NONEMPTINESS OF SEVERI VARIETIES ON ENRIQUES SURFACES

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Abstract. Let (S, L) be a general polarized Enriques surface, with L not numerically 2–divisible. We prove the existence of regular components of all Severi varieties of irreducible nodal curves in the linear system $|L|$, i.e., for any number of nodes $\delta = 0, \ldots, p_a(L) - 1$. This solves a classical open problem and gives a positive answer to a recent conjecture of Pandharipande–Schmitt, under the additional condition of non–2–divisibility.

1. Introduction

Let S be a smooth, projective complex surface and L a line bundle on S. Let $p_a(L) = \frac{1}{2}L \cdot (L + K_S) + 1$ denote the arithmetic (or sectional) genus of L. For any integer δ satisfying $0 \leq \delta \leq p_a(L)$ we denote by $V_{|L|,\delta}(S)$ the *Severi variety* parametrizing irreducible δ-nodal curves in $|L|$. A heuristic count shows that the *expected dimension* of $V_{|L|,\delta}(S)$ is $\dim(|L|) - \delta$.

Severi varieties were introduced by Severi in [30, Anhang F], where he proved that all Severi varieties of irreducible δ-nodal curves of degree d in $\mathbb{P}^2$ are nonempty and smooth of the expected dimension. Severi also claimed irreducibility of such varieties, but his proof contains a gap. The irreducibility was proved by Harris in [21].

Severi varieties on other surfaces have received much attention in recent years, especially in connection with enumerative formulas computing their degrees (see [1, 2, 3, 14, 19, 22, 26, 36, 37]). Nonemptiness, smoothness, dimension and irreducibility for Severi varieties have been widely investigated on various rational surfaces (see, e.g., [20, 31, 33, 34, 35]), as well as K3 and abelian surfaces (see, e.g., [5, 24, 25, 26, 27, 32, 39]). Extremely little is known on other surfaces. In particular, Severi varieties may have unexpected behaviour: examples are given in [7] of surfaces of general type with reducible Severi varieties, and also with components of dimension different from the expected one.

In this paper we consider the case of Enriques surfaces. If S is an Enriques surface, it is known (cf. [9, Prop. 1]) that $V_{|L|,\delta}(S)$, *if nonempty*, is smooth and every irreducible component has dimension either $p_a(L) - \delta - 1$ or $p_a(L) - \delta$. Moreover, if S is general in moduli, the latter case can only occur if L is 2-divisible in $\mathrm{Pic}(S)$. Any component of dimension $p_a(L) - \delta - 1$ is called *regular*, and these components can only be nonempty for $\delta < p_a(L) - 1$, that is, they parametrize nodal curves of genus at least one. The nonemptiness problem has been open until now.

For any integer $g \geq 2$, let $\mathcal{E}_g$ denote the moduli space of complex polarized Enriques surfaces (S, L) of *(sectional) genus* g, that is, S is an Enriques surface and L is an ample line bundle on S such that $L^2 = 2g - 2$. Thus, g is the arithmetic genus of all curves

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in the linear system $|L|$. The spaces $\mathcal{E}_g$ have many irreducible components. A way to determine these has recently been given in [23], after partial results were obtained in [10], cf. Theorem 5.7 below.

Denote by $\mathcal{E}_g[2]$ the locus in $\mathcal{E}_g$ parametrizing pairs (S, L) such that L is 2-divisible in $\mathrm{Num}(S)$. The main result of this paper settles the existence of regular components of all Severi varieties on general polarized Enriques surfaces outside $\mathcal{E}_g[2]$:

Theorem 1.1. *Let (S, L) be a general element of any irreducible component of $\mathcal{E}_g \setminus \mathcal{E}_g[2]$. Then $V_{|L|,\delta}(S)$ is nonempty and has a regular component, of dimension $g-1-\delta$, for all $0 \leq \delta < g$.*

By [9, Cor. 1], the theorem follows as soon as one proves the case of maximal δ, that is, $\delta = g-1$, in which case the parametrized curves are elliptic.

We note that Theorem 1.1 implies a conjecture due to Pandharipande and Schmitt regarding smooth curves of genus $g \geq 2$ on Enriques surfaces (see [28, Conj. 5.1]). Our result implies this conjecture for curves whose classes are not 2–divisible (see [28, Prop. 2.2 and text after Conj. 5.1]).

We shall prove Theorem 1.1 by degenerating a general Enriques surface to the union of two surfaces R and P, birational to the symmetric square of a general elliptic curve and the projective plane respectively, and glued along a smooth elliptic curve T numerically anticanonical on each surface. We introduce such degenerations in §2. On such a semi–stable limit we identify suitable curves that deform to elliptic nodal curves on the general Enriques surface that are rigid, i.e., they do not move in a positive dimensional family. As remarked above, this suffices to prove the theorem. The aforementioned suitable curves consist, apart from some (-1)–curves as components, of an irreducible nodal elliptic curve on R and an irreducible nodal rational curve on P intersecting at one single point on T where they both are smooth and have a contact of high order. Such curves are members of so–called logarithmic Severi varieties on the surfaces on which they lie. We develop all necessary tools and results on such varieties on the two types of surfaces in question in §3. The analysis of the conditions under which the limit curves actually deform to rigid nodal elliptic curves on the general Enriques surface is performed in the crucial §4. These conditions boil down to numerical properties to be verified by the line bundles determined on each component of the limit surfaces. An important ingredient next is the description of all components of moduli spaces of polarized Enriques surfaces in terms of decompositions of the polarizing line bundles into effective isotropic divisors as developed recently in [10, 23], which we review in §5. The corresponding identification of suitable isotropic Cartier divisors on the limit surfaces is done in §6. Finally, §7 is devoted to exhibiting, for each component of the moduli spaces of polarized Enriques surfaces, a suitable isotropic decomposition of the limit polarising line bundle, such that its restriction on each component verifies the conditions for deforming the curves mentioned above.

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2. Flat limits of Enriques surfaces

In this section we will introduce the semi–stable degenerations of general Enriques surfaces that we will use in our proof of Theorem 1.1.

Let E be a smooth elliptic curve. Denote by $\oplus$ (and $\ominus$) the group operation on E and by e_0 the neutral element. Let $R := \mathrm{Sym}^2(E)$ and $\pi : R \to E$ be the (Albanese) projection map sending $x + y$ to $x \oplus y$. We denote the fiber of π over a point $e \in E$ by

$$\mathfrak{f}_e := \pi^{-1}(e) = \{x + y \in \mathrm{Sym}^2(E) \mid x \oplus y = e \text{ (equivalently, } x + y \sim e + e_0)\},$$

which is the $\mathbb{P}^1$ defined by the linear system $|e + e_0|$. (Here, and throughout the paper, $\sim$ denotes linear equivalence of divisors.) We denote the algebraic equivalence class of the fibers by $\mathfrak{f}$. Symmetric products of elliptic curves have been studied in detail in [4], to which we will frequently refer in this paragraph.

For each $e \in E$, we define the curve $\mathfrak{s}_e$ (called D_e in [4]) as the image of the section $E \to R$ mapping x to $e + (x \ominus e)$. We let $\mathfrak{s}$ denote the algebraic equivalence class of these sections, which are the ones with minimal self-intersection, namely 1, cf. [4]. We note that $\mathrm{Sym}^2(E)$ is the $\mathbb{P}^1$–bundle on E with invariant -1. We observe for later use that for $x \neq y$ we have

$$\mathfrak{s}_x \cap \mathfrak{s}_y = \{x + y\}. \tag{1}$$

We also note that

$$K_R \sim -2\mathfrak{s}_{e_0} + \mathfrak{f}_{e_0}.$$

Let η be any of the three nonzero 2-torsion points of E. The map $E \to R$ defined by $e \mapsto e + (e \oplus \eta)$ realizes E as an unramified double cover of its image curve $T := \{e + (e \oplus \eta) \mid e \in E\}$, which is a smooth elliptic curve. We have

$$T \sim -K_R + \mathfrak{f}_\eta - \mathfrak{f}_{e_0} \sim 2\mathfrak{s}_{e_0} - 2\mathfrak{f}_{e_0} + \mathfrak{f}_\eta,$$

by [4, (2.10)]. In particular, $T \not\sim -K_R$ and $2T \sim -2K_R$.

Embed T as a cubic in $P := \mathbb{P}^2$. Consider nine (possibly coinciding) points $y_1, \ldots, y_9 \in T$. Divide the nine points in two subsets, say of i and $9 - i$ points, with $0 \leq i \leq 9$. Let $\widetilde{R} \to R$ and $\widetilde{P} \to P$, respectively, denote the blow–ups at the schemes on T determined by these two subsets of i and $9 - i$ points, respectively. Denote by $\mathfrak{e}_R$ and $\mathfrak{e}_P$ the total exceptional divisors on $\widetilde{R}$ and $\widetilde{P}$, respectively, and denote the strict transforms of $T, \mathfrak{s}, \mathfrak{f}$ with the same symbols. We have

$$T \sim 2\mathfrak{s}_{e_0} - 2\mathfrak{f}_{e_0} + \mathfrak{f}_\eta - \mathfrak{e}_R \not\sim -K_{\widetilde{R}} \sim 2\mathfrak{s}_{e_0} - \mathfrak{f}_{e_0} - \mathfrak{e}_R \quad \text{on } \widetilde{R}, \tag{2}$$

$$2T \sim -2K_{\widetilde{R}} \quad \text{on } \widetilde{R}, \tag{3}$$

$$T \sim 3\ell - \mathfrak{e}_P \sim -K_{\widetilde{P}} \quad \text{on } \widetilde{P} \tag{4}$$

where ℓ is the pull–back on $\widetilde{P}$ of a general line in P. Define $X = \widetilde{R} \cup_T \widetilde{P}$ as the surface obtained by gluing $\widetilde{R}$ and $\widetilde{P}$ along T. Denote by $\mathcal{D}_{[i]}$ the family of such surfaces. It is easy to see that $\mathcal{D}_{[i]}$ is irreducible of dimension 10 (when one also allows E to vary in moduli). We define $\mathcal{D} := \cup_{i=0}^{9} \mathcal{D}_{[i]}$.

Let X be a member of $\mathcal{D}$. The *first cotangent sheaf* $T^1_X := \mathfrak{ext}^1_{\mathcal{O}_X}(\Omega_X, \mathcal{O}_X)$ of X (cf. [29, Cor. 1.1.11] or [16, §2]) satisfies

$$T^1_X \simeq \mathcal{N}_{T/\widetilde{R}} \otimes \mathcal{N}_{T/\widetilde{P}}$$

by [16, Prop. 2.3], which is trivial if and only if the nine points satisfy the condition

$$y_1 + \cdots + y_9 \in |\mathcal{N}_{T/R} \otimes \mathcal{N}_{T/P}|. \tag{5}$$

Thus, X is *semi-stable* if and only if (5) holds, cf. [16, Def. (1.13)] and [17, (0.4)]. We will denote by $\mathcal{D}^*_{[i]}$ the subfamily of $\mathcal{D}_{[i]}$ consisting of semi–stable surfaces. It is easy to see that $\mathcal{D}^*_{[i]}$ is irreducible of dimension 9. We define $\mathcal{D}^* := \cup_{i=0}^9 \mathcal{D}^*_{[i]}$.

We recall that a Cartier divisor, or a line bundle, $\mathcal{L}$ in $\mathrm{Pic}(X)$, is a pair (L', L'') such that $[L'] \in \mathrm{Pic}(\widetilde{R})$, $[L''] \in \mathrm{Pic}(\widetilde{P})$ and $L'|_T \simeq L''|_T$. Since T is numerically equivalent to the anticanonical divisor on both $\widetilde{R}$ and $\widetilde{P}$, we have

$$\mathcal{L}^2 = (L')^2 + (L'')^2 = 2p_a(L') - 2 + 2p_a(L'') - 2 + 2d, \quad d := L' \cdot T = L'' \cdot T.$$

The canonical divisor K_X is represented by

$$K_X = (K_{\widetilde{R}} + T, 0) = (\mathfrak{f}_\eta - \mathfrak{f}_{e_0}, 0) \ \text{ in } \ \mathrm{Pic}(\widetilde{R}) \times \mathrm{Pic}(\widetilde{P}).$$

In particular, by (2)-(4) we have

$$K_X \not\sim 0 \ \text{ and } \ 2K_X \sim 0.$$

By [23, Lemma 3.5] the Cartier divisor K_X is the only nonzero torsion element of $\mathrm{Pic}(X)$. (The proof is for $i = 2$, but carries over to the general case.)

Remark 2.1. For any $[L'] \in \mathrm{Pic}(\widetilde{R})$ and $[L''] \in \mathrm{Pic}(\widetilde{P})$ such that $L' \cdot T = L'' \cdot T$, there are finitely many line bundles $\overline{L}'$ on $\widetilde{R}$ numerically equivalent to L', such that $(\overline{L}', L'')$ is a line bundle on X. Since K_X is the only non–trivial torsion element of $\mathrm{Pic}(X)$, there are actually only two such line bundles $\overline{L}'$, and the resulting pairs $(\overline{L}', L'')$ differ by K_X. These line bundles are numerically equivalent and we will denote by $[L', L'']$ their numerical equivalence class.

By (5), if X is semi–stable it also carries the Cartier divisor ξ represented by the pair

$$\xi = (T, -T) \sim (2\mathfrak{s}_{e_0} - 2\mathfrak{f}_{e_0} + \mathfrak{f}_\eta - \mathfrak{e}_R, -3\ell + \mathfrak{e}_P) \tag{6}$$

in $\mathrm{Pic}(\widetilde{R}) \times \mathrm{Pic}(\widetilde{P})$ (see [17, (3.3)]).

The central result for our purposes is:

Theorem 2.2. *Let $y_1, \ldots, y_9 \in T$ be general such that $X = \widetilde{R} \cup_T \widetilde{P}$ is a member of $\mathcal{D}^*$.*

There is a flat family $\pi : \mathfrak{X} \to \mathbb{D}$ over the unit disc such that $\mathfrak{X}$ is smooth and, setting $S_t := \pi^{-1}(t)$, we have that

- *$S_0 = X$, and*
- *S_t is a smooth general Enriques surface for $t \neq 0$.*

Furthermore, denoting by $\iota_t : S_t \subset \mathfrak{X}$ the inclusion, there is a short exact sequence

$$0 \longrightarrow \mathbb{Z}[\xi] \longrightarrow \mathrm{Pic}(X) \simeq H^2(\mathfrak{X}, \mathbb{Z}) \xrightarrow{\iota_t^*} H^2(S_t, \mathbb{Z}) \simeq \mathrm{Pic}(S_t) \longrightarrow 0.$$

Proof. This follows from [23, Prop. 3.7, Thm. 3.10 and Cor. 3.11] in the case where X lies in $\mathcal{D}^*_{[2]}$. Once we have the statement in this case, we can prove it in the other cases by making a birational transformation of $\mathfrak{X}$ to flip any of the exceptional curves between $\widetilde{P}$ and $\widetilde{R}$ (see for example [12, §4.1], where the flip is called a *1–throw*). □

3. Logarithmic Severi varieties

Theorem 1.1 will be proved by degenerating a general Enriques surface to a surface $\widetilde{R} \cup_T \widetilde{P}$ in $\mathcal{D}^*$. It will be essential to construct curves on $\widetilde{R} \cup_T \widetilde{P}$ that will deform to nodal irreducible elliptic curves on the general Enriques surface. As we will see in §4, the good limit curves on $\widetilde{R}$ and $\widetilde{P}$ are nodal curves with high order tangency with T at the same point on each component. These are members of so-called *logarithmic Severi varieties*, parametrizing nodal curves with given tangency conditions to a fixed curve. This will be the topic of this section. We start with some general definitions and results:

Definition 3.1. Let S be a smooth projective surface, $T \subset S$ a smooth, irreducible curve and L a line bundle or a divisor class on S. Let g be an integer satisfying $0 \leq g \leq p_a(L)$.

For any effective divisor $\mathfrak{d} = m_1p_1 + \cdots + m_lp_l$ on T, where the p_i are pairwise distinct, we denote by $V_{g,\mathfrak{d}}(S, T, L)$ the locus of curves in S such that

- C is irreducible of geometric genus g and algebraically equivalent to L,
- denoting by $\nu : \widetilde{C} \to S$ the normalization of C composed with the inclusion $C \subset S$, there exists $q_i \in \nu^{-1}(p_i)$ such that ν^*T contains m_iq_i, for all $i \in \{1, \ldots, l\}$.

For any integer m satisfying $0 < m \leq L \cdot T$ we let $V_{g,m}(S, T, L)$ denote the locus of curves contained in some $V_{g,mp}(S, T, L)$ for some (non-fixed) $p \in T$.

We denote by $V^*_{g,m}(S, T, L)$ the open sublocus of $V_{g,m}(S, T, L)$ parametrizing curves that are *smooth* at the intersection points with T and otherwise *nodal*.

In the sequel $\equiv$ will denote numerical equivalence of divisors. We will need:

Proposition 3.2. *Let $S, T, L, \mathfrak{d}, g$ and m be as in Definition 3.1. Assume that $T \equiv -K_S$.*

(i) If $L \cdot T > \sum_{i=1}^{l} m_i$, then all irreducible components of $V_{g,\mathfrak{d}}(S, T, L)$ have dimension $g - 1 + L \cdot T - \sum_{i=1}^{l} m_i$.
(ii) All irreducible components of $V_{g,m}(S, T, L)$ have dimension $g + L \cdot T - m$.
(iii) If $m \leq L \cdot T - 2$, then the general member $[C]$ in any component of $V_{g,m}(S, T, L)$ is smooth at its intersection points with T, and for all curve $G \subset S$ not having T as an irreducible component, and all finite set $\Gamma \subset S$, if $[C]$ is general with respect to G and Γ then C is tranverse to G and does not intersect Γ.
(iv) If $m \leq L \cdot T - 3$, then the general member in any component of $V_{g,m}(S, T, L)$ is nodal.

Proof. The result follows from [3, §2], as outlined in [15, Thm. (1.4)]. □

3.1. **Families of blown-up surfaces.** We will also need to work in families in the following way. For $S = R$ or P containing T as above and for any nonnegative integer n we consider the family $\mathcal{S}^{[n]} \to T^n$ with fibers $\mathrm{Bl}_{y_1,\ldots,y_n}(S)$, the blow–up of S at $y_1, \ldots, y_n$, over $(y_1, \ldots, y_n) \in T^n$ (when the points are coinciding, this has to be interpreted as blowing up curvilinear schemes on T). To be precise, the fibers are *marked*, in the sense that their (total) exceptional divisors are labelled with $1, \ldots, n$. Whenever we have a line bundle on a single surface $\mathrm{Bl}_{y_1,\ldots,y_n}(S)$, we can write it in terms of the generators of $\mathrm{Pic}(S)$ and of the exceptional divisors over each y_i, and thus we can extend it to a relative line bundle on the whole family $\mathcal{S}^{[n]}$ in the obvious way. We will therefore mostly not distinguish notationally between a relative line bundle L and its restriction to any surface in the family.

Similarly, there are n relative (total) exceptional divisors $\mathfrak{e}_i$ on $\mathcal{S}^{[n]}$ whose fiber over a point $(y_1,\dots,y_n)\in T^n$ is the exceptional divisor on $\mathrm{Bl}_{y_1,\dots,y_n}(S)$ over the point y_i, which we by abuse of notation still denote by $\mathfrak{e}_i$.

Definition 3.3. Let L be a relative line bundle on $\mathcal{S}^{[n]}$. The *value of L on the ith exceptional divisor* is the number $L\cdot\mathfrak{e}_i$ on any fiber $\mathrm{Bl}_{y_1,\dots,y_n}(S)$. We say that L is *positive on the ith exceptional divisor* if $L\cdot\mathfrak{e}_i>0$.

We shall consider the relative Hilbert scheme

$$\mathcal{H}^{[n]}_{S,L}\longrightarrow T^n$$

whose fibers are the Hilbert schemes of curves on $\mathrm{Bl}_{y_1,\dots,y_n}(S)$ algebraically (or equivalently numerically) equivalent to L. We have a (possibly empty) scheme

$$\mathcal{V}^{[n]}_{g,m}(S,T,L)\longrightarrow T^n$$

whose fibers are $V_{g,m}(\mathrm{Bl}_{y_1,\dots,y_n}(S),T,L)\subset\mathcal{H}^{[n]}_{S,L}$ (here, as usual, we denote by T its strict transform on the blow–up). Taking the closure in $\mathcal{H}^{[n]}_{S,L}$, we obtain a (possibly empty) scheme with a morphism

$$\nu=\nu^{[n]}_{g,m}(S,T,L):\overline{\mathcal{V}^{[n]}_{g,m}(S,T,L)}\longrightarrow T^n, \tag{7}$$

whose fibers we denote by

$$\nu^{-1}(y_1,\dots,y_n):=\overline{V}^{[n]}_{g,m}(\mathrm{Bl}_{y_1,\dots,y_n}(S),T,L).$$

Note that for any $(y_1,\dots,y_n)\in T^n$ one has

$$\overline{V_{g,m}(\mathrm{Bl}_{y_1,\dots,y_n}(S),T,L)}\subseteq\overline{V}^{[n]}_{g,m}(\mathrm{Bl}_{y_1,\dots,y_n}(S),T,L).$$

3.2. **Logarithmic Severi varieties on blow–ups of the symmetric square of an elliptic curve.** Let $T\subset R=\mathrm{Sym}^2(E)$ as defined in §2. Let $y_1,\dots,y_n\in T$ and let $\widetilde{R}:=\mathrm{Bl}_{y_1,\dots,y_n}(R)$ denote the blow–up of R at $y_1,\dots,y_n$, with (total) exceptional divisors $\mathfrak{e}_i$ over y_i. We denote the strict transforms of $\mathfrak{s}$, $\mathfrak{f}$ and T on $\widetilde{R}$ by the same names. We also still denote by $\pi:\widetilde{R}\to E$ the composition of the blow–up $\widetilde{R}\to R$ with the Albanese morphism $R\to E$ (cf. beginning of §2). By (2)–(3) we have

$$T\equiv -K_{\widetilde{R}}\equiv 2\mathfrak{s}-\mathfrak{f}-\mathfrak{e}_1-\cdots-\mathfrak{e}_n.$$

Definition 3.4. A line bundle or Cartier divisor L on $\widetilde{R}$ is *odd* if $L\cdot\mathfrak{f}$ is odd.

Definition 3.5. We denote by $\mathrm{Sym}^n(T)_m\subset\mathrm{Sym}^n(T)$ the subscheme consisting of divisors with a point of multiplicity $\geq m$.

Lemma 3.6. *Let L be an odd line bundle or Cartier divisor on $\widetilde{R}$. Let m be any integer satisfying $1\leq m\leq L\cdot T$. Then the following hold:*

(i) No curve C in $\overline{V}_{1,m}(\widetilde{R},T,L)$ contains T.
(ii) For any component $V\subset\overline{V}_{1,m}(\widetilde{R},T,L)$ the restriction map

$$\begin{array}{ccc} V & \longrightarrow & \mathrm{Sym}^{L\cdot T}(T)_m\\ C & \mapsto & C\cap T\end{array}$$

is well-defined, finite and surjective. In particular,

$$\dim(V)=L\cdot T-m+1.$$

(iii) For a general curve C in any component of $V^*_{1,m}(\widetilde{R},T,L)$, let N be the reduced subscheme of $\widetilde{R}$ supported at the nodes of C, and Z any subscheme of $C\cap T$ of degree $C\cdot T-1$. Then the linear system $|\mathcal{O}_{\widetilde{R}}(C)\otimes \mathcal{J}_{N\cup Z}|$ consists only of C.

Proof. Assume that we have $C=hT+C'$ in $\overline{V}_{1,m}(\widetilde{R},T,L)$ for some $h>0$, with C' not containing T. We have $L\cdot\mathfrak{f}=C\cdot\mathfrak{f}=2h+C'\cdot\mathfrak{f}$, whence $C'\cdot\mathfrak{f}>0$ since L is assumed to be odd. Hence C' has at least one component dominating E via $\pi:\widetilde{R}\to E$, and therefore C cannot be a limit of an elliptic curve. Thus (i) follows.

It also follows that the restriction map in (ii) is everywhere defined. The fiber over a $Z\in \mathrm{Sym}^{L\cdot T}(T)$ consists of all curves C in V such that $C\cap T=Z$. This must be finite, for otherwise we would find a member of the fiber passing through an additional general point $p\in T$, a contradiction (using again that no curve in V contains T). Hence the restriction morphism in (ii) is finite. We have $\dim(V)\geq L\cdot T+1-m$ by Proposition 3.2(ii) and semicontinuity, which equals $\dim\big(\mathrm{Sym}^{L\cdot T}(T)_m\big)$. The morphism is therefore surjective and equality holds for the dimension. This proves (ii).

Let now C be a curve in $V^*_{1,m}(\widetilde{R},T,L)$ and Z be any subscheme of $C\cap T$ of degree $C\cdot T-1$. Let $\widehat{R}\to\widetilde{R}$ denote the blow–up of $\widetilde{R}$ along Z, considered as a curvilinear subscheme of T, and let $\widehat{C}$ and $\widehat{T}$ denote the strict transforms of C and T, respectively, and $\widehat{L}:=\mathcal{O}_{\widehat{R}}(\widehat{C})$. Then $\widehat{C}$ is a member of $V^*_{1,1}(\widehat{R},\widehat{T},\widehat{L})$. To prove (iii) we may reduce to proving that if X is a general member of a component of $V^*_{1,1}(\widehat{R},\widehat{T},\widehat{L})$, and N is the subscheme of its nodes, then the linear system $|\mathcal{O}_{\widehat{R}}(X)\otimes\mathcal{J}_N|$ consists only of X.

Let $\delta=p_a(C)-1$. The variety $V^*_{1,1}(\widehat{R},\widehat{T},\widehat{L})$ is the open subset of the *Severi variety of δ-nodal curves algebraically equivalent to* $\widehat{L}$ consisting of curves with nodes off $\widehat{T}$. All of its components have dimension $\widehat{L}\cdot\widehat{T}=1$ by (ii) (or Proposition 3.2(ii)), and it is smooth by standard arguments (see, e.g., [11, Prop. 2.2]). Let W be any component of $V^*_{1,1}(\widehat{R},\widehat{T},\widehat{L})$. Then W is fibered over $\mathrm{Pic}^0(E)\simeq E$ in subvarieties $W_{\widehat{L}'}$ parametrizing δ-nodal curves in $|\widehat{L}'|$, where $\widehat{L}'$ is any line bundle numerically equivalent to $\widehat{L}$. By (ii) the linear equivalence classes of the curves in W vary. Thus $W_{\widehat{L}'}$ is nonempty for general $\widehat{L}'$, whence smooth and zero-dimensional. The tangent space to $W_{\widehat{L}'}$ at any point $[X]$ is isomorphic to $H^0(\widehat{L}'\otimes\mathcal{J}_N)/\mathbb{C}$, where N is the scheme of nodes of X (see, e.g., [8, §1]). In particular, for a general X in W we have

$$\dim\big(|\mathcal{O}_{\widehat{R}}(X)\otimes\mathcal{J}_N|\big)=h^0(\mathcal{O}_{\widehat{R}}(X)\otimes\mathcal{J}_N)-1=\dim(W_{\widehat{L}'})=0,$$

whence $|\mathcal{O}_{\widehat{R}}(X)\otimes\mathcal{J}_N|$ consists only of X, as desired. This proves (iii). □

In view of part (iii) of the previous result, we make the following:

Definition 3.7. We let $V^{**}_{1,m}(\widetilde{R},T,L)$ denote the open subvariety of $V^*_{1,m}(\widetilde{R},T,L)$ parametrizing curves C such that, for N its scheme of nodes and for every subscheme Z of $C\cap T$ of degree $C\cdot T-1$, the linear system $|\mathcal{O}_{\widetilde{R}}(C)\otimes\mathcal{J}_{N\cup Z}|$ consists only of C.

The main existence result of this subsection is Proposition 3.9 right below. To state it we need a definition:

Definition 3.8. A line bundle or Cartier divisor L on $\widetilde{R}$ is said to verify condition $(\star)$ if it is of the form $L\equiv\alpha\mathfrak{s}+\beta\mathfrak{f}-\sum_{i=1}^n\gamma_i\mathfrak{e}_i$ such that:
(i) $\alpha\geq 1$ and $\beta\geq 0$;

(ii) $\alpha \geq \gamma_i$ for $i = 1, \ldots, n$;
(iii) $\alpha + \beta \geq \sum_{i=1}^{n} \gamma_i$;
(iv) $\alpha + 2\beta \geq \sum_{i=1}^{n} \gamma_i + 4$ (equivalently, $-L \cdot K_{\widetilde{R}} \geq 4$).

Proposition 3.9. *Let E and $y_1, \ldots, y_n \in T$ be general. Assume that L is a line bundle on $\widetilde{R}$ that is odd (cf. Definition 3.4) and satisfies condition ($\star$) (cf. Definition 3.8). Then, if $0 < m \leq L \cdot T - 3$, the variety $V^{**}_{1,m}(\widetilde{R}, T, L)$ (cf. Definitions 3.1 and 3.7) has pure dimension $L \cdot T - m + 1$. Moreover, for all curve $G \subset \widetilde{R}$ not having T as an irreducible component, the general member of $V^{**}_{1,m}(\widetilde{R}, T, L)$ intersects G transversely.*

Proof. By Proposition 3.2(ii)–(iv) and Lemma 3.6(iii) we only need to prove non–emptiness of $V_{1,m}(\widetilde{R}, T, L)$. Following an idea in the proof of [6, Thm. 3.10], we will prove this by induction on m. The base case $m = 1$ follows from [11, Prop. 2.3].

Assume that we have proved non–emptiness of $V_{1,m}(\widetilde{R}, T, L)$ for some $1 \leq m \leq L \cdot T - 4$. By Lemma 3.6(ii) its general member C satisfies

$$C \cap T = mp_0 + p_1 + \cdots + p_l + p_{l+1}, \quad l = L \cdot T - m - 1 \geq 3,$$

where $p_0, \ldots, p_{l+1}$ are pairwise distinct, general points on T. Set $\mathfrak{d} = mp_0 + p_1 + \cdots + p_l$. Then $V_{1,\mathfrak{d}}(\widetilde{R}, T, L) \neq \emptyset$ and all its components are one-dimensional, by Proposition 3.2(i). The general member in any component intersects T in $mp_0 + p_1 + \cdots + p_l + q$, where the point q varies in the family, by Proposition 3.2(iii). Pick a component $\overline{V}$ of its closure inside the component of the Hilbert scheme of $\widetilde{R}$ containing $|L|$. After a finite base change, we find a smooth projective curve B, a surjective morphism $B \to \overline{V}$ and a family

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{f} & \widetilde{R} \\ {\scriptstyle g}\downarrow & & \\ B & & \end{array}$$

of stable maps of genus one such that, setting $\mathcal{C}_b := g^*b$ for any $b \in B$, the curve $f_*\mathcal{C}_b$ is a member of $\overline{V}$, and such that

$$f^*T = mP_0 + P_1 + \cdots + P_l + Q + W,$$

where

(I) P_i and Q are sections of g, for $i \in \{0, \ldots, l\}$,
(II) $f(P_i) = p_i$, for $i \in \{0, \ldots, l\}$,
(III) $f(Q) = T$,
(IV) $g_*W = 0$,
(V) $f_*W = 0$;

the latter property follows from the fact that no member of the family contains T, by Lemma 3.6(i).

Property (III) implies that $f^{-1}(T)$ is connected as follows. Consider the Stein factorization $\mathcal{C} \xrightarrow{f'} R' \xrightarrow{h} \tilde{R}$ of f. Then $h^{-1}(T)$ is of pure dimension 1. Since all irreducible components of f^*T except Q are contracted by f, it follows that $h^{-1}(T) = f'(Q)$, in particular it is irreducible. Eventually, since f' has connected fibers, $f^{-1}(T) = (f')^{-1}(h^{-1}(T))$ is connected.

In particular, P_0 and Q are connected by an effective divisor $W' \subset f^{-1}(p_0) \cap \mathcal{C}_{b_0} \subset W$ for some $b_0 \in B$. Thus,

$$f_*\mathcal{C}_{b_0} \cap T = (m+1)p_0 + p_1 + \cdots + p_l, \;\; l \geq 3. \tag{8}$$

By the generality of the points $p_0, \ldots, p_l$, they cannot be contained in any (-1)-curve on $\widetilde{R}$, nor can any two of them lie in a fiber of $\pi : \widetilde{R} \to E$. Consequently, $f_*\mathcal{C}_{b_0}$ cannot contain any rational component. Moreover, $f_*\mathcal{C}_{b_0}$ must be a reduced curve by (8). Therefore $f_*\mathcal{C}_{b_0} = C$ is an irreducible elliptic curve, hence $\mathcal{C}_{b_0}$ consists of one smooth elliptic curve $\widetilde{C}$ such that $f(\widetilde{C}) = C$ and otherwise chains of rational curves contracted by f and attached to $\widetilde{C}$ at one single point each. Therefore $f^{-1}(p_0) \cap \widetilde{C}$ is a single (smooth) point of $\widetilde{C}$, hence $[C] \in V_{1,(m+1)p_0+p_1+\cdots+p_l}(\widetilde{R}, T, L)$ by (8), which implies $[C] \in V_{1,m+1}(\widetilde{R}, T, L)$. □

3.3. **Logarithmic Severi varieties on blown up planes.** Fix a smooth cubic curve $T \subset P = \mathbb{P}^2$. Let $y_1, \ldots, y_n \in T$, for $n \geq 0$, and consider the blow–up $\widetilde{P} := \mathrm{Bl}_{y_1,\ldots,y_n}(P) \to P$ at $y_1, \ldots, y_n$. We denote the strict transforms of the general line on P by ℓ and by $\mathfrak{e}_i$ the (total) exceptional divisor over y_i. We denote still by T the strict transform of T. Note that $T \sim -K_{\widetilde{P}} \sim 3\ell - \mathfrak{e}_1 - \cdots - \mathfrak{e}_n$.

The next result parallels Lemma 3.6.

Lemma 3.10. *Let L be a line bundle or Cartier divisor on $\widetilde{P}$. Let m be any integer satisfying $1 \leq m \leq L \cdot T$. Then the following hold:*

(i) No curve C in $\overline{V}_{0,m}(\widetilde{P}, T, L)$ contains T.

(ii) For any component $V \subset \overline{V}_{0,m}(\widetilde{P}, T, L)$ the restriction map

$$\begin{array}{ccc} V & \longrightarrow & \mathrm{Sym}^{L\cdot T}(T)_m \\ C & \mapsto & C \cap T \end{array}$$

is well-defined and finite, with image $|L|_T| \cap \mathrm{Sym}^{L\cdot T}(T)_m$, which has codimension one. In particular,

$$\dim(V) = L \cdot T - m.$$

Proof. Since the members of $\overline{V}_{0,m}(\widetilde{P}, T, L)$ are limits of rational curves, none of them can contain T as a component, which proves (i). As in the proof of Lemma 3.6(ii), the restriction map is everywhere defined and finite. Its image lies in $|L|_T| \cap \mathrm{Sym}^{L\cdot T}(T)_m$. Since, by Proposition 3.2(ii) and semicontinuity, $\dim(V) \geq L \cdot T - m$, which equals $\dim\left(|L|_T| \cap \mathrm{Sym}^{L\cdot T}(T)_m\right)$, the latter is in fact the image. This proves (ii). □

The next result is about the relative version $\nu : \overline{\mathcal{V}^{[n]}_{g,m}(P, T, L)} \longrightarrow T^n$ of the logarithmic Severi variety $\overline{V}_{0,m}(\widetilde{P}, T, L)$ considered in Lemma 3.10 above (see subsection 3.1).

Lemma 3.11. *(i) Assume that $n > 0$ and L is a relative line bundle that is positive on the i-th exceptional divisor. Fix a point $(y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n) \in T^{n-1}$. Let $\mathcal{V}$ be any component of*

$$\{\nu^{-1}(y_1, \ldots, y_{i-1}, p, y_{i+1}, \ldots, y_n), \;\; p \in T\}.$$

Then the restriction map

$$\mathcal{V} \longrightarrow \mathrm{Sym}^{L\cdot T}(T)_m$$

is finite and surjective.

(ii) Assume furthermore that $n \geq 2$ and L is positive, with two different values, on the i-th and j-th exceptional divisor, $i < j$. Fix any linear series $\mathfrak{g}$ of type g^1_2 on T and any point $(y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_{j-1}, y_{j+1}, \ldots, y_n) \in T^{n-2}$. Let $\mathcal{V}$ be any component of the subset

$$\{\nu^{-1}(y_1, \ldots, y_{i-1}, p, y_{i+1}, \ldots, y_{j-1}, q, y_{j+1}, \ldots, y_n) \,|p + q \in \mathfrak{g}\}.$$

Then the restriction map

$$\mathcal{V} \longrightarrow \mathrm{Sym}^{L\cdot T}(T)_m$$

is finite and surjective.

Proof. Assume $L \cdot \mathfrak{e}_i > 0$ for some i. Varying p, we obtain a one-dimensional nontrivial family of surfaces $\mathrm{Bl}_{y_1,\ldots,y_{i-1},p,y_{i+1},\ldots,y_n}(P)$ and a one-dimensional non–constant family of line bundles whose restrictions to T yield a one-dimensional non–constant family of line bundles. This together with Lemma 3.10(ii) yields (ii).

Finally, assume $L \cdot \mathfrak{e}_i = a_i > 0$ and $L \cdot \mathfrak{e}_j = a_j > 0$, with $a_i \neq a_j$. Varying $p+q \in \mathfrak{g}$, we get a one-dimensional nontrivial family of surfaces $\mathrm{Bl}_{y_1,\ldots,y_{i-1},p,y_{i+1},\ldots,y_{j-1},q,y_{j+1},\ldots,y_n}(P)$ as above and a one-dimensional family of line bundles, all of the form $L' - a_i\mathfrak{e}_i - a_j\mathfrak{e}_j$, with L' fixed (on $\mathrm{Bl}_{y_1,\ldots,y_{i-1},y_{i+1},\ldots,y_{j-1},y_{j+1},\ldots,y_n}(P)$) and $\mathfrak{e}_i$ and $\mathfrak{e}_j$ varying with p and q. To prove (iv), we will prove that the family of restrictions

$$\left\{L'|_T - a_ip - a_jq\right\}_{p+q\in\mathfrak{g}}$$

to T is non–constant. Assume that

$$L'|_T - a_ip - a_jq \sim L'|_T - a_ip' - a_jq' \text{ for } p + q \neq p' + q' \in \mathfrak{g}. \tag{9}$$

This yields $(a_i - a_j)p \sim (a_i - a_j)p'$ and $(a_i - a_j)q \sim (a_i - a_j)q'$. For fixed $x \in T$, there are only finitely many points $y \in T$ such that $(a_i - a_j)x \sim (a_i - a_j)y$. For general $p+q, p'+q' \in \mathfrak{g}$, condition (9) is therefore not fulfilled. This finishes the proof of (ii). □

The main existence result of this subsection is the following:

Proposition 3.12. *Let $y_1, \ldots, y_n \in T$ be general, $n \leq 8$, and L be big and nef on $\widetilde{P}$. If $0 < m \leq L\cdot T - 3$, the variety $V^*_{0,m}(\widetilde{P}, T, L)$ is nonempty of dimension $T\cdot L - m$. Moreover, its general member intersects any fixed curve on $\widetilde{P}$ different from T transversely.*

Proof. This is an application of [6, Cor. 3.11]; there are some conditions to check, so we give a proof for completeness.

The statements about dimension and transversal intersection follow from Proposition 3.2(ii) and (iii), respectively, once non–emptiness is proved. By Proposition 3.2(iii)–(iv) we have that $V^*_{0,m}(\widetilde{P}, L) \neq \emptyset$ as soon as $V_{0,m}(\widetilde{P}, T, L) \neq \emptyset$, because of the condition $m \leq L \cdot T - 3$. We therefore have left to prove nonemptiness of $V_{0,m}(\widetilde{P}, T, L)$. We will prove this by induction on m, as in the proof of Proposition 3.9, again following an idea in the proof of [6, Thm. 3.10].

Since $y_1 \ldots, y_n \in T$ are general and $n \leq 8$, we may take $y_1, \ldots, y_n$ to be general points of $\mathbb{P}^2$ and T a general plane cubic containing them. Hence $\widetilde{P}$ is a Del Pezzo surface, so that T is ample on it. It is then well-known, by [20, Thms. 3-4], that $V_{0,1}(\widetilde{P}, T, L) \neq \emptyset$.

Assume now that we have proved non–emptiness of $V_{0,m}(\widetilde{P}, T, L)$ for some $1 \leq m \leq L \cdot T - 4$. By Lemma 3.10(ii) its general member C satisfies

$$C \cap T = mp_0 + p_1 + \cdots + p_l + p_{l+1} + p_{l+2}, \;\; l = L \cdot T - m - 2 \geq 2,$$

where $p_0, \dots, p_{l+2}$ are distinct, and we may take $p_0, \dots, p_{l+1}$ general on T.

For later purposes, we observe that, since there are only finitely many divisor classes D such that $C - D > 0$ (hence $C \cdot T > D \cdot T$), and for each such class the image of the restriction morphism $|D| \to \mathrm{Sym}^{D \cdot T}(T)$ has codimension one, the generality of the points implies that

$$\text{there is no effective divisor } D \neq T \text{ such that} \quad C - D > 0 \text{ and } D \cap T \subset \{p_0, p_1, \dots, p_l\}. \tag{10}$$

Set $\mathfrak{d} = mp_0 + p_1 + \cdots + p_l$. Then $V_{0,\mathfrak{d}}(\widetilde{P}, T, L) \neq \emptyset$ and all its components are one-dimensional, by Proposition 3.2(i). The general member in any component intersects T in $mp_0 + p_1 + \cdots + p_l + q_1 + q_2$, where the points q_1, q_2 vary in the family, by Proposition 3.2(iii). Pick a component $\overline{V}$ of its closure inside the component of the Hilbert scheme of $\widetilde{P}$ containing $|L|$. After a finite base change, we find a smooth projective curve B, a surjection $B \to \overline{V}$ and a family

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{f} & \widetilde{P} \\ {\scriptstyle g}\downarrow & & \\ B & & \end{array}$$

of stable maps of genus zero such that, setting $\mathcal{C}_b := g^*b$ for any $b \in B$, the curve $f_*\mathcal{C}_b$ is a member of $\overline{V}$, and such that

$$f^*T = mP_0 + P_1 + \cdots + P_l + Q_1 + Q_2 + W,$$

where

- (I) P_i and Q_j are sections of g, for $i \in \{0, \dots, l\}$, $j \in \{1, 2\}$,
- (II) $f(P_i) = p_i$, for $i \in \{0, \dots, l\}$,
- (III) $f(Q_j) = T$, for $j \in \{1, 2\}$,
- (IV) $g_*W = 0$,
- (V) $f_*W = 0$;

the latter property follows from the fact that no member of $\overline{V}$ contains T, by Lemma 3.10(i). Property (III) implies as in the proof of Proposition 3.9 that $f^{-1}(T)$ is either connected, or has two connected components containing Q_1 and Q_2 respectively. In particular, we may assume that P_0 and Q_2 are connected by a chain $W' \subset W$ such that $W' \subset \mathcal{C}_{b_0}$ for some $b_0 \in B$, and $f(W') = p_0$. Thus,

$$f_*\mathcal{C}_{b_0} \cap T = (m+1)p_0 + p_1 + \cdots + p_l + q_1, \quad q_1 = f(Q_1), \quad l \geq 2. \tag{11}$$

Since T is ample, all components of $f_*\mathcal{C}_{b_0}$ intersect T. By (10) and (11), $f_*\mathcal{C}_{b_0}$ must be reduced and irreducible, say $f_*\mathcal{C}_{b_0} = C$, an irreducible rational curve. Since $\mathcal{C}_{b_0}$ has arithmetic genus 0, it consists of a tree of smooth rational curves, with one component $\widetilde{C}$ such that $f(\widetilde{C}) = C$, and the other components contracted by f. Therefore, $f^{-1}(p_0) \cap \widetilde{C} = (W' + P_0) \cap \widetilde{C}$ is a single (smooth) point of $\widetilde{C}$. It follows that $[C] \in V_{0,(m+1)p_0+p_1+\cdots+p_l}(\widetilde{P}, T, L)$, which implies $[C] \in V_{0,m+1}(\widetilde{P}, T, L)$. □

4. Deforming to rigid elliptic curves

As mentioned in the introduction, to prove Theorem 1.1 it will suffice by [9, Cor. 1] to prove that $V_{|L|,g-1}(S)$ has a 0-dimensional component. We will call any element of such a 0-dimensional component a *rigid nodal elliptic curve*. We will prove the existence of such a curve on a general (S, L) in any component of $\mathcal{E}_g \setminus \mathcal{E}_g[2]$ by degeneration, using Theorem 2.2, constructing suitable curves on limit surfaces in $\mathcal{D}^*$ that will deform to rigid curves in $V_{|L|,g-1}(S)$. In this section we will identify numerical conditions on limit line bundles under which deformations to such curves can be achieved. The general strategy of proof is given in the following:

Proposition 4.1. *Let $X = \widetilde{R} \cup_T \widetilde{P}$ be a general member of a component of $\mathcal{D}^*$ and $Y = C \cup_T D$ a curve on X, with $C \subset \widetilde{R}$ and $D \subset \widetilde{P}$, having the following properties: there are distinct points $x, p_1, \ldots, p_k, q_1, \ldots, q_l$ on T, for nonnegative integers k, l, and a positive integer m such that*

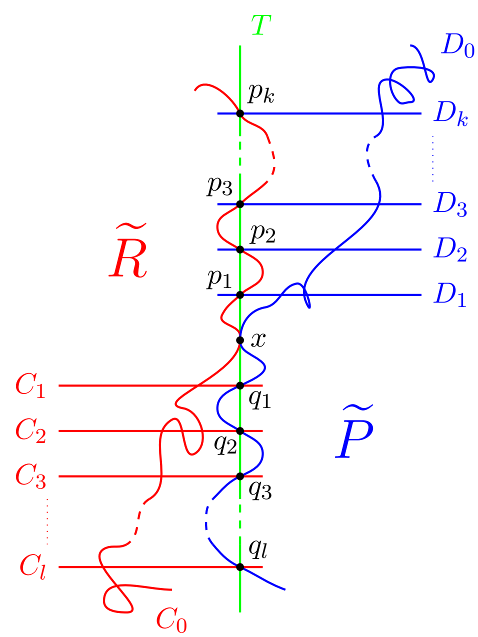
- *$C = C_0 + C_1 + \cdots + C_l$ is nodal, with $[C_0] \in V^{**}_{1,m}(\widetilde{R}, T, C_0)$ and C_i a (-1)-curve, $i \in \{1, \ldots, l\}$,*
- *$C_0 \cap T = mx + p_1 + \cdots + p_k$,*
- *C_0 is odd,*
- *$C_i \cap T = \{q_i\}$, $i \in \{1, \ldots, l\}$,*

and

- *$D = D_0 + D_1 + \cdots + D_k$ is nodal, with $[D_0] \in V^{*}_{0,m}(\widetilde{P}, T, D_0)$ and D_i a (-1)-curve, $i \in \{1, \ldots, k\}$,*
- *$D_0 \cap T = mx + q_1 + \cdots + q_l$,*
- *$D_i \cap T = \{p_i\}$, $i \in \{1, \ldots, k\}$.*

Then Y deforms to an irreducible rigid nodal elliptic curve on the general deformation S of X.

Proof. Here is a picture of what Y looks like:



We note that Y is Cartier, has an m–tacnode at x and is otherwise nodal. Moreover, an easy computation as in [23, p. 119] shows that

$$\dim(|Y|) = \frac{1}{2}Y^2 = p_a(Y) - 1. \tag{12}$$

We define

- N_{C_0} the scheme of nodes of C_0 and γ_0 its degree,
- N_C the scheme of intersection points between components of C and γ its degree,
- N_{D_0} the scheme of nodes of D_0 and δ_0 its degree,
- N_D the scheme of intersection points between components of D and δ its degree.

Then $N_{C_0} \cup N_C \cup N_{D_0} \cup N_D$ is the set of nodes of Y off T. Since

$$\gamma_0 = p_a(C_0) - 1 = \frac{1}{2} C_0 \cdot (C_0 + K_{\widetilde{R}}) = \frac{1}{2}\left(C_0^2 - C_0 \cdot T\right) = \frac{1}{2}\left(C_0^2 - m - k\right)$$

and, similarly,

$$\delta_0 = p_a(D_0) = \frac{1}{2}\left(D_0^2 - m - l\right) + 1,$$

we compute

$$\begin{aligned} (13)\quad p_a(Y) &= \frac{1}{2}Y^2 + 1 = \frac{1}{2}\left(C^2 + D^2\right) + 1 = \frac{1}{2}\left(C_0^2 - l + 2\gamma + D_0^2 - k + 2\delta\right) + 1 \\ &= \frac{1}{2}\left(C_0^2 - m - k\right) + \frac{1}{2}\left(D_0^2 - m - l\right) + m + \gamma + \delta + 1 \\ &= \gamma_0 + (\delta_0 - 1) + m + \gamma + \delta + 1 = \gamma_0 + \gamma + \delta_0 + \delta + m. \end{aligned}$$

Let $\mathfrak{X} \to \mathbb{D}$ be the deformation of X to a general smooth Enriques surface S_t in Theorem 2.2. Then Y deforms to a Cartier divisor Y_t on S_t by the same theorem. By (12) we have $\dim(|Y|) = \dim(|Y_t|)$. Let $\mathfrak{D}$ be the sublinear system of $|Y|$ of curves with an $(m-1)$–tacnode at x and passing through $N_{C_0} \cup N_C \cup N_{D_0} \cup N_D$. We claim that

$$(14)\qquad \mathfrak{D} \text{ consists only of } Y \text{ itself.}[1]$$

Granting this for the moment, (12)-(14) yields that the codimension of $\mathfrak{D}$ is $m - 1 + \gamma_0 + \gamma + \delta_0 + \delta$. Thus, the hypotheses of [18, Thm. 3.3, Cor. 3.12 and proof of Thm. 1.1] are fulfilled[2] and we conclude that, under the deformation of X to S_t, we may deform Y deforming the m-tacnode of Y at x to $m-1$ nodes, and simultaneously preserving the $\gamma_0 + \gamma + \delta_0 + \delta$ nodes of Y in the smooth locus of X, whereas the nodes of Y on T automatically smooth. Thus Y deforms to a nodal curve $Y_t \subset S_t$ with a total of

$$\gamma_0 + \gamma + \delta_0 + \delta + m - 1 = p_a(Y) - 1 = p_a(Y_t) - 1$$

nodes (using (13)). Since one easily sees that no subcurve of Y is Cartier, Y_t is irreducible, whence Y_t is nodal and elliptic, as desired. It is rigid, as Y is rigid on X.

We have left to prove (14). To this end, let $A \cup_T B \in |Y|$ be a curve with an $(m-1)$–tacnode at x and passing through $N_{C_0} \cup N_C \cup N_{D_0} \cup N_D$, where $A \subset \widetilde{R}$ and $B \subset \widetilde{P}$. Then both A and B must intersect T in a scheme containing $(m-1)x$ and moreover $N_{C_0} \cup N_C \subset A$ and $N_{D_0} \cup N_D \subset B$.

The fact that $N_D \subset B$ implies that B must contain all (-1)–curves $D_1, \ldots, D_k$. Hence $B = B_0 + D_1 + \cdots + D_k$, with $B_0 \sim D_0$. Similarly, the fact that $N_C \subset A$ implies that A must contain all (-1)–curves $C_1, \ldots, C_k$. Hence $A = A_0 + C_1 + \cdots + C_k$, with

[1] From a deformation-theoretic point of view, (14) implies that the *equisingular deformation locus of Y in $\mathfrak{X}$* is smooth and zero-dimensional (cf. [18, Lemma 3.4]), thus consisting only of the point $[Y]$.

[2] We remark that the hypothesis in [18] that both components of X are regular is not necessary; it suffices that $h^1(\mathcal{O}_X) = 0$, which is proved as in [23, Lemma 3.4].

$A_0 \sim C_0$. Since $A \cup_T B$ is Cartier, A_0 must pass through $p_1, \ldots, p_k$ and B_0 must pass through $q_1, \ldots, q_l$. Thus

$$A_0 \cap T \supset (m-1)x + p_1 + \cdots + p_k =: Z_C \text{ and } B_0 \cap T \supset (m-1)x + q_1 + \cdots + q_l =: Z_D.$$

Hence $A_0 \in |\mathcal{O}_{\widetilde{R}}(C_0) \otimes \mathcal{J}_{N_{C_0} \cup Z_C}|$. Since $\deg(Z_C) = C_0 \cdot T - 1$ and $[C_0] \in V^{**}_{1,m}(\widetilde{R}, T, C_0)$, this implies $A_0 = C_0$ (recall Definition 3.7), whence $A = C$.

Similarly, $B_0 \in |\mathcal{O}_{\widetilde{P}}(D_0) \otimes \mathcal{J}_{N_{D_0} \cup Z_D}|$, whence, if $B_0 \neq D_0$, we would get

$$\begin{aligned} D_0^2 &= B_0 \cdot D_0 \geq \deg(Z_D) + 2\deg(N_{D_0}) = (D_0 \cdot T - 1) + 2p_a(D_0) \\ &= -D_0 \cdot K_{\widetilde{P}} - 1 + D_0^2 + D_0 \cdot K_{\widetilde{P}} + 2 > D_0^2, \end{aligned}$$

a contradiction. Thus $B_0 = D_0$, whence $B = D$. This proves (14). □

The next two results form the basis for our proof of Theorem 1.1. We henceforth assume that E is a *general* elliptic curve.

Proposition 4.2. *Let R' (respectively, P') be a blow–up of R (resp., P) at $s \geq 0$ (resp,. $t \geq 1$) general points of T. Assume L' (resp., L'') is a line bundle (or Cartier divisor) on R' (resp., P') and k is an integer such that the following conditions are satisfied:*

- *(i) $s + t - 5 \leq k \leq \min\{3, s, t-1\}$,*
- *(ii) $L' \cdot T = L'' \cdot T$,*
- *(iii) $L' \equiv L'_0 + C_1 + \cdots + C_k$, where the C_i are disjoint (-1)-curves and L'_0 satisfies condition $(\star)$ and is odd,*
- *(iv) $L'' \sim L''_0 + D_1 + \cdots + D_k$, where the D_i are disjoint (-1)-curves and L''_0 is big and nef,*
- *(v) there are $t-k$ additional (-1)-curves on P', mutually disjoint and disjoint from $D_1, \ldots, D_k$, such that L'' is positive on at least one of them.*

Then there are blow–ups $\widetilde{R} \to R'$ and $\widetilde{P} \to P'$ at distinct point of T such that $\widetilde{R} \cup_T \widetilde{P}$ is general in a component of $\mathcal{D}^*$ and, denoting by $\widetilde{L}'$ and $\widetilde{L}''$ the pull–backs of L' and L'' to $\widetilde{R}$ and $\widetilde{P}$ respectively, there is a line bundle $\widetilde{L} \in [\widetilde{L}', \widetilde{L}'']$ (cf. Remark 2.1) such that $(\widetilde{R} \cup_T \widetilde{P}, \widetilde{L})$ deforms to a smooth polarized Enriques surface (S, L) with S containing an irreducible, rigid nodal elliptic curve in $|L|$.

Proof. Set $m := L'_0 \cdot T - 3 = L''_0 \cdot T - 3$ (equality follows from assumptions (ii)–(iv)). By condition (iv) in $(\star)$, we have $m > 0$. The line bundle L'_0 satisfies the conditions of Proposition 3.9. Hence $V^{**}_{1,m}(R', T, L'_0) \neq \emptyset$ and its general member intersects $C_1, \ldots, C_k$ transversely. Let C_0 be such a general member; we have $L' \equiv C_0 + C_1 + \cdots + C_k$ by assumption (iii).

The surface R' is a blow–up of R at s general points $y_1, \ldots, y_s$ of T, where $k \leq s \leq 4$ by assumption (i). Denoting the exceptional divisor over y_i by $\mathfrak{e}_i$, the surface R' contains precisely s additional (-1)-curves $\mathfrak{e}'_i$, $i = 1, \ldots, s$, such that $\mathfrak{e}_i \cdot \mathfrak{e}'_i = 1$ and each $\mathfrak{e}_i + \mathfrak{e}'_i$ is a fiber of the projection $R' \to E$. Set $y'_i := \mathfrak{e}'_i \cap T$. Since $C_1, \ldots, C_k$ are disjoint (-1)-curves by assumption (iii), we have, after renumbering, that $C_i = \mathfrak{e}_i$ or $\mathfrak{e}'_i$. We will in the following for simplicity assume that $C_i = \mathfrak{e}_i$; the other cases can be treated in the same way by substituting y_i with y'_i at the appropriate places. Thus we have

$$\begin{aligned} C_0 \cap T &= mp + p_1 + p_2 + p_3, \text{ for } p, p_1, p_2, p_3 \in T, \\ C_i \cap T &= y_i, \ i \in \{1, \ldots, k\}. \end{aligned}$$

By Lemma 3.6(ii) the points p, p_1, p_2, p_3 are general on T (even for fixed $y_1, \ldots, y_k$). The points $y_1, \ldots, y_k$ are general on T by the assumption that R' is a blow–up of R at *general* points on T.

The surface P' is a blow–up of $\mathbb{P}^2$ at t general points on T, where $k < t \leq 5$ by assumption (i). The line bundle L''_0 is big and nef by assumption (iv), whence by Proposition 3.12 we have $V^*_{0,m}(P', T, L''_0) \neq \emptyset$ and its general member intersects $D_1, \ldots, D_k$ transversely. Let D_0 be such a general member; we have $L'' \sim D_0 + D_1 + \cdots + D_k$ by assumption (iv).

By assumption (iv)-(v), $D_1, \ldots, D_k$ are disjoint (-1)-curves belonging to a set of t disjoint (-1)-curves, which we may therefore take as an exceptional set for a blowdown $P' \to \mathbb{P}^2$ centered at points that we denote by $x_1, \ldots, x_t \in T$. Furthermore, there is an exceptional curve, different from $D_1, \ldots, D_k$, call it D_{k+1}, such that $D_{k+1} \cdot L''_0 > 0$. We have

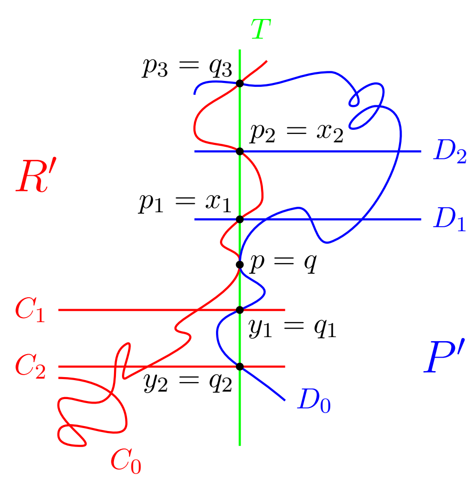
$$\begin{aligned} D_0 \cap T &= mq + q_1 + q_2 + q_3, \text{ for } q, q_1, q_2, q_3 \in T, \\ D_i \cap T &= x_i, \ i \in \{1, \ldots, k\}. \end{aligned}$$

By Lemma 3.11(i), using the existence of D_{k+1}, the points q, q_1, q_2, q_3 are general on T (even for fixed $x_1, \ldots, x_k$). The points $x_1, \ldots, x_k$ are general on T by the assumption that P' is a blow–up of P at *general* points on T.

Since the points $p, p_1, p_2, p_3, y_1, \ldots, y_k$ are general on T, and likewise the points $q, q_1, q_2, q_3, x_1, \ldots, x_k$, we may assume that our choices of C_0 and D_0 come with the identifications

$$\begin{gathered} p = q \\ x_i = p_i, \ \ y_i = q_i \ \text{ for } \ 1 \leq i \leq k \quad (\text{recall that } k \leq 3), \\ p_i = q_i \ \text{ for } \ k+1 \leq i \leq 3 \quad (\text{if } k = 3, \text{ this condition is empty}), \end{gathered}$$

so that we can glue R' and P' along T in such a way that $(C_0 + C_1 + \cdots + C_k, D_0 + D_1 + \cdots + D_k)$ is Cartier on $R' \cup_T P'$. The following picture shows the case $k = 2$:



By assumption (i) we have that $s + t \leq k + 5 \leq 8$. The set of points

$$y_1, \ldots, y_s, x_1, \ldots, x_t, (p_{k+1} = q_{k+1}), \ldots, (p_3 = q_3)$$

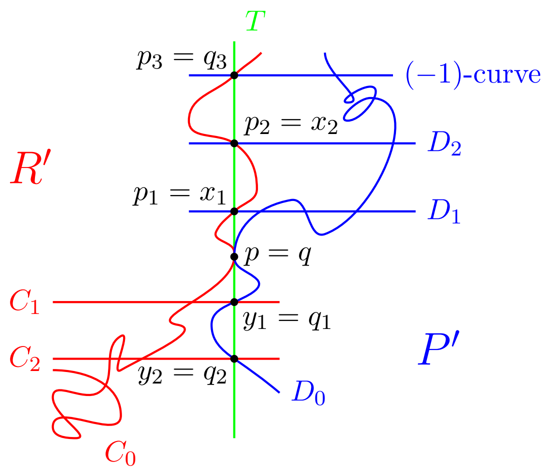
is therefore a set of $s+t+3-k \leq 8$ general points on T. Pick now points $w_1, \ldots, w_{k+6-s-t}$ on T, general under the condition that

$$y_1 + \cdots + y_s + x_1 + \cdots + x_t + p_{k+1} + \cdots + p_3 + w_1 + \cdots + w_{k+6-s-t} \in |\mathcal{N}_{T/R} \otimes \mathcal{N}_{T/P}|.$$

Note that in this way we get a general divisor of $|\mathcal{N}_{T/R} \otimes \mathcal{N}_{T/P}|$. Then blow up either R' or P' at the points

$$(p_{k+1} = q_{k+1}), \ldots, (p_3 = q_3), w_1, \ldots, w_{k+6-s-t}$$

to obtain $\widetilde{R} \cup_T \widetilde{P}$, which is a general member of a component of $\mathcal{D}^*$, containing the inverse image Y of the curve $(C_0 + C_1 + \cdots + C_k) \cup_T (D_0 + D_1 + \cdots + D_k)$, which satisfies the conditions of Proposition 4.1. The result therefore follows by Proposition 4.1. (The following picture shows the inverse image of the curve in the previous image in the case $k = 2$: an additional (-1)-curve appears over $p_3 = q_3$.)



□

Proposition 4.3. *Let $\widetilde{R}$ (respectively, $\widetilde{P}$) be a blow–up of R (resp., P) at 4 (resp., 5) general points of T. Assume L' (resp., L'') is a line bundle (or Cartier divisor) on $\widetilde{R}$ (resp., $\widetilde{P}$) such that the following conditions are satisfied:*

- *(i)* $L' \cdot T = L'' \cdot T$,
- *(ii)* $L' \equiv L'_0 + C_1 + C_2 + C_3$, *where the* C_i *are disjoint* (-1)*-curves and* L'_0 *satisfies condition* $(\star)$ *and is odd,*
- *(iii)* $L'' \sim L''_0 + D_1 + D_2 + D_3$, *where the* D_i *are disjoint* (-1)*-curves and* L''_0 *is big and nef,*
- *(iv) there are two additional* (-1)*-curves* D_4, D_5 *on* $\widetilde{P}$*, mutually disjoint and disjoint from* D_1, D_2, D_3*, satisfying* $L'' \cdot D_4 \neq L'' \cdot D_5$.

Then there exist $\overline{R}$ (respectively, $\overline{P}$) a blow–up of R (resp., P) at 4 (resp., 5) points of T and a line bundle $\overline{L}'$ on $\overline{R}$ (resp. $\overline{L}''$ on $\overline{P}$) such that:

- *(a) the pair* $(\overline{R}, \overline{L}')$ *(resp.* $(\overline{P}, \overline{L}'')$*) is a deformation of* $(\widetilde{R}, L')$ *(resp.* $(\widetilde{P}, L'')$*),*
- *(b) the surface* $\overline{R} \cup_T \overline{P}$ *is a general member of* $\mathcal{D}^*_{[4]}$ *and* $(\overline{L}', \overline{L}'')$ *is a line bundle on it,*
- *(c)* $(\overline{R} \cup_T \overline{P}, (\overline{L}', \overline{L}''))$ *deforms to a smooth polarized Enriques surface* (S, L) *such that* $|L|$ *contains an irreducible, rigid nodal elliptic curve.*

Proof. We argue as in the beginning of the previous proof with $k = 3$, $s = 4$ and $t = 5$, noting that L'' is positive on at least one of the two curves D_4, D_5. We find, as before, a Cartier divisor in a surface

$$Y := C \cup_T D \subset \widetilde{R} \cup_T \widetilde{P} =: X, \quad \widetilde{R} = \mathrm{Bl}_{y_1,y_2,y_3,y_4}(R), \quad \widetilde{P} = \mathrm{Bl}_{y_5,\ldots,y_9}(P), \tag{15}$$

for general $y_i \in T$, $i \in \{1, \ldots, 9\}$, such that, denoting by $\mathfrak{e}_i$ the exceptional divisor over y_i, one has

$$C = C_0 + \mathfrak{e}_1 + \mathfrak{e}_2 + \mathfrak{e}_3, \quad D = D_0 + \mathfrak{e}_5 + \mathfrak{e}_6 + \mathfrak{e}_7,$$

with $C \equiv L'$ and $D \sim L''$ both nodal, $C_0 \in V^{**}_{1,m}(\widetilde{R}, T, L'_0)$, $D_0 \in V^{*}_{0,m}(\widetilde{P}, T, L''_0)$, and we set $C_i = \mathfrak{e}_i$ for $i \in \{1, 2, 3\}$ and $D_i = \mathfrak{e}_{i+4}$ for $i \in \{1, 2, 3, 4, 5\}$. Moreover, setting $m := C_0 \cdot T - 3 = D_0 \cdot T - 3$, there is a point $x \in T$ such that

$$C_0 \cap T = mx + y_5 + y_6 + y_7, \quad D_0 \cap T = mx + y_1 + y_2 + y_3.$$

As in the previous proof, the points x, y_i are general on T.

The problem now is that we cannot a priori guarantee that $\sum_{i=1}^{9} y_i \in |\mathcal{N}_{T/R} \otimes \mathcal{N}_{T/P}|$ to ensure that X is a member of $\mathcal{D}^*$ and therefore conclude as in the previous proof; we only know that X is a member of $\mathcal{D}$. We will prove that we can create such a member of $\mathcal{D}^*$ without losing the "nice" properties of Y.

Note first that condition (iv) says that $L'' \cdot \mathfrak{e}_8 \neq L'' \cdot \mathfrak{e}_9$. If $L'' \cdot \mathfrak{e}_i = 0$ for $i = 8$ or 9, we may contract $\mathfrak{e}_i$ and reduce to the case studied in the previous proposition. We will therefore assume that

$$L'' \cdot \mathfrak{e}_8 > 0, \;\; L'' \cdot \mathfrak{e}_9 > 0 \;\text{ and }\; L'' \cdot \mathfrak{e}_8 \neq L'' \cdot \mathfrak{e}_9. \tag{16}$$

Fix now general $y_1, y_2, y_3, y_5, y_6, y_7 \in T$. Varying (x_4, x_8, x_9) in T^3 we obtain a 3–dimensional family of surfaces of the form

$$\mathrm{Bl}_{y_1,y_2,y_3,x_4}(R) \cup_T \mathrm{Bl}_{y_5,y_6,y_7,x_8,x_9}(P)$$

together with a family of lines bundles (L', L''). There exists a relative Hilbert scheme $\mathcal{H}$ of effective Cartier divisors $C' \cup_T D'$ on these surfaces, such that on each surface the line bundle (C', D') is in the numerical equivalence class $[L', L'']$, cf. Remark 2.1 (considering L' and L'' as relative line bundles, see subsection 3.1). By our assumptions we have a nonempty subscheme $\mathcal{W}$ of $\mathcal{H}$ with a dominating morphism $\mathcal{W} \to T^3$ whose fiber over a point (x_4, x_8, x_9) consists of pairs (X', Y') such that

- $X' = R' \cup_T P'$ with $R' = \mathrm{Bl}_{y_1,y_2,y_3,x_4}(R)$ and $P' = \mathrm{Bl}_{y_5,y_6,y_7,x_8,x_9}(P)$,
- $Y' = C' \cup_T D'$, where

$$\begin{aligned} C' &= C'_0 + \mathfrak{e}_1 + \mathfrak{e}_2 + \mathfrak{e}_3 \equiv L' \\ D' &= D'_0 + \mathfrak{e}_5 + \mathfrak{e}_6 + \mathfrak{e}_7 \sim L'', \end{aligned}$$

with $C'_0 \in V^{**}_{1,m}(R', T, L'_0)$, $D'_0 \in V^{*}_{0,m}(P', T, L''_0)$ intersecting all $\mathfrak{e}_i$ transversely,

$$\begin{aligned} C'_0 \cap T &= mx' + y_5 + y_6 + y_7, \\ D'_0 \cap T &= mx' + y_1 + y_2 + y_3, \end{aligned}$$

for a point $x' \in T \setminus \{y_1, y_2, y_3, y_5, y_6, y_7\}$.

We once and for all substitute $\mathcal{W}$ with a dominating component containing $[(X, Y)]$ and will henceforth assume $\mathcal{W}$ is irreducible. Taking the closure in $\mathcal{H}$, we obtain a closed scheme with a surjective morphism

$$g : \overline{\mathcal{W}} \to T^3.$$

If $[(X',Y')] \in \overline{\mathcal{W}}$, then Y' looks like $C' \cup_T D'$ as above, except that $C'_0 \in \overline{V}_{1,m}(R',T,L'_0)$ and $D'_0 \in \overline{V}_{0,m}(P',T,L''_0)$, the intersection with the $\mathfrak{e}_i$s need not be transversal and x' may coincide with one of the points y_is (this follows from Lemmas 3.6(i) and 3.10(i)). In any event (X',Y') comes equipped with a point $x' \in T$, which is the only point of intersection between the non-exceptional members of Y'. We will call this the *tacnodal point* of $[(X',Y')]$ (although it may be a worse singularity of Y' for special pairs). We therefore have a natural map

$$p : \overline{\mathcal{W}} \to T$$

sending a pair to its tacnodal point. For $x' \in T$ we set $\overline{\mathcal{W}}_{x'} := p^{-1}(x')$; this is the locus of pairs $[(X',Y')]$ of $\overline{\mathcal{W}}$ with tacnodal point x'.

Claim 4.4. *The map g is finite and* $\dim(\overline{\mathcal{W}}) = 3$.

Proof of claim. Since g is surjective, we only need to prove that g is finite. Fix any $X' = R' \cup_T P'$ as above, with $R' = \mathrm{Bl}_{y_1,y_2,y_3,x_4}(R)$ and $P' = \mathrm{Bl}_{y_5,y_6,y_7,x_8,x_9}(P)$, and assume $[(X',Y')] \in \overline{\mathcal{W}}$. Let $\overline{C_0} \subset R'$ and $\overline{D_0} \subset P'$ be the non-exceptional irreducible components of Y', elliptic and rational, respectively. They intersect only at the tacnodal point $x' \in T$, and otherwise they intersect T in fixed points, as the (-1)-curves are fixed on each component of X'. Therefore, by Lemma 3.10(ii), there are finitely many possibilities for the point x' and the curve $\overline{D_0}$, and consequently finitely many possibilities for the intersection $\overline{C_0} \cap T$. As $\overline{C_0}$ is odd by assumption (ii), there are by Lemma 3.6(ii) only finitely many possibilities for $\overline{C_0}$ as well. This proves that g is finite. □

Let $\left(T^3\right)^*$ denote the two-dimensional subset of triples $(x_4,x_8,x_9) \in T^3$ such that

$$y_1+y_2+y_3+x_4+y_5+y_6+y_7+x_8+x_9 \in |\mathcal{N}_{T/R} \otimes \mathcal{N}_{T/P}|, \tag{17}$$

and let $\overline{\mathcal{W}}^* = g^{-1}\left(T^3\right)^*$. This is the locus of pairs $[(X',Y')]$ of $\overline{\mathcal{W}}$ such that X' is semi-stable. For $x' \in T$ we set $\overline{\mathcal{W}}^*_{x'} := \overline{\mathcal{W}}^* \cap \overline{\mathcal{W}}_{x'}$.

We write $g_1 : \overline{\mathcal{W}} \to T$ and $g_2 : \overline{\mathcal{W}} \to T \times T$ for the composition of g with the projections onto the first factor and onto the product of the second and third factors, respectively. In other words g_1 maps a pair $(X' = R' \cup_T P', Y')$ as above to x_4, whereas g_2 maps it to (x_8,x_9).

Claim 4.5. *For all $x' \in T$ the following hold:*

(i) $\overline{\mathcal{W}}^*_{x'} \neq \emptyset$ (hence $\overline{\mathcal{W}}_{x'} \neq \emptyset$),

(ii) $(g_1)|_{\overline{\mathcal{W}}^*_{x'}}$ is surjective (hence also $(g_1)|_{\overline{\mathcal{W}}_{x'}}$ is surjective).

Proof of claim. Fix any $x_4 \in T$. By Lemma 3.11(ii), taking into account property (16), and by Lemma 3.6(ii), $g_2(\overline{\mathcal{W}}_{x'} \cap g_1^{-1}(x_4))$ has non–empty intersection with the locus of points (x_8,x_9) such that

$$x_8+x_9 \in |\mathcal{N}_{T/R} \otimes \mathcal{N}_{T/P}(-y_1-y_2-y_3-x_4-y_5-y_6-y_7)|,$$

(equivalently, (17) is satisfied). This implies (i) and (ii). □

Note that Claim 4.5 also implies that p is surjective, whence all fibers $\overline{\mathcal{W}}_{x'}$ are two–dimensional by Claim 4.4.

Now consider the map

$$\begin{array}{rcl} \sigma : T \times T & \longrightarrow & \mathrm{Sym}^2(T), \\ (x,y) & \mapsto & x+y \end{array}$$

and recall that there is a fibration $u : \mathrm{Sym}^2(T) \to T$ with fibers being the g^1_2s on T. For each $x_4 \in T$, let $\mathfrak{g}(x_4) \subset \mathrm{Sym}^2(T)$ be the g^1_2 given by

$$\mathfrak{g}(x_4) := |\mathcal{N}_{T/R} \otimes \mathcal{N}_{T/P}(-y_1 - y_2 - y_3 - x_4 - y_5 - y_6 - y_7)|.$$

As $x_4 \in T$ varies, the one-dimensional family of $\mathfrak{g}(x_4)$s are precisely the fibers of u.

Claim 4.6. *For all $x' \in T$ one has $u(\sigma(g_2(\overline{\mathcal{W}}_{x'}))) = T$, that is, $\sigma(g_2(\overline{\mathcal{W}}_{x'}))$ is not a union of fibers of u.*

Proof of claim. Suppose to the contrary that $\sigma(g_2(\overline{\mathcal{W}}_{x'}))$ is a union of fibers of u. Then, for general $x_4 \in T$ we would have $\sigma(g_2(\overline{\mathcal{W}}_{x'})) \cap \mathfrak{g}(x_4) = \emptyset$, contradicting Claim 4.5(ii). □

Let now $x' \in T$ be general. Set $\mathcal{W}_{x'} = \overline{\mathcal{W}}_{x'} \cap \mathcal{W}$, which is nonempty, as $[(X,Y)] \in \mathcal{W}_x$. Since $\overline{\mathcal{W}}_{x'}$ is a general fiber of p and $\overline{\mathcal{W}}$ is irreducible, $\mathcal{W}_{x'}$ is dense in any component of $\overline{\mathcal{W}}_{x'}$. It follows that $g_2(\mathcal{W}_{x'})$ is dense in any component of $g_2(\overline{\mathcal{W}}_{x'})$. By the last claim, $\sigma(g_2(\mathcal{W}_{x'})) \cap \mathfrak{g}(x_4) \neq \emptyset$ for general $x_4 \in T$. Pick any $(x_8, x_9) \in g_2(\mathcal{W}_{x'}) \cap \sigma^{-1}(\mathfrak{g}(x_4))$. Then by definition there exists a pair $[(X^1, Y^1 = C^1 \cup_T D^1)] \in \mathcal{W}_{x'}$ such that

$$X^1 = \mathrm{Bl}_{y_1,y_2,y_3,x'_4}(R) \cup_T \mathrm{Bl}_{y_5,y_6,y_7,x_8,x_9}(P) \ \text{ for some } \ x'_4 \in T, \quad \text{and}$$

$$y_1 + y_2 + y_3 + x_4 + y_5 + y_6 + y_7 + x_8 + x_9 \in |\mathcal{N}_{T/R} \otimes \mathcal{N}_{T/P}|. \tag{18}$$

In particular, $D^1 \subset \mathrm{Bl}_{y_5,y_6,y_7,x_8,x_9}(P)$ satisfies the conditions of Proposition 4.1 and

$$D^1 = D^1_0 + \mathfrak{e}_5 + \mathfrak{e}_6 + \mathfrak{e}_7, \ \ D^1_0 \cap T = mx' + y_1 + y_2 + y_3. \tag{19}$$

As x_4 is general in T, one has $x_4 \in g_1(\mathcal{W}_{x'})$ (since $(g_1)\,|_{\overline{\mathcal{W}}_{x'}}$ is surjective, by Claim 4.5(ii)). Then by definition there exists a pair $[(X^2, Y^2 = C^2 \cup_T D^2)] \in \mathcal{W}_{x'}$ such that

$$X^2 = \mathrm{Bl}_{y_1,y_2,y_3,x_4}(R) \cup_T \mathrm{Bl}_{y_5,y_6,y_7,x'_8,x'_9}(P) \ \text{ for some } \ x'_8, x'_9 \in T.$$

In particular, $C^2 \subset \mathrm{Bl}_{y_1,y_2,y_3,x_4}(R)$ satisfies the conditions of Proposition 4.1 and

$$C^2 = C^2_0 + \mathfrak{e}_1 + \mathfrak{e}_2 + \mathfrak{e}_3, \ \ C^2_0 \cap T = mx' + y_5 + y_6 + y_7. \tag{20}$$

Consider the pair

$$(\overline{X} = \mathrm{Bl}_{y_1,y_2,y_3,x_4}(R) \cup_T \mathrm{Bl}_{y_5,y_6,y_7,x_8,x_9}(P), \overline{Y} = C^2 \cup_T D^1).$$

Recall that $y_1, y_2, y_3, y_5, y_6, y_7$ were chosen general to start with, and x_4 and x' are also general by construction. Lemma 3.11(ii) implies that we may choose $x_8 + x_9$ general in $\mathfrak{g}(x_4)$. It follows that $\overline{X}$ is a general member of $\mathcal{D}^*_{[4]}$, i.e., (b) holds. Properties (19) and (20) imply that the pair $(\overline{X}, \overline{Y})$ satisfies the conditions of Proposition 4.1 and therefore (c) holds. It is moreover clear that also (a) holds. □

5. Isotropic 10-sequences and simple isotropic decompositions

An important tool for identifying the various components of the moduli spaces of polarized Enriques surfaces is the decomposition of line bundles as sums of effective isotropic divisors. In this section we will recall some notions and results from [10, 23].

Definition 5.1. ([13, p. 122]) An *isotropic* 10*-sequence* on an Enriques surface S is a sequence of isotropic effective divisors $\{E_1, \ldots, E_{10}\}$ such that $E_i \cdot E_j = 1$ for $i \neq j$.

It is well-known that any Enriques surface contains isotropic 10-sequences. Note that we, contrary to [13], require the divisors to be *effective*, which can always be arranged by changing signs. We also recall the following result:

Lemma 5.2. ([10, Lemma 3.4(a)],[13, Cor. 2.5.5]) *Let* $\{E_1,\dots,E_{10}\}$ *be an isotropic* 10-*sequence. Then there exists a divisor* D *on* S *such that* $D^2=10$ *and* $3D\sim E_1+\cdots+E_{10}$*. Furthermore, for any* $i\neq j$*, we have*

$$D\sim E_i+E_j+E_{i,j},\ \ \textit{with } E_{i,j} \textit{ effective isotropic, } E_i\cdot E_{i,j}=E_j\cdot E_{i,j}=2, \tag{21}$$

and $E_k\cdot E_{i,j}=1$ *for* $k\neq i,j$*. Moreover,* $E_{i,j}\cdot E_{k,l}=\begin{cases}1, & \textit{if } \{i,j\}\cap\{k,l\}\neq\emptyset,\\ 2, & \textit{if } \{i,j\}\cap\{k,l\}=\emptyset.\end{cases}$

The next result yields a "canonical" way of decomposing line bundles:

Proposition 5.3. ([23, Thm. 5.7]) *Let* L *be an effective line bundle on an Enriques surface* S *such that* $L^2>0$*. Then there are unique nonnegative integers* $a_0,a_1,\dots,a_7,a_9$*,* a_{10}*, depending on* L*, satisfying*

$$a_1\geq\cdots\geq a_7, \qquad \textit{and} \tag{22}$$

$$a_9+a_{10}\geq a_0\geq a_9\geq a_{10} \tag{23}$$

such that L *can be written as*

$$L\sim a_1E_1+\cdots+a_7E_7+a_9E_9+a_{10}E_{10}+a_0E_{9,10}+\varepsilon_LK_S, \tag{24}$$

for an isotropic 10*-sequence* $\{E_1,\dots,E_{10}\}$ *(depending on* L*) and*

$$\varepsilon_L=\begin{cases}0, & \textit{if } L+K_S \textit{ is not } 2\textit{-divisible in } \mathrm{Pic}(S),\\ 1, & \textit{if } L+K_S \textit{ is } 2\textit{-divisible in } \mathrm{Pic}(S).\end{cases} \tag{25}$$

Remark 5.4. Although the coefficients a_i are unique, the isotropic 10-sequence in Proposition 5.3 is not unique, not even up to numerical equivalence or permutation, and nor is the presentation (24). See [23, Rem. 5.6].

Definition 5.5. ([23, Def. 5.1, 5.8]) Let L be any effective line bundle on an Enriques surface S such that $L^2>0$. A decomposition of the form (24) with coefficients satisfying (22), (23) and (25) is called a *fundamental presentation of* L. The coefficients $a_i=a_i(L)$, $i\in\{0,1,\dots,7,9,10\}$ and ε_L appearing in any fundamental presentation are called *fundamental coefficients of* L or *of* (S,L).

Remark 5.6. By [10, Lemma 4.8] (or [23, Thm. 1.3(f) and Prop. 5.5]) a line bundle L is 2-divisible in $\mathrm{Num}(S)$ if and only all $a_i=a_i(L)$ are even, $i\in\{0,1,\dots,7,9,10\}$. In particular, by (25) or [10, Cor. 4.7], the number $\varepsilon=\varepsilon_L$ satisfies

$$\varepsilon=\begin{cases}0, & \text{if some } a_i \text{ is odd},\\ 0 \text{ or } 1, & \text{if all } a_i \text{ are even}.\end{cases} \tag{26}$$

This means that any 11-tuple $(a_0,a_1,\dots,a_7,a_9,a_{10},\varepsilon)$ occuring as fundamental coefficients satisfies the conditions (22), (23) and (26). Conversely, for any such 11-tuple we can choose any isotropic 10-sequence on any Enriques surface and write down the line bundle (24) having this 11-tuple as fundamental coefficients.

For any integer $g\geq 2$, let $\mathcal{E}_g$ denote the moduli space of complex polarized Enriques surfaces (S,L) of genus g, which is a quasi-projective variety by [38, Thm. 1.13]. Its irreducible components are determined by the fundamental coefficients, by the following:

Theorem 5.7. ([23, Thm. 5.9]) *Given an irreducible component $\mathcal{E}$ of $\mathcal{E}_g$, all pairs (S,L) in $\mathcal{E}$ have the same fundamental coefficients. Different components correspond to different fundamental coefficients.*

The following technical result will be useful for our purposes:

Lemma 5.8. *Let (S,L) be an element of $\mathcal{E}_g \setminus \mathcal{E}_g[2]$. Set $a_i = a_i(L)$. Then one of the following holds:*

(i) There are three distinct $k,l,m \in \{1,\dots,7\}$ such that $a_i+a_k+a_l+a_m$ is odd for $i=9$ or 10.

(ii) $a_0>0$ is odd and all a_i for $i\neq 0$ are even.

(iii) $a_0>0$ and all a_i for $i \neq 0$ are odd.

Proof. We first show that if $a_0=0$, then we end up in case (i). We have $a_9=a_{10}=0$ by condition (23). Moreover, by Remark 5.6, the set $I:=\{i\in\{1,\dots,7\} \mid a_i \text{ is odd}\}$ is nonempty. If $\sharp I \geq 3$, then for any distinct $k,l,m\in I$ we have that $a_i+a_k+a_l+a_m = a_k+a_l+a_m$ is odd for $i=9$ and 10. If $\sharp I \leq 2$, we may pick $k\in I$ and two distinct $l,m\in\{1,\dots,7\}\setminus I$; then again $a_i+a_k+a_l+a_m=a_k+a_l+a_m$ is odd for $i=9$ and 10.

We may therefore assume that $a_0>0$.

Assume next that (i) does not hold; then we have

$$a_i+a_k+a_l+a_m \text{ is even for all } i\in\{9,10\} \text{ and distinct } k,l,m\in\{1,\dots,7\}. \tag{27}$$

This clearly implies that a_9 and a_{10} have the same parity.

Assume that a_9 and a_{10} are even. Then (27) implies that $a_k+a_l+a_m$ is even for all distinct $k,l,m\in\{1,\dots,7\}$. Hence a_i is even for all $i\in\{1,\dots,7\}$. Then a_0 is odd by Remark 5.6 and we end up in case (ii).

Assume that a_9 and a_{10} are odd. Then (27) implies that $a_k+a_l+a_m$ is odd for all distinct $k,l,m\in\{1,\dots,7\}$. Hence a_i is odd for all $i\in\{1,\dots,7\}$, yielding case (iii). □

6. Isotropic 10-sequences on members of $\mathcal{D}$

The notions of isotropic divisors and isotropic 10-sequences can be extended in the obvious way to all members of $\mathcal{D}$. Referring to [23, §3] for more details, we will in Example 6.1 below construct one such 10–sequence that we will use in the proof of Theorem 1.1 in the next section.

Recall that we have the points $y_1,\dots,y_9\in T$, which are the blown up points on either R or P. We will now assume that $y_1,\dots,y_9$ are distinct, though the case of coinciding points can be treated similarly. Denote by $\mathfrak{e}_j$ the exceptional divisor over y_j, without fixing whether it lies on $\widetilde{R}$ or $\widetilde{P}$.

View $y_j\in T\subset P$. The linear system of lines in P through y_j is a pencil inducing a g^1_2 on T, which has, by Riemann-Hurwitz, two members that also belong to a fiber of $\pi_{|T}: T\to E$. In other words, there are two fibers $\mathfrak{f}_{\alpha_j}$ and $\mathfrak{f}_{\alpha'_j}$ of $\pi: R\to E$ such that the intersection divisors $\mathfrak{f}_{\alpha_j}\cap T$ and $\mathfrak{f}_{\alpha'_j}\cap T$ belong to this g^1_2. One may verify that $\alpha'_j=\alpha_j\oplus\eta$. In particular, there are two uniquely defined points α_j and $\alpha_j\oplus\eta$ on E such that the pairs

$$(\mathfrak{f}_{\alpha_j}+\mathfrak{e}_j,\ell) \text{ and } (\mathfrak{f}_{\alpha_j\oplus\eta}+\mathfrak{e}_j,\ell), \text{ if } \mathfrak{e}_j\subset\widetilde{R},$$
$$(\mathfrak{f}_{\alpha_j},\ell-\mathfrak{e}_j) \text{ and } (\mathfrak{f}_{\alpha_j\oplus\eta},\ell-\mathfrak{e}_j), \text{ if } \mathfrak{e}_j\subset\widetilde{P},$$

define Cartier divisors on $X := \widetilde{R} \cup_T \widetilde{P}$. One may check that their difference is K_X.

Similarly, for four distinct (general) $y_i, y_j, y_k, y_l \in T$, the linear system of plane conics through y_i, y_j, y_k, y_l is again a pencil inducing a g^1_2 on T. As above, there are two fibers $\mathfrak{f}_{\alpha_{ijkl}}$ and $\mathfrak{f}_{\alpha_{ijkl}\oplus\eta}$ of $\pi : R \to E$ such that the divisors $\mathfrak{f}_{\alpha_{ijkl}} \cap T$ and $\mathfrak{f}_{\alpha_{ijkl}\oplus\eta} \cap T$ belong to this g^1_2. In particular, the pairs

$$(\mathfrak{f}_{\alpha_{ijkl}} + \mathfrak{e}_i + \mathfrak{e}_j + \mathfrak{e}_k + \mathfrak{e}_l, 2\ell) \text{ and } (\mathfrak{f}_{\alpha_{ijkl}\oplus\eta} + \mathfrak{e}_i + \mathfrak{e}_j + \mathfrak{e}_k + \mathfrak{e}_l, 2\ell), \text{ if } \mathfrak{e}_i, \mathfrak{e}_j, \mathfrak{e}_k, \mathfrak{e}_l \subset \widetilde{R},$$
$$(\mathfrak{f}_{\alpha_{ijkl}}, 2\ell - \mathfrak{e}_i - \mathfrak{e}_j - \mathfrak{e}_k - \mathfrak{e}_l) \text{ and } (\mathfrak{f}_{\alpha_{ijkl}\oplus\eta}, 2\ell - \mathfrak{e}_i - \mathfrak{e}_j - \mathfrak{e}_k - \mathfrak{e}_l), \text{ if } \mathfrak{e}_i, \mathfrak{e}_j, \mathfrak{e}_k, \mathfrak{e}_l \subset \widetilde{P},$$

together with similar pairs when $\mathfrak{e}_i, \mathfrak{e}_j, \mathfrak{e}_k, \mathfrak{e}_l$ are distributed differently, define Cartier divisors on X. One may again check that their difference is K_X.

Considering instead $y_i \in T \subset R = \mathrm{Sym}^2(E)$ we may write $y_i = p_i + (p_i \oplus \eta)$, for some $p_i \in E$. There are two sections in R passing through y_i, namely $\mathfrak{s}_{p_i}$ and $\mathfrak{s}_{p_i\oplus\eta}$, cf. (1). Thus, the pairs

$$(\mathfrak{s}_{p_i} - \mathfrak{e}_i, 0) \text{ and } (\mathfrak{s}_{p_i\oplus\eta} - \mathfrak{e}_i, 0), \text{ if } \mathfrak{e}_i \subset \widetilde{R},$$
$$(\mathfrak{s}_{p_i}, \mathfrak{e}_i) \text{ and } (\mathfrak{s}_{p_i\oplus\eta}, \mathfrak{e}_i), \text{ if } \mathfrak{e}_i \subset \widetilde{P}$$

define Cartier divisors on X. Again one may check that their difference is K_X.

Example 6.1. We consider $\widetilde{R} = \mathrm{Bl}_{y_1,y_2,y_3,y_4}(R)$ and $\widetilde{P} = \mathrm{Bl}_{y_5,\dots,y_9}(\mathbb{P}^2)$. Define

$$\begin{aligned} E^0_i &:= (\mathfrak{s}_{p_i} - \mathfrak{e}_i, 0) \text{ for } i \in \{1,2,3,4\}, \\ E^0_i &:= (\mathfrak{f}_{\alpha_i}, \ell - \mathfrak{e}_i) \text{ for } i \in \{5,6,7,8\}, \\ E^0_9 &:= (\mathfrak{s}_{p_9}, \mathfrak{e}_9), \\ E^0_{10} &:= (\mathfrak{f}_{\alpha_{5678}}, 2\ell - \mathfrak{e}_5 - \mathfrak{e}_6 - \mathfrak{e}_7 - \mathfrak{e}_8). \end{aligned}$$

These are all Cartier divisors on $X = \widetilde{R} \cup_T \widetilde{P}$ by the above considerations. One may check that $(E^0_i)^2 = 0$ for all i and $E^0_i \cdot E^0_j = 1$ for all $i \neq j$. If X is a member of $\mathcal{D}^*$, then, arguing as in the proof of [23, Lemma 3.6], one may show that

$$E^0_1 + \cdots + E^0_{10} - \xi \sim 3(E^0_9 + E^0_{10} + E^0_{9,10}),$$

with ξ as in (6) and

$$E^0_{9,10} = (\mathfrak{f}_{\alpha_9}, \ell - \mathfrak{e}_9).$$

Thus, we may similarly to (21) define

$$E^0_{i,j} := \frac{1}{3}\left(E^0_1 + \cdots + E^0_{10} - \xi\right) - E^0_i - E^0_j \text{ for each } i \neq j.$$

In particular, letting $y_{78} \in T \subset \mathbb{P}^2$ be the third intersection point of the line through y_7 and y_8 with T, and writing $y_{78} = p_{78} + (p_{78} \oplus \eta) \in T \subset \mathrm{Sym}^2(E)$ for some $p_{78} \in E$, we will use that

$$E^0_{5,6} \sim (\mathfrak{s}_{p_{78}}, \ell - \mathfrak{e}_7 - \mathfrak{e}_8).$$

Note that $E^0_{9,10}$ and $E^0_{5,6}$ are Cartier divisors on any X in $\mathcal{D}$.

Remark 6.2. If $X = \widetilde{R} \cup_T \widetilde{P}$ belongs to $\mathcal{D}^*$, then, by Theorem 2.2, as it deforms to a general Enriques surface S, the sequence $(E^0_1, \dots, E^0_{10})$ deforms to an isotropic 10-sequence $(E_1, \dots, E_{10})$ on S and each $E^0_{i,j}$ deforms to $E_{i,j}$ satisfying (21).

7. Proof of Theorem 1.1

We are now ready to finish the proof of our main result Theorem 1.1. Keeping in mind that the various irreducible components of $\mathcal{E}_g$ are determined by the fundamental coefficients of the line bundles they parametrize (cf. Theorem 5.7), the proof will be divided in various cases depending on parity properties of the fundamental coefficients. Also recall that since we assume that we are not in $\mathcal{E}_g[2]$, at least one of the fundamental coefficients a_i is odd and $\varepsilon = 0$ (cf. Remark 5.6). In particular, the components we consider contain both (S, L) and $(S, L + K_S)$, so there is no need to distinguish between linear and numerical equivalence classes (cf. also [23, Thm. 1.1]).

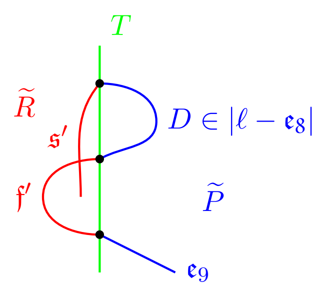
The proof strategy will be as follows: given fundamental coefficients a_i, find a suitable line bundle with the same fundamental decomposition on some limit surface in terms of the isotropic divisors in Example 6.1, and apply Proposition 4.2 or 4.3 (and Remark 6.2). As mentioned in the beginning of §4, the existence of a rigid nodal elliptic curve will prove Theorem 1.1.

We will first treat three special cases in §7.1-7.3 and then the three cases of Lemma 5.8 in §7.4-7.6.

7.1. **The case $a_1 = a_2 = 1$ and $a_i = 0$ otherwise.** Consider the limit line bundle $L^0 = E_8^0 + E_9^0$, with E_8^0 and E_9^0 as in Example 6.1 (where we consider a general surface $\widetilde{R} \cup_T \widetilde{P}$ in $\mathcal{D}^*$). Then

$$L^0|_{\widetilde{R}} \equiv \mathfrak{f} + \mathfrak{s} \quad \text{and} \quad L^0|_{\widetilde{P}} \sim (\ell - \mathfrak{e}_8) + \mathfrak{e}_9.$$

In this case there is no need to invoke Proposition 4.2 or 4.3: indeed, the linear system $|L|$ contains the following curve:



Here $\mathfrak{f}'$ is the unique fiber passing through the point $y_9 = \mathfrak{e}_9 \cap T$ and D is the unique element of $|\ell - \mathfrak{e}_8|$ passing through the point y_9' such that $y_9 + y_9' = T \cap \mathfrak{f}'$; finally $\mathfrak{s}'$ is one of the two sections $\mathfrak{s}$ passing through the remaining intersection point of D with T.

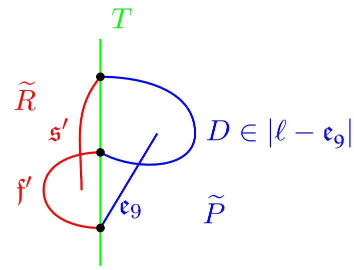
Arguing as in the proof of Proposition 4.1, this curve can be deformed to a one-nodal rigid elliptic curve of arithmetic genus 2 in the linear system $|L|$ as $(\widetilde{R} \cup_T \widetilde{P}, L^0)$ deforms to (S, L).

7.2. **The case $a_0 = a_9$ and $a_i = 0$ otherwise.** Both a_0 and a_9 are odd.

7.2.1. *Subcase* $a_0 = a_9 = 1$. Consider the limit line bundle $L^0 = E_{9,10}^0 + E_9^0$, with $E_{9,10}^0$ and E_9^0 as in Example 6.1 (as above). Then

$$L^0|_{\widetilde{R}} \equiv \mathfrak{f} + \mathfrak{s} \quad \text{and} \quad L^0|_{\widetilde{P}} \sim (\ell - \mathfrak{e}_9) + \mathfrak{e}_9.$$

There is again no need to invoke Proposition 4.2 or 4.3: indeed, the linear system $|L|$ contains the following curve, constructed as in the previous case:



Arguing again as in the proof of Proposition 4.1, this curve can be deformed to a rigid elliptic two-nodal curve of arithmetic genus 3 in the linear system $|L|$ as $(\widetilde{R} \cup_T \widetilde{P}, L^0)$ deforms to (S, L).

7.2.2. *Subcase* $a_0 = a_9 \geq 3$. Consider the limit line bundle $L^0 = a_0(E^0_{5,6} + E^0_5)$, with $E^0_{5,6}$ and E^0_5 as in Example 6.1. Then

$$L' := L^0|_{\widetilde{R}} \equiv a_0\mathfrak{s} + a_0\mathfrak{f} \text{ and } L'' := L^0|_{\widetilde{P}} \sim a_0(\ell - \mathfrak{e}_7 - \mathfrak{e}_8) + a_0(\ell - \mathfrak{e}_5) = a_0(2\ell - \mathfrak{e}_5 - \mathfrak{e}_7 - \mathfrak{e}_8).$$

We see that we might as well substitute $\widetilde{R}$ with R and $\widetilde{P}$ with $P' = \mathrm{Bl}_{y_5,y_7,y_8}(P)$ and consider $L^0 = (L', L'')$ as a line bundle on $R \cup_T P'$. We apply Proposition 4.2 with $k = 0$, $s = 0$ and $t = 3$.

Clearly L' satisfies condition $(\star)$ and is odd and L'' is big and nef. Moreover, $\mathfrak{e}_i \cdot L'' > 0$ for $i \in \{5, 7, 8\}$. The conditions of Proposition 4.2 are satisfied and we are done.

7.3. **The cases** $a_7 = a_9 = a_{10} = a_0 = 0$**.** At least one of the a_i is odd. Pick the minimal such and call it c_1. A limit line bundle is of type

$$L^0 \equiv c_1 E^0_1 + \sum_{i=5}^{8} c_i E^0_i + c_{10} E^0_{10},$$

with the E^0_i as in Example 6.1, where c_1 is the minimal odd coefficient and we may also assume that $c_5 > 0$. Then

$$L' := L^0|_{\widetilde{R}} \equiv c_1(\mathfrak{s} - \mathfrak{e}_1) + \sum_{i=5}^{8} c_i\mathfrak{f} + c_{10}\mathfrak{f} = c_1\mathfrak{s} + (c_5 + c_6 + c_7 + c_8 + c_{10})\mathfrak{f} - c_1\mathfrak{e}_1,$$

$$L'' := L^0|_{\widetilde{P}} \sim \sum_{i=5}^{8} c_i(\ell - \mathfrak{e}_i) + c_{10}(2\ell - \mathfrak{e}_5 - \mathfrak{e}_6 - \mathfrak{e}_7 - \mathfrak{e}_8).$$

We see that we might as well substitute $\widetilde{R}$ with $R' = \mathrm{Bl}_{y_1}(R)$ and $\widetilde{P}$ with $P' = \mathrm{Bl}_{y_5,y_6,y_7,y_8}(P)$ and consider $L^0 = (L', L'')$ as a line bundle on $R' \cup_T P'$. We apply Proposition 4.2 with $s = 1$, $t = 4$ and $k = 0$. Conditions (i)–(iii) of $(\star)$ are verified by L'; condition (iv) is equivalent to $c_5 + c_6 + c_7 + c_8 + c_{10} \geq 2$, which is verified unless $c_5 = 1$ and $c_6 = c_7 = c_8 = c_{10} = 0$. Since c_1 was assumed to be a minimal odd fundamental coefficient, we must have $c_1 = 1$ as well. This case is the one treated in §7.1. We may therefore assume that L' satisfies $(\star)$. One readily checks that L' is odd (since c_1 is odd) and that L'' is big and nef (all components have square zero, intersect and can be represented by irreducible curves). This verifies conditions (i)–(iv) in Proposition 4.2. Since for instance $\mathfrak{e}_5 \cdot L'' = c_5 + c_{10} \geq c_5 > 0$, also condition (v) therein is satisfied. Hence we are done by Proposition 4.2.

7.4. **Case where there are three distinct $k, l, m \in \{1, \dots, 7\}$ such that $a_i + a_k + a_l + a_m$ is odd for $i = 9$ or 10 (case (i) in Lemma 5.8).** Note that the cases among these with $a_0 = 0$ (whence also $a_9 = a_{10} = 0$) and $a_7 = 0$ fall into the cases treated in §7.3. We can therefore assume that

$$a_7 > 0 \ \text{(whence } a_i > 0 \text{ for all } i \in \{1, \dots, 7\}\text{), if } a_0 = 0. \tag{28}$$

Similarly, the cases among these with $a_0 = a_9$ and all remaining $a_i = 0$ fall into the cases treated in §7.2. We can therefore assume that

$$a_0 \neq a_9, \ \text{ if } \ a_i = 0 \ \text{ for all } \ i \in \{1, \dots, 7, 10\}. \tag{29}$$

We will pick indices k, l, m so that $a_i + a_k + a_l + a_m$ is odd for $i = 9$ or 10 and $a_k + a_l + a_m$ is minimal with respect to this property. Subsequently, if both $a_9 + a_k + a_l + a_m$ and $a_{10} + a_k + a_l + a_m$ are odd, we will pick $i \in \{9, 10\}$ so that $a_i + a_k + a_l + a_m$ is minimal. We rename these coefficients a_i, a_k, a_l, a_m as c_9, c_2, c_3, c_4, set $c_0 = a_0$, $c_{10} = \begin{cases} a_{10}, & \text{if } i = 9 \\ a_9, & \text{if } i = 10 \end{cases}$, and rename the remaining a_i as c_5, c_6, c_7, c_8. We thus have a limit line bundle

$$L^0 \equiv c_0 E^0_{9,10} + c_9 E^0_9 + c_{10} E^0_{10} + \sum_{i=2}^{8} c_i E^0_i,$$

with the E^0_i and $E^0_{9,10}$ as in Example 6.1, where

$$c_9 + c_{10} \geq c_0 \geq \max\{c_9, c_{10}\}, \tag{30}$$

$$c_2 \geq c_3 \geq c_4, \tag{31}$$

$$c_5 \geq c_6 \geq c_7 \geq c_8, \tag{32}$$

$$c_9 + c_2 + c_3 + c_4 \ \text{ is odd,} \tag{33}$$

$$\text{there are no } i \in \{9, 10\},\ k, l, m \in \{2, \dots, 8\} \text{ such that} \tag{34}$$

$$c_i + c_k + c_l + c_m \ \text{ is odd and}$$

$$\begin{cases} c_k + c_l + c_m < c_2 + c_3 + c_4, \text{or} \\ c_k + c_l + c_m = c_2 + c_3 + c_4 \ \text{ and } \ c_i + c_k + c_l + c_m < c_9 + c_2 + c_3 + c_4. \end{cases}$$

Furthermore, (28) gives

$$c_i > 0 \ \text{ for all } \ i \in \{2, \dots, 8\}, \ \text{ if } \ c_0 = 0, \tag{35}$$

and (29) yields

$$c_0 \neq c_9 \ \text{ if } \ c_i = 0 \ \text{ for all } \ i \in \{2, \dots, 8, 10\}. \tag{36}$$

We define

$$\kappa := \sharp\{j \in \{2, 3, 4\} \mid c_j > 0\} \quad \text{and} \quad \lambda := \sharp\{j \in \{5, 6, 7, 8\} \mid c_j > 0\}.$$

Claim 7.1. *The following hold:*

(i) If $c_0 = 0$, then $(\kappa, \lambda) = (3, 4)$.
(ii) If $\lambda \leq 2$, then $\kappa \leq 1$; moreover, $\kappa = 1$ implies $c_{10} \geq 2$.
(iii) If $(\kappa, \lambda, c_{10}) = (0, 0, 0)$, then $c_0 \neq c_9$.

Proof. Property (i) follows from condition (35).

Next assume $\lambda \leq 2$, that is, $c_7 = c_8 = 0$. Then properties (33) and (34) yield that $c_3 = c_4 = 0$, that is, $\kappa \leq 1$, as we now explain.

Indeed, if c_4 is even and positive we have that $c_9+c_2+c_3+c_8$ is odd and $c_2+c_3+c_8 < c_2+c_3+c_4$, contradicting (34). Similarly for the cases where c_i is even and positive with $i=2,3$. From this it follows that none among c_2, c_3, c_4 can be even and positive.

If c_3 and c_4 are odd, then $c_9+c_2+c_7+c_8$ is odd and $c_2+c_7+c_8 < c_2+c_3+c_4$, contradicting (34). Similarly if c_2 and c_3 or c_2 and c_4 are odd. Hence at most one among c_2, c_3, c_4 is odd.

In conclusion at least two among c_2, c_3, c_4 must be zero, hence $c_3 = c_4 = 0$ by (31), as we claimed.

If $\kappa = 1$, we have $c_2 > 0$ and $c_9 + c_2$ is odd by (33). Condition (34) yields that c_9 is even (whence c_2 is odd), for otherwise $c_9 = c_9 + c_3 + c_4 + c_8$ would be odd with $0 = c_3 + c_4 + c_8 < c_2 = c_2 + c_3 + c_4$. For the same reason, c_{10} is even, and condition (34) yields that $c_{10} \geq c_9$. By (30) and the fact that $c_0 > 0$ from (i), we must have $c_{10} > 0$. This proves (ii).

Finally, (iii) is a reformulation of property (36). □

Consider

$$\begin{aligned} L' := L^0|_{\widetilde{R}} &\equiv c_0\mathfrak{f} + c_9\mathfrak{s} + c_{10}\mathfrak{f} + \sum_{i=2}^{4} c_i(\mathfrak{s} - \mathfrak{e}_i) + \sum_{i=5}^{8} c_i\mathfrak{f} \\ &= (c_2+c_3+c_4+c_9)\mathfrak{s} + (c_0+c_5+c_6+c_7+c_8+c_{10})\mathfrak{f} - \sum_{i=2}^{4} c_i\mathfrak{e}_i \\ &= L'_0 + \sum_{i=2}^{\kappa+1}(\mathfrak{f} - \mathfrak{e}_i), \end{aligned}$$

where

$$L'_0 := (c_2+c_3+c_4+c_9)\mathfrak{s} + (c_0+c_5+c_6+c_7+c_8+c_{10}-\kappa)\mathfrak{f} - \sum_{i=2}^{\kappa+1}(c_i-1)\mathfrak{e}_i$$

and $\sum_{i=2}^{\kappa+1}(\mathfrak{f} - \mathfrak{e}_i)$ is the sum of κ disjoint (-1)-curves. We note that we may consider L' as a line bundle on the blow–up of R at κ points. Hence we will eventually apply Proposition 4.2 with $k = s = \kappa$.

Claim 7.2. *L'_0 verifies condition $(\star)$ and is odd.*

Proof. Oddness is equivalent to condition (33). Conditions (i)–(iii) of $(\star)$ are easily checked. Condition (iv) is equivalent to

$$2c_0 + c_9 + 2c_{10} + 2\sum_{i=5}^{8} c_i - \kappa \geq 4. \tag{37}$$

If $c_0 = 0$, then $c_9 = c_{10} = 0$ by (30) and $(\kappa, \lambda) = (3, 4)$ by Claim 7.1(i), whence the left hand side of (37) equals $2\sum_{i=5}^{8} c_i - \kappa \geq 8 - 3 = 5$, and we are done. Hence we may assume that $c_0 > 0$ for the rest of the proof.

We note that, by (30),

$$2c_0 + c_9 + 2c_{10} + 2\sum_{i=5}^{8} c_i - \kappa \geq 3c_0 + c_{10} + 2\lambda - \kappa \geq 3 + c_{10} + 2\lambda - \kappa.$$

This, together with Claim 7.1(ii) tells us that (37) is always satisfied if $\lambda \geq 1$. Assume therefore that $\lambda = 0$. Then $\kappa \leq 1$ by Claim 7.1(ii). If $\kappa = 1$, then $c_{10} \geq 2$ by Claim 7.1(ii), and (37) is again satisfied. If $\kappa = 0$, we have that c_9 is odd by (33). If $c_{10} > 0$, (37) is satisfied. Otherwise Claim 7.1(iii) yields $c_0 \geq 2$, whence (37) is again satisfied. □

Consider

$$\begin{aligned} L'' := L^0|_{\widetilde{P}} &\sim c_0(\ell - \mathfrak{e}_9) + c_9\mathfrak{e}_9 + c_{10}(2\ell - \mathfrak{e}_5 - \mathfrak{e}_6 - \mathfrak{e}_7 - \mathfrak{e}_8) + \sum_{i=5}^{8} c_i(\ell - \mathfrak{e}_i) \\ &= (c_0 - c_9)(\ell - \mathfrak{e}_9) + c_9\ell + c_{10}(2\ell - \mathfrak{e}_5 - \mathfrak{e}_6 - \mathfrak{e}_7 - \mathfrak{e}_8) + \sum_{i=5}^{8} c_i(\ell - \mathfrak{e}_i). \end{aligned}$$

The idea is now to apply Proposition 4.2 with $k = \kappa$.

7.4.1. *Subcase* $\lambda = 3, 4$. We have $c_5 \geq c_6 \geq c_7 > 0$ by (32). Define

$$\begin{aligned} L''_0(3) &:= (c_0 - c_9)(\ell - \mathfrak{e}_9) + c_9\ell + c_{10}(2\ell - \mathfrak{e}_5 - \mathfrak{e}_6 - \mathfrak{e}_7 - \mathfrak{e}_8) + \\ &+ \sum_{i=5}^{7}(c_i - 1)(\ell - \mathfrak{e}_i) + c_8(\ell - \mathfrak{e}_8) + (\ell - \mathfrak{e}_5 - \mathfrak{e}_6) + (\ell - \mathfrak{e}_6 - \mathfrak{e}_7) + (\ell - \mathfrak{e}_7 - \mathfrak{e}_8), \\ L''_0(2) &= L''_0(3) + \mathfrak{e}_6, \\ L''_0(1) &= L''_0(3) + \mathfrak{e}_6 + \mathfrak{e}_7, \\ L''_0(0) &= L''_0(3) + \mathfrak{e}_6 + \mathfrak{e}_7 + \mathfrak{e}_8. \end{aligned} \tag{38}$$

Then one may check that

$$L'' = L''_0(\kappa) + \begin{cases} 0, & \text{if } \kappa = 0, \\ \mathfrak{e}_8, & \text{if } \kappa = 1, \\ \mathfrak{e}_7 + \mathfrak{e}_8, & \text{if } \kappa = 2, \\ \mathfrak{e}_6 + \mathfrak{e}_7 + \mathfrak{e}_8, & \text{if } \kappa = 3. \end{cases}$$

Claim 7.3. *$L''_0(\kappa)$ is big and nef for all $\kappa \in \{0, 1, 2, 3\}$.*

Proof. Since $\mathfrak{e}_i \cdot L''_0(3) > 0$, for $i \in \{6, 7, 8\}$, it suffices to verify that $L''_0(3)$ is big and nef. All divisors in the sum (38) are of nonnegative square, except for the last three. Nefness follows if the latter three intersect $L''_0(3)$ nonnegatively. We have

$$\begin{aligned} L''_0(3) \cdot (\ell - \mathfrak{e}_5 - \mathfrak{e}_6) &= c_0 + (c_7 - 1) + c_8 - 1 + 0 + 1 \geq 0, \\ L''_0(3) \cdot (\ell - \mathfrak{e}_7 - \mathfrak{e}_8) &= c_0 + (c_5 - 1) + (c_6 - 1) + 1 + 0 - 1 \geq 0. \end{aligned}$$

Finally,

$$L''_0(3) \cdot (\ell - \mathfrak{e}_6 - \mathfrak{e}_7) = c_0 + (c_5 - 1) + c_8 + 0 - 1 + 0 \geq c_0 + c_8 - 1,$$

which is nonnegative, since by Claim 7.1(i), either $c_0 > 0$, or $\lambda = 4$ (whence $c_8 > 0$). This proves nefness. Bigness is easily checked. □

We apply Proposition 4.2 with $k = s = \kappa$, $t = 5$ and $L_0'' = L_0(\kappa)$. What is left to be checked is condition (v). The set of additional $t - k = 5 - \kappa$ disjoint (-1)-curves on $\widetilde{P}$ is

$$\begin{cases} \mathfrak{e}_5, \mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8, \mathfrak{e}_9, & \text{if } \kappa = 0, \\ \mathfrak{e}_5, \mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_9, & \text{if } \kappa = 1, \\ \mathfrak{e}_5, \mathfrak{e}_6, \mathfrak{e}_9 & \text{if } \kappa = 2, \\ \mathfrak{e}_5, \mathfrak{e}_9 & \text{if } \kappa = 3, \end{cases}$$

and $\mathfrak{e}_5 \cdot L''(\kappa) = c_{10} + c_5 \geq c_5 > 0$, as $\lambda > 0$, verifying condition (v) in Proposition 4.2.

7.4.2. *Subcase* $\lambda \leq 2$. We have $c_0 > 0$, $c_7 = c_8 = 0$, and $\kappa \leq 1$ by Claim 7.1(i)–(ii).

If $\kappa = 0$ we apply Proposition 4.2 with $k = s = 0$ and $t = 5$. Condition (v) therein is satisfied, as for instance $\mathfrak{e}_5 \cdot L'' = c_{10} + c_5$ and $\mathfrak{e}_9 \cdot L'' = c_0 - c_9$; indeed, if $\mathfrak{e}_5 \cdot L'' = 0$, then $c_{10} = c_5 = 0$, whence $\lambda = 0$, so Claim 7.1(iii) yields $\mathfrak{e}_9 \cdot L'' > 0$.

If $\kappa = 1$, then $c_{10} \geq 2$ by Claim 7.1(ii). Write $L'' = L_0'' + \mathfrak{e}_9$, with

$$L_0'' := (c_0 - c_9)(\ell - \mathfrak{e}_9) + c_9\ell + (c_{10} - 1)(2\ell - \mathfrak{e}_5 - \mathfrak{e}_6 - \mathfrak{e}_7 - \mathfrak{e}_8) + \sum_{i=5}^{6} c_i(\ell - \mathfrak{e}_i) + (2\ell - \mathfrak{e}_5 - \mathfrak{e}_6 - \mathfrak{e}_7 - \mathfrak{e}_8 - \mathfrak{e}_9),$$

which is big and nef, since the only term with negative square is the last one, and one checks that $(2\ell - \mathfrak{e}_5 - \mathfrak{e}_6 - \mathfrak{e}_7 - \mathfrak{e}_8 - \mathfrak{e}_9) \cdot L_0'' = c_0 + c_9 + c_5 + c_6 - 1 \geq 0$. We apply Proposition 4.2 with $k = s = 1$ and $t = 5$. Condition (v) therein is satisfied, as for instance $\mathfrak{e}_5 \cdot L'' = c_{10} + c_5 \geq c_{10} \geq 2$.

7.5. **Case where $a_0 > 0$ is odd and all remaining a_i are even (case (ii) in Lemma 5.8).** Since a_9, a_{10} are even and a_0 is odd, we have

$$a_9 + a_{10} > a_0 > a_9 \geq a_{10},$$

which implies $a_0 \geq 3$ and $a_9, a_{10} \geq 2$. Rearranging indices, we have a limit line bundle

$$L^0 \equiv c_0 E^0_{5,6} + \sum_{i=1}^{8} c_i E^0_i + c_{10} E^0_{10},$$

with the E^0_i and $E^0_{5,6}$ as in Example 6.1, where

$$c_5 + c_6 > c_0 > c_5 \geq c_6 \geq 2, \ \ c_0 \text{ is odd, } c_5, c_6 \text{ are even,} \tag{39}$$

$$c_4 \leq c_3 \leq c_2 \leq c_1 \leq c_7 \leq c_8 \leq c_{10}, \ \ \text{all even.} \tag{40}$$

We define

$$\kappa := \sharp\{j \in \{1, 2, 3, 4\} \mid c_j > 0\}.$$

Consider

$$\begin{aligned} L' := L^0|_{\widetilde{R}} &\equiv c_0\mathfrak{s} + \sum_{i=1}^{4} c_i(\mathfrak{s} - \mathfrak{e}_i) + \sum_{i=5}^{8} c_i\mathfrak{f} + c_{10}\mathfrak{f} \\ &= (c_0 + c_1 + c_2 + c_3 + c_4)\mathfrak{s} + (c_5 + c_6 + c_7 + c_8 + c_{10})\,\mathfrak{f} - \sum_{i=1}^{\kappa} c_i\mathfrak{e}_i \\ &= L_0' + \sum_{i=2}^{\kappa} (\mathfrak{f} - \mathfrak{e}_i), \end{aligned}$$

where

$$L'_0 := (c_0+c_1+c_2+c_3+c_4)\mathfrak{s}+(c_5+c_6+c_7+c_8+c_{10}-\kappa+1)\,\mathfrak{f}-c_1\mathfrak{e}_1-\sum_{i=2}^{\kappa}(c_i-1)\mathfrak{e}_i$$

and $\sum_{i=2}^{\kappa}(\mathfrak{f}-\mathfrak{e}_i)$ is the sum of $\max\{0,\kappa-1\}$ disjoint (-1)-curves. We note that we may consider L' as a line bundle on the blow–up of R at κ points. Hence we will eventually apply Proposition 4.2 with $s=\kappa$ and $k=\max\{0,\kappa-1\}$.

Claim 7.4. *L'_0 verifies condition $(\star)$ and is odd.*

Proof. Oddness follows since $c_0+c_1+c_2+c_3+c_4$ is odd by our assumptions (39) and (40). Conditions (i)–(iii) of $(\star)$ are easily checked. Condition (iv) is equivalent to

$$c_0+2(c_5+c_6+c_7+c_8+c_{10})\geq \kappa+2.$$

This is verified since, by (39), the left hand side is $\geq c_0+2c_5+2c_6\geq 3+4+4=11$. □

We have

$$L'' := L^0|_{\widetilde{P}} \sim c_0(\ell-\mathfrak{e}_7-\mathfrak{e}_8)+\sum_{i=5}^{8}c_i(\ell-\mathfrak{e}_i)+c_{10}(2\ell-\mathfrak{e}_5-\mathfrak{e}_6-\mathfrak{e}_7-\mathfrak{e}_8).$$

We can view L'' as a line bundle on $\mathrm{Bl}_{y_5,y_6,y_7,y_8}(P)$. The idea is now to apply Proposition 4.2 with $k=\max\{0,\kappa-1\}$, $s=\kappa$ and $t=4$.

7.5.1. *Subcase $c_7=0$.* By (40) we have $c_1=c_2=c_3=c_4=0$, whence $\kappa=0$. We apply Proposition 4.2 with $s=k=0$ and $t=4$. Condition (v) therein is satisfied, as for instance $\mathfrak{e}_5\cdot L''=c_5+c_{10}\geq c_5>0$ by (39).

7.5.2. *Subcase $c_7>0$.* By (40) we have $c_7,c_8,c_{10}\geq 2$.
Define

(41) $$L''_0(3) := c_0(\ell-\mathfrak{e}_7-\mathfrak{e}_8)+c_5(\ell-\mathfrak{e}_5)+\sum_{i=6}^{8}(c_i-1)(\ell-\mathfrak{e}_i)+$$
$$+c_{10}(2\ell-\mathfrak{e}_5-\mathfrak{e}_6-\mathfrak{e}_7-\mathfrak{e}_8)+(\ell-\mathfrak{e}_6-\mathfrak{e}_7)+(\ell-\mathfrak{e}_7-\mathfrak{e}_5)+(\ell-\mathfrak{e}_8-\mathfrak{e}_6),$$

$$\begin{aligned} L''_0(2) &= L''(3)+\mathfrak{e}_5,\\ L''_0(1) &= L''(3)+\mathfrak{e}_5+\mathfrak{e}_6,\\ L''_0(0) &= L''(3)+\mathfrak{e}_5+\mathfrak{e}_6+\mathfrak{e}_7.\end{aligned}$$

Then one may check that, for $j\in\{0,1,2,3\}$:

$$L''=L''_0(j)+\begin{cases}0, & \text{if } j=0,\\ \mathfrak{e}_7, & \text{if } j=1,\\ \mathfrak{e}_6+\mathfrak{e}_7, & \text{if } j=2,\\ \mathfrak{e}_5+\mathfrak{e}_6+\mathfrak{e}_7, & \text{if } j=3.\end{cases}$$

Claim 7.5. *$L''_0(j)$ is big and nef for all $j\in\{0,1,2,3\}$.*

Proof. Since $\mathfrak{e}_i \cdot L_0''(3) > 0$, for $i \in \{5,6,7\}$, it suffices to verify that $L_0''(3)$ is big and nef. All divisors in the sum (41) are of nonnegative square, except for the first and the last three. We have, using (39) and the fact that $c_5, c_6, c_7, c_8, c_{10} \geq 2$:

$$\begin{aligned}
L_0''(3) \cdot (\ell - \mathfrak{e}_7 - \mathfrak{e}_8) &= -c_0 + c_5 + (c_6 - 1) \geq 0, \\
L_0''(3) \cdot (\ell - \mathfrak{e}_6 - \mathfrak{e}_7) &= c_5 + (c_8 - 1) - 1 \geq 2 + 1 - 1 = 2, \\
L_0''(3) \cdot (\ell - \mathfrak{e}_7 - \mathfrak{e}_5) &= (c_6 - 1) + (c_8 - 1) - 1 + 1 \geq 1 + 1 + 0 \geq 2, \\
L_0''(3) \cdot (\ell - \mathfrak{e}_8 - \mathfrak{e}_6) &= c_5 + (c_7 - 1) + 1 - 1 \geq 2 + 1 + 0 = 3,
\end{aligned}$$

which proves that $L_0''(3)$ is nef. It is easily verified that it is big. □

Now we apply Proposition 4.2 with $k = \max\{0, \kappa - 1\} \leq 3$, $s = \kappa$, $t = 4$ and $L_0'' = L_0(k)$. What is left to be checked is condition (v). This is satisfied because $\mathfrak{e}_8 \cdot L_0''(k) = c_0 + c_8 - 1 + c_{10} + 1 = c_0 + c_8 + c_{10} > 0$.

7.6. **Case where $a_0 > 0$ and all remaining a_i are odd (case (iii) in Lemma 5.8).** Rearranging indices, we have a limit line bundle

$$L^0 \equiv c_0 E^0_{9,10} + c_9 E^0_9 + c_{10} E^0_{10} + \sum_{i=1}^{7} c_i E^0_i,$$

with the E^0_i and $E^0_{9,10}$ as in Example 6.1, where

$$c_9 + c_{10} \geq c_0 \geq c_9 \geq c_{10} > 0, \quad c_9, c_{10} \text{ odd}, \tag{42}$$

$$0 < c_1 \leq c_2 \leq \cdots \leq c_6 \leq c_7, \quad \text{all odd}. \tag{43}$$

Consider

$$\begin{aligned}
L' &:= L^0|_{\widetilde{R}} \equiv c_0\mathfrak{f} + c_9\mathfrak{s} + c_{10}\mathfrak{f} + \sum_{i=1}^{4} c_i(\mathfrak{s} - \mathfrak{e}_i) + \sum_{i=5}^{7} c_i\mathfrak{f} \\
&= (c_1 + c_2 + c_3 + c_4 + c_9)\mathfrak{s} + (c_0 + c_5 + c_6 + c_7 + c_{10})\mathfrak{f} - \sum_{i=1}^{4} c_i\mathfrak{e}_i \\
&= L_0' + \sum_{i=2}^{4} (\mathfrak{f} - \mathfrak{e}_i),
\end{aligned}$$

where

$$L_0' := (c_1 + c_2 + c_3 + c_4 + c_9)\mathfrak{s} + (c_0 + c_5 + c_6 + c_7 + c_{10} - 3)\,\mathfrak{f} - c_1\mathfrak{e}_1 - \sum_{i=2}^{4}(c_i - 1)\mathfrak{e}_i$$

and $\sum_{i=2}^{4}(\mathfrak{f} - \mathfrak{e}_i)$ is the sum of three disjoint (-1)-curves. We will eventually apply Proposition 4.3.

Claim 7.6. *L_0' verifies condition $(\star)$ and is odd.*

Proof. Oddness follows since $c_9 + c_1 + c_2 + c_3 + c_4$ is odd by our assumptions (42)-(43). Conditions (i)–(iv) of $(\star)$ easily follow from properties (42)–(43). □

We have

$$\begin{aligned} L'' := L^0|_{\widetilde{P}} &\sim c_0(\ell - \mathfrak{e}_9) + c_9\mathfrak{e}_9 + c_{10}(2\ell - \mathfrak{e}_5 - \mathfrak{e}_6 - \mathfrak{e}_7 - \mathfrak{e}_8) + \sum_{i=5}^{7} c_i(\ell - \mathfrak{e}_i) \\ &= L''_0 + \mathfrak{e}_6 + \mathfrak{e}_7 + \mathfrak{e}_8, \end{aligned}$$

with

$$\begin{aligned} L''_0 := &(c_0 - c_9)(\ell - \mathfrak{e}_9) + c_9\ell + c_{10}(2\ell - \mathfrak{e}_5 - \mathfrak{e}_6 - \mathfrak{e}_7 - \mathfrak{e}_8) + \sum_{i=5}^{7}(c_i - 1)(\ell - \mathfrak{e}_i) + \\ &+(\ell - \mathfrak{e}_5 - \mathfrak{e}_6) + (\ell - \mathfrak{e}_6 - \mathfrak{e}_7) + (\ell - \mathfrak{e}_7 - \mathfrak{e}_8). \end{aligned}$$

Claim 7.7. *L''_0 is big and nef.*

Proof. All terms in the expression of L''_0 right above have nonnegative square except for the last three. One computes

$$\begin{aligned} L''_0 \cdot (\ell - \mathfrak{e}_5 - \mathfrak{e}_6) &= c_0 + (c_7 - 1) - 1 + 0 + 1 \geq c_0 > 0, \\ L''_0 \cdot (\ell - \mathfrak{e}_6 - \mathfrak{e}_7) &= c_0 + (c_5 - 1) - 1 \geq 0, \\ L''_0 \cdot (\ell - \mathfrak{e}_7 - \mathfrak{e}_8) &= c_0 + (c_5 - 1) + (c_6 - 1) + 1 - 1 \geq c_0 > 0, \end{aligned}$$

which shows that L''_0 is nef. One easily computes that it is big. □

We apply Proposition 4.3. What is left to be checked is condition (iv): The additional disjoint (-1)-curves on $\widetilde{P}$ are $\mathfrak{e}_5$ and $\mathfrak{e}_9$, and we have

$$\mathfrak{e}_5 \cdot L''_0 = c_{10} + c_5 > c_{10} \geq c_0 - c_9 = \mathfrak{e}_9 \cdot L''_0,$$

by (42) and (43).

This concludes the proof of Theorem 1.1.

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