δ-very ample, then the general δ-dimensional linear subsystem $\mathbf{P}^\delta \subseteq |L|$ contains a finite number of δ-nodal curves, and all other members are reduced curves of geometric genus $p_g > p(L) - \delta$.*

This has the following immediate corollary: in the setting of the proposition, if V is an irreducible variety of codimension $\leqslant \delta$ in $|L|$ parametrizing curves of geometric genus $p_g \leqslant p(L) - \delta$, then the general member of V is in fact a δ-nodal curve.

Corollary 6. *If L is δ-very ample and $(2\delta - 1)$-spanned, then the Severi variety of nodal curves $V^\delta(L)$ is non-empty and irreducible of codimension δ in $|L|$.*

Proof. Every irreducible component of $V^\delta(L)$ is contained in $\Sigma_\delta(L)$. On the other hand, $\Sigma_\delta(L)$ is irreducible of codimension δ by Prop. 4, and has an open subset contained in $V^\delta(L)$ by Prop. 5. □

In the cases of Theorem 1, one has the following estimates on the dimensions of the Severi varieties.

Proposition 7. *Let S be a $K3$ (resp. Enriques, resp. Abelian) surface, L an effective line bundle on S, and $g \leqslant p(L)$ an integer. Every irreducible component of the equigeneric Severi variety $V_g(L)$ has dimension $= g$ (resp. $\geqslant g - 1$, resp. $= g - 2$).*

For Enriques surfaces, the estimate follows from [8, Lem. 2.3 and ineq. (2.6)]. For $K3$ and Abelian surfaces a well known extra argument is needed, see [8, Prop. 4.5 and 4.13].

Proof of Theorem 1. As we have observed above, under the assumptions of Theorem 1, L is actually $(2p(L)-2g-1)$-very ample, hence also $(p(L)-g)$-very ample, so that both Propositions 4 and 5 apply for $\delta = p(L) - g$. It follows that $V^\delta(L)$ is an irreducible, dense, non-empty, open subset of $\Sigma_\delta(L)$.

On the other hand, let V be an irreducible component of $V_g(L)$. By Prop. 7, V has codimension $\leqslant \delta$ in $|L|$. It thus follows from Prop. 5 that the general member of V is a δ-nodal curve, hence V is contained, and actually dense in $\Sigma_\delta(L)$. This concludes the proof. □

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