THE FUNDAMENTAL GROUP OF QUOTIENTS OF A PRODUCT OF CURVES

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This article is dedicated to the memory of Fritz Grunewald

Abstract. We prove a structure theorem for the fundamental group of the quotient X of a product of curves by the action of a finite group G, hence for that of any resolution of the singularities of X.

1. Introduction

The study of varieties isogenous to a product of curves was initiated by Catanese in [Cat00], inspired by a construction of Beauville. These varieties are quotients of a product of smooth projective curves $C_1 \times \cdots \times C_n$ by the free action of a finite group G.

Much of the work in this area has been focused in the $n = 2$ case. Surfaces isogenous to a product of curves provide a wide class of surfaces quite manageable to work with, since they are determined by discrete combinatorial data. They were used successfully to address various questions (see e.g. the survey paper [BCP06]), and in particular to obtain substantial information about various moduli spaces of surfaces of general type (see e.g. [BC04, BCG08, BCGP09]).

In the case of a variety isogenous to a product, the action of G is free, and $X := (C_1 \times \cdots \times C_n)/G$ is smooth. Furthermore, we have the following natural description of the fundamental group of X.

Proposition 1.1. [Cat00] *If $X := (C_1 \times \cdots \times C_n)/G$ is the quotient of a product of curves by the free action of a finite group, then the fundamental group of X sits in an exact sequence*

$$1 \to \Pi_{g_1} \times \cdots \times \Pi_{g_n} \to \pi_1(X) \to G \to 1, \tag{1.1}$$

where each Π_{g_i} is the fundamental group of C_i. This extension, in the unmixed case where each Π_{g_i} is a normal subgroup, is determined by the associated maps $G \to \mathrm{Out}\,(\Pi_{g_i})$ to the respective Teichmüller modular groups.

In the recent paper [BCGP09], Bauer, Catanese, Grunewald and Pignatelli prove that a similar statement still holds under weaker assumptions.

Theorem 1.2. [BCGP09, Thm. 0.10 and Thm. 4.1] *Assume that G acts faithfully on each curve C_i as a group of automorphisms, and let $X := (C_1 \times \cdots \times C_n)/G$ be the (possibly singular) quotient by the diagonal action of G. Then the fundamental group $\pi_1(X)$ has a normal subgroup of finite index isomorphic to the product of n surface groups. We call G' the quotient group.*

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Here, by a surface group we mean a group isomorphic to the fundamental group of a compact Riemann surface. Note that, unlike in Proposition 1.1, the surface groups in Theorem 1.2 above are not necessarily isomorphic to the fundamental groups of the curves $C_1, \ldots, C_n$, and furthermore that the corresponding quotient G' of $\pi_1(X)$ is not necessarily isomorphic to G.

The first step of the proof of Theorem 1.2 consists in showing that $\pi_1(X)$ is isomorphic to the quotient of the fibre product $\mathbf{T} := \mathbf{T}_1 \times_G \cdots \times_G \mathbf{T}_n$ of n orbifold surface groups (see Subsection 2.1) by its torsion subgroup $\mathrm{Tors}(\mathbf{T})$. Whereas this first part rests upon geometrical considerations, the rest of the proof relies on an abstract group theoretic argument showing that this quotient necessarily contains a normal subgroup as described in Theorem 1.2. In particular, the relation occurring between the groups G and G' is not well understood.

Using a suitable resolution of the singularities of X, Bauer, Catanese, Grunewald and Pignatelli show in addition that the fundamental group of any resolution Y of X is isomorphic to the fundamental group of X, so that the same description holds for $\pi_1(Y)$.

In [BCGP09], as an important application of Theorem 1.2, many new families of algebraic surfaces S of general type with $p_g(S) = 0$ are constructed, and several new examples of groups are realized as the fundamental group of an algebraic surface S of general type with $p_g(S) = 0$. This increases notably our knowledge on algebraic surfaces. In fact the authors consider and classify all the surfaces whose canonical models arise as quotients $X := (C_1 \times C_2)/G$ of the product of two curves of genera $g(C_1), g(C_2) \geq 2$ by the action of a finite group G such that $p_g(X) = q(X) = 0$.

In the present paper, we drop the assumption that the actions of G on $C_1, \ldots, C_n$ are faithful. We obtain the following expected strengthening of Theorem 1.2.

Theorem 1.3. *Let $C_1, \ldots, C_n$ be smooth projective curves, and let G be a finite group acting on each C_i as a group of automorphisms. Then the fundamental group of the quotient $X := C_1 \times \cdots \times C_n/G$ by the diagonal action of G has a normal subgroup of finite index that is isomorphic to the product of n surface groups.*

This result should allow in the future the realization of interesting groups as fundamental groups of higher dimensional algebraic varieties, following the method developed in [BCGP09] for surfaces. Notice that, in the case where the G-actions are faithful, X can only have isolated cyclic-quotient singularities, while if the actions are not faithful, then the singular locus of X can have components of positive dimension, and the singularities are abelian-quotient singularities.

Again, one shows that any desingularization of the quotient X has a fundamental group isomorphic to that of X. This time however, we have to rely on a strong result of Kollár [K93].

The proof follows then closely the one of Theorem 1.2 of [BCGP09]. The main new difficulty one has to overcome is to find a natural counterpart to the fibered product $\mathbf{T}_1 \times_G \cdots \times_G \mathbf{T}_n$, acting discontinuously on the product $\tilde{C}_1 \times \cdots \times \tilde{C}_n$ of the universal covers of $C_1, \ldots, C_n$. After that similar group theoretic arguments work with some slight modifications.

It has already been observed in [Cat00] that Theorem 1.3 follows directly from Theorem 1.2 when $n = 2$, by performing the quotient $(C_1 \times C_2)/G$ in successive steps. For $n > 2$ however, this procedure does not apply.

The paper is organized as follows. In Section 2, we fix notations and collect some basic facts about group actions on compact Riemann surfaces. Section 3 is devoted to the proof of Theorem 1.3: the proof itself is given in Subsection 3.1, using intermediate results proven in Subsections 3.2 and 3.3.

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2. Notations and basic results

2.1. **Notations.** We work over the field of complex numbers $\mathbf{C}$.

Let G be a group, and consider a subset $H \subset G$. We write $H \leq G$ when H is a subgroup of G, and $H \trianglelefteq G$ when H is a normal subgroup of G. If $A \subset G$ is any subset, then $\langle\langle A \rangle\rangle_G$ denotes the normal subgroup of G generated by A.

Let g be a non negative integer. We call Π_g the *surface group* of genus g, defined as

$$\Pi_g := \left\langle a_1, b_1, \ldots, a_g, b_g \,\middle|\, \prod\nolimits_{i=1}^{g} [a_i, b_i] = 1 \right\rangle .$$

This is the fundamental group of compact Riemann surfaces of genus g. On the other hand, letting in addition $m_1, \ldots, m_r$ be positive integers, we denote by $\mathbf{T}(g; m_1, \ldots, m_r)$ the *orbifold surface group* of signature $(g; m_1, \ldots, m_r)$, defined as

$$\mathbf{T}(g; m_1, \ldots, m_r) := \Big\langle a_1, b_1, \ldots, a_g, b_g, c_1, \ldots, c_r \,\Big|\, c_1^{m_1} = \cdots = c_r^{m_r} = \prod\nolimits_{i=1}^{g} [a_i, b_i] \cdot c_1 \cdot \cdots \cdot c_r = 1 \Big\rangle .$$

It is obtained from the fundamental group of the complement of a set of r distinct points in a compact Riemann surface of genus g, by quotienting by the normal subgroup generated by $\gamma_1^{m_1}, \ldots, \gamma_r^{m_r}$, where each γ_i is a simple geometric counter-clockwise loop around the i-th removed point.

Let G be a finite group. An *appropriate orbifold homomorphism* is a surjective homomorphism $\varphi : \mathbf{T}(g; m_1, \ldots, m_r) \to G$ such that $\varphi(c_i)$ has order m_i for $i = 1, \ldots, r$.

The action of a group G as a group of homeomorphisms on a topological space X is said to be *discontinuous* if the following two conditions are satisfied: (i) the stabilizer of each point of X is finite; (ii) each point $x \in X$ has a neighbourhood U such that $g(U) \cap U = \emptyset$, for each $g \in G$ such that $gx \neq x$.

2.2. **Basic results.** The following result is essentially a reformulation of Riemann's existence theorem (see [BCGP09, Thm. 2.1]).

Theorem 2.1. *A finite group G acts faithfully as a group of automorphisms on a compact Riemann surface of genus g if and only if there are natural numbers $g', m_1, \ldots, m_r$ and an appropriate orbifold homomorphism*

$$\varphi : \mathbf{T}(g'; m_1, \ldots, m_r) \to G$$

such that the Riemann-Hurwitz relation holds:

$$2g-2 = |G| \left(2g' - 2 + \sum_{i=1}^{r} \left(1 - \frac{1}{m_i}\right)\right).$$

Remark 2.2. As already remarked in [BCGP09], under the above hypotheses, g' is the geometric genus of $C' := C/G$, and $m_1, \ldots, m_r$ are the branching indices at the branching points of the G-cover $p\colon C \to C'$. The appropriate orbifold homomorphism φ is induced by the monodromy of the Galois étale G-covering $p^\circ : C^\circ \to C'^\circ$ induced by p, where C'° is the Riemann surface obtained from C' by removing the branch points of p, and $C^\circ := p^{-1}(C'^\circ)$. In particular, $\varphi(c_i)$ generates the stabilizer of the corresponding ramification point.

Furthermore, the kernel of φ is isomorphic to the fundamental group $\pi_1(C)$, and the action of $\pi_1(C)$ on the universal cover $\tilde{C}$ of C extends to a discontinuous action of $\mathbf{T} := \mathbf{T}(g'; m_1, \ldots, m_r)$. Let $u\colon \tilde{C} \to C$ be the covering map. It is φ-equivariant, and $C/G \cong \tilde{C}/\mathbf{T}$.

We now give two elementary facts that will be used in the following.

Lemma 2.3. (i) *Let $x \in \tilde{C}$. Then the restriction of φ to the stabilizer St_x of x (with respect to the action of $\mathbf{T}$ on $\tilde{C}$) is injective.*
(ii) *Let $t \in \mathrm{St}_x$. Then t is conjugated to c_i^m, for some $i \in \{1, \ldots, r\}$ and $m \in \mathbf{N}$.*

Proof. The $\pi_1(C)$-action on $\tilde{C}$ is free, so $\pi_1(C) \cap \mathrm{St}_x = \{1\}$. This yields (i), because $\pi_1(C)$ is the kernel of φ.

To prove (ii), let $y = u(x)$. If $t = 1$, then the result is clear. Else, there exists an integer $i \in \{1, \ldots, m\}$ and a point $x' \in u^{-1}(y)$ that is fixed by c_i. It then follows from (i) that $\mathrm{St}_{x'} = \langle c_i \rangle$, hence that t is conjugated to a power of c_i.

□

3. Main Theorem

The main result of the paper is the following

Theorem 3.1. *Let $C_1, \ldots, C_n$ be compact Riemann surfaces, and let G be a finite group that acts as a group of automorphisms on each C_i. We consider the quotient of the product $C_1 \times \cdots \times C_n$ by the diagonal action of G. Then there is a normal subgroup of finite index Π in the fundamental group*

$$\pi_1 \left(\frac{C_1 \times \cdots \times C_n}{G} \right),$$

such that Π is isomorphic to the product of n surface groups.

Notice that, according to notations 2.1, a surface group is a group isomorphic to the fundamental group of a compact Riemann surface of genus a non negative integer g, in particular we admit also the "degenerate cases" where $g = 0, 1$.

The proof of this theorem follows closely that of [BCGP09, Thm. 4.1], and is given in the next subsections. Before we move on to this proof, let us give the following important consequence of Theorem 3.1.

Corollary 3.2. *Let $C_1, \dots, C_n$ and G be as in the statement of Theorem 3.1, and let Y be a resolution of the singularities of $X := (C_1 \times \cdots \times C_n)/G$. Then, the fundamental group of Y is isomorphic to the fundamental group of X, and moreover it has a normal subgroup of finite index isomorphic to the product of n surface groups.*

Proof. The natural morphism

$$f_* : \pi_1(Y) \longrightarrow \pi_1(X)$$

induced by the resolution $f : Y \to X$ is an isomorphism. This follows directly from [K93, Sec. 7]: since X is normal and only has quotient singularities, Y is locally simply connected by [K93, Thm. 7.2], hence f_* is an isomorphism by [K93, Lem. 7.2]. The second claim is now a direct consequence of Theorem 3.1. □

3.1. **Proof of the main theorem.** For $i = 1, \dots, n$, we let

$$K_i = \ker(G \to \mathrm{Aut}(C_i)) \quad \text{and} \quad H_i = G/K_i\,,$$

where $G \to \mathrm{Aut}(C_i)$ is the morphism associated to the action of G on C_i. We call p_i the projection $G \to H_i$. Now H_i acts faithfully on C_i, so we have (see Remark 2.2) a short exact sequence

$$1 \to \pi_1(C_i) \to \mathbf{T}_i \xrightarrow{\varphi_i} H_i \to 1, \tag{3.1}$$

where $\mathbf{T}_i$ is an orbifold surface group, and φ_i is an appropriate orbifold homomorphism. Let $\Sigma_i := G \times_{H_i} \mathbf{T}_i$ be the fibered product corresponding to the Cartesian diagram

$$\begin{tikzcd} \Sigma_i \arrow[r] \arrow[d, "\psi_i"'] \arrow[dr, phantom, "\square"] & \mathbf{T}_i \arrow[d, "\varphi_i"] \\ G \arrow[r, "p_i"'] & H_i. \end{tikzcd}$$

We call $\psi_i : \Sigma_i \to G$ the projection on the first factor. Pulling-back (3.1) by $p_i : G \to H_i$, we obtain a short exact sequence

$$1 \to \pi_1(C_i) \to \Sigma_i \xrightarrow{\psi_i} G \to 1, \tag{3.2}$$

where the left-hand side map is $\gamma \in \pi_1(C_i) \mapsto (1, \gamma) \in \Sigma_i$.

Next, we define $\tilde{G} := \Sigma_1 \times_G \cdots \times_G \Sigma_n$ as the fibered product correponding to the Cartesian diagram below.

$$\begin{tikzcd} & & \tilde{G} \arrow[dll] \arrow[dl] \arrow[dr] \arrow[drr] & & \\ \Sigma_1 \arrow[drr] & \Sigma_2 \arrow[dr] & \cdots & \Sigma_{n-1} \arrow[dl] & \Sigma_n \arrow[dll] \\ & & G & & \end{tikzcd}$$

Let $\Delta : G \to G \times \cdots \times G$ be the diagonal morphism. Then $\tilde{G}$ can also be seen as the fibered product $G \times_{(G \times \cdots \times G)} (\Sigma_1 \times \cdots \times \Sigma_n) \to G$ with respect to the two morphisms Δ and $(\psi_1, \dots, \psi_n)$. Therefore, the pull-back by Δ of the product of the n exact sequences (3.2) for $i = 1, \dots, n$ is a short exact sequence

$$1 \to \prod_{i=1}^{n} \pi_1(C_i) \to \tilde{G} \xrightarrow{\Psi} G \to 1, \tag{3.3}$$

where Ψ is the first projection $G \times_{(G \times \cdots \times G)} (\Sigma_1 \times \cdots \times \Sigma_n) \to G$.

Now we have the following, coming from the fact that $\tilde{G}$ acts discontinuously on the universal cover of $C_1 \times \cdots \times C_n$.

Proposition 3.3. *Let $\tilde{G}' \trianglelefteq \tilde{G}$ be the normal subgroup of $\tilde{G}$ generated by those elements which have non-empty fixed-point set. Then*

$$\pi_1 \left(\frac{C_1 \times \cdots \times C_n}{G} \right) \quad \cong \quad \frac{\tilde{G}}{\tilde{G}'}.$$

Proof. For $i = 1, \dots, n$, the action of $\mathbf{T}_i$ on the universal covering $\tilde{C}_i$ of C_i (see Remark 2.2) induces an action of Σ_i on $\tilde{C}_i$ via the projection of Σ_i on its second factor $\mathbf{T}_i$. We obtain in this way an action of $\tilde{G}$ on the product $\tilde{C}_1 \times \cdots \times \tilde{C}_n$.

This action is discontinuous: let St_x be the stabilizer of a point $x \in \tilde{C}_1 \times \cdots \times \tilde{C}_n$ with respect to the action of $\tilde{G}$. Then the same argument as that in the proof of Lemma 2.3 shows that $\Psi_{|\mathrm{St}_x}$ is injective, from which it follows that St_x is finite because G is finite. On the other hand, condition (ii) in the definition of a discontinuous action is a consequence of the fact that the $\mathbf{T}_i$-actions are themselves discontinuous.

Then, the main theorem in [Arm68] applies to our situation, and gives a group isomorphism

$$\pi_1 \left(\frac{\tilde{C}_1 \times \cdots \times \tilde{C}_n}{\tilde{G}} \right) \cong \frac{\tilde{G}}{\tilde{G}'}.$$

Eventually, since the universal covering $\mathcal{U} \colon \tilde{C}_1 \times \cdots \times \tilde{C}_n \to C_1 \times \cdots \times C_n$ is Ψ-equivariant, we have an isomorphism

$$\frac{C_1 \times \cdots \times C_n}{G} \cong \frac{\tilde{C}_1 \times \cdots \times \tilde{C}_n}{\tilde{G}},$$

and the proposition follows.

□

Remark 3.4. The elements of $\tilde{G}$ which have fixed-points are precisely those elements of finite order. Therefore $\tilde{G}'$ is the torsion subgroup of $\tilde{G}$.

Now the proof of Theorem 3.1 relies on the following result, the proof of which we postpone to Subsection 3.3.

Proposition 3.5. *The quotient $\tilde{G}/\tilde{G}'$ is an extension*

$$1 \to E \to \tilde{G}/\tilde{G}' \xrightarrow{\theta} T \to 1$$

of a finite group E by a group T that is a finite-index subgroup of a product of n orbifold surface groups.

Using the results of [GJZ08], the latter fact enables one to show that there is a finite index normal subgroup $\Gamma \trianglelefteq \tilde{G}/\tilde{G}'$ that injects in T:

Lemma 3.6. *Let S be a group sitting in an exact sequence*

$$1 \to E \to S \to T \to 1,$$

where E is a finite group, and T is a finite index subgroup of a product of n orbifold surface groups. Then S is residually finite. In particular, there exists a finite index normal subgroup $\Gamma \trianglelefteq S$ such that $\Gamma \cap E = \{1\}$.

Proof. By [GJZ08, Prop. 6.1], an extension of a finite group by a group that is residually finite and good in the sense of [Ser94] is residually finite. It therefore suffices to show that T enjoys the two aforementioned properties.

An orbifold surface group is residually finite. Therefore T is itself residually finite, being a finite index subgroup of a product of orbifold surface groups.

By [GJZ08, Lem. 3.2], it is enough to show that a product of orbifold surface groups is good to prove that T is good. But [GJZ08, Prop. 3.7] tells us that an orbifold surface group is good, and [GJZ08, Prop. 3.4] that a product of good groups is good.

□

We are now in a position to complete the proof of our main theorem:

Proof of Theorem 3.1. Let $\mathbf{T}_1 \times \cdots \times \mathbf{T}_n$ be a product of n orbifold surface groups containing T as a finite index subgroup, and let us consider $\Gamma \trianglelefteq \tilde{G}/\tilde{G}'$ a normal subgroup of finite index such that $E \cap \Gamma = \{1\}$. Then $\theta(\Gamma) \leq \mathbf{T}_1 \times \cdots \times \mathbf{T}_n$ has finite index.

Now every orbifold surface group contains a surface group as a finite index subgroup (see e.g. [Bea95]), so let Π_i be a finite index surface group in $\mathbf{T}_i$ for each $i = 1, \ldots, n$.

For each i, we consider

$$\theta(\Gamma)_i := \theta(\Gamma) \cap (\{1\} \times \cdots \times \mathbf{T}_i \times \cdots \times \{1\})$$

as a subgroup $\theta(\Gamma)_i \leq \mathbf{T}_i$, and set

$$\Pi_i' := \bigcap_{g \in \mathbf{T}_i} g\left(\theta(\Gamma)_i \cap \Pi_i\right) g^{-1},$$

the biggest normal subgroup of $\mathbf{T}_i$ contained in $\theta(\Gamma)_i \cap \Pi_i$. Then Π_i' has finite index in Π_i, and thus is itself a surface group. Eventually, $\Pi := \Pi_1' \times \cdots \times \Pi_n'$ is a subgroup of $\theta(\Gamma)$, which is normal and of finite index in T. Therefore, $\theta^{-1}(\Pi) \cap \Gamma$ is a normal subgroup of $\tilde{G}/\tilde{G}'$, with finite index, and isomorphic to Π.

□

3.2. **Results in group theory.** In this subsection, we prove some technical results that are needed for the proof of Proposition 3.5.

Let Σ be any group, $R \trianglelefteq \Sigma$ be a normal subgroup, and $L \subset \Sigma$ be a subset. We define

$$N(R, L) := \langle\langle \{hkh^{-1}k^{-1} \mid h \in L\,,\; k \in R\}\rangle\rangle_{\Sigma} \tag{3.4}$$

and

$$\hat{\Sigma} := \hat{\Sigma}(R, L) := \Sigma/N(R, L). \tag{3.5}$$

We call $\hat{R}$ and $\hat{L}$ the images of R and L respectively by the projection $\Sigma \to \hat{\Sigma}$. There is an isomorphism: $\hat{\Sigma}/\langle\langle \hat{L}\rangle\rangle_{\hat{\Sigma}} \cong \Sigma/\langle\langle L\rangle\rangle_{\Sigma}$. Notice also that $N(R, L) \trianglelefteq R$ and $N(R, L) \trianglelefteq \langle\langle L\rangle\rangle_{\Sigma}$, which implies that $\hat{R}$ is a normal subgroup of $\hat{\Sigma}$.

Lemma 3.7. *If $R \trianglelefteq \Sigma$ has finite index, and if $L \subset \Sigma$ is a finite subset consisting of elements of finite order, then $\langle\langle \hat{L} \rangle\rangle_{\hat{\Sigma}}$ is finite.*

Proof. The subgroup $\langle\langle \hat{L} \rangle\rangle_{\hat{\Sigma}}$ is the image of $\langle\langle L \rangle\rangle_{\Sigma}$ under the projection $\Sigma \to \hat{\Sigma}$. Since R has finite index in Σ, and L is finite, it follows that $\langle\langle \hat{L} \rangle\rangle_{\hat{\Sigma}}$ is generated by finitely many elements which are conjugated to those of $\hat{L}$. Since the elements of L have finite order, these generators have finite order as well.

The center $\mathrm{Z}\left(\langle\langle \hat{L} \rangle\rangle_{\hat{\Sigma}}\right)$ of $\langle\langle \hat{L} \rangle\rangle_{\hat{\Sigma}}$ contains $\hat{R} \cap \langle\langle \hat{L} \rangle\rangle_{\hat{\Sigma}}$, and hence has finite index in $\langle\langle \hat{L} \rangle\rangle$. Now, by [BCGP09, Lem. 4.6], if a group S is generated by finitely many elements of finite order, and if its centre has finite index in S, then S is finite. From this we conclude that $\langle\langle \hat{L} \rangle\rangle_{\hat{\Sigma}}$ is finite.

□

We now consider the particular case when Σ is a group constructed as in Subsection 3.1: $\Sigma = G \times_H \mathbf{T}$, where G is a finite group, H is a quotient of G, and $\mathbf{T}$ is any group coming with a surjective morphism $\varphi : \mathbf{T} \to H$.

Lemma 3.8. *The projection on the second factor $q : \Sigma \to \mathbf{T}$ induces a morphism*

$$\bar{q} : \frac{\Sigma}{\langle\langle L \rangle\rangle_{\Sigma}} \to \frac{\mathbf{T}}{\langle\langle q(L) \rangle\rangle_{\mathbf{T}}} \tag{3.6}$$

in a natural way. It is surjective, and has finite kernel.

Proof. We have $q\left(\langle\langle L \rangle\rangle_{\Sigma}\right) = \langle\langle q(L) \rangle\rangle_{\mathbf{T}}$. The map $\bar{q}$ is therefore induced by the composition $\Sigma \xrightarrow{q} \mathbf{T} \to \mathbf{T}/\langle\langle q(L) \rangle\rangle_{\mathbf{T}}$, which is clearly surjective. To prove the finiteness of its kernel, notice that for any $(g,t) \in q^{-1}\left(\langle\langle q(L) \rangle\rangle_{\mathbf{T}}\right)$, there exists $h \in G$ with $(h,t) \in \langle\langle L \rangle\rangle_{\Sigma}$, hence $gh^{-1} \in K := \ker(G \to H)$. It follows that $q^{-1}(\langle\langle q(L) \rangle\rangle_{\mathbf{T}}) = K\langle\langle L \rangle\rangle_{\Sigma}$, where K is seen as a subgroup in Σ via the injection $k \in K \mapsto (k,1) \in \Sigma$. Eventually, $\ker(\bar{q}) \cong K \subset G$, which is finite.

□

3.3. **Realization of the fundamental group as a suitable extension.** In this subsection, we give a full proof of Proposition 3.5. We use the basic results in group theory established in Subsection 3.2 above.

For $i = 1, \ldots, n$, we fix the following presentation for the orbifold groups $\mathbf{T}_i$ in (3.1):

$$\mathbf{T}_i = \Big\langle a_{i1}, b_{i1}, \ldots, a_{ig_i}, b_{ig_i}, c_{i1}, \ldots, c_{ir_i} \;\Big|\; c_{i1}^{m_{i1}} = \cdots = c_{ir_i}^{m_{ir_i}} = \prod_{j=1}^{g_i} [a_{ij}, b_{ij}] \cdot c_{i1} \cdot \cdots \cdot c_{ir_i} = 1 \Big\rangle,$$

and set $R_i = \pi_1(C_i)$. We write the elements of $\tilde{G}$ as $(g, z_1, \ldots, z_n)$, with $(g, z_i) \in \Sigma_i$ for $i = 1, \ldots, n$. Then we have:

Lemma 3.9. *For each $i = 1, \ldots, n$, there exists a finite subset $\mathcal{N}_i \subset \tilde{G}$, such that*

$$\langle\langle \mathcal{N}_i \rangle\rangle_{\tilde{G}} = \tilde{G}',$$

and whose elements are of the form

$$(g, z_1 d_1^{\ell_1} z_1^{-1}, \ldots, d_i^{\ell_i}, \ldots, z_n d_n^{\ell_n} z_n^{-1})$$

for some $g \in G$, some $d_j \in \{c_{j1}, \ldots, c_{jr_j}\}$ and $\ell_j \in \mathbf{N}$ for $j = 1, \ldots, n$, and some $z_j \in \mathbf{T}_j$ for $j \neq i$.

Remark 3.10. As a direct consequence of Lemma 3.9, if

$$(g, z_1 d_1^{\ell_1} z_1^{-1}, \ldots, d_i^{\ell_i}, \ldots, z_n d_n^{\ell_n} z_n^{-1}) \in \mathcal{N}_i$$

for some i, then for any $j \neq i$, there exists

$$(h, y_1, \ldots, \widehat{y_j}, \ldots, y_n) \in \Sigma_1 \times_G \cdots \times_G \widehat{\Sigma_j} \times_G \cdots \Sigma_n$$

(where a hat means that the corresponding factor is omitted), such that

$$(hgh^{-1}, y_1 z_1 d_1^{\ell_1} z_1^{-1} y_1^{-1}, \ldots, y_i d_i^{\ell_i} y_i^{-1}, \ldots, d_j^{\ell_j}, \ldots, y_n z_n d_n^{\ell_n} z_n^{-1} y_n^{-1}) \in \mathcal{N}_j.$$

Proof of Lemma 3.9. Let $s \in \tilde{G}$ be an element with non empty fixed-point set, and let us fix $i \in \{1, \ldots, n\}$. By Lemma 2.3 (ii), s writes

$$s = (g, z_1 d_1^{\ell_1} z_1^{-1}, \ldots, z_n d_n^{\ell_n} z_n^{-1}),$$

with notations as in the statement of the Lemma. Obviously, one can find $h \in G$, and $\zeta_j \in \mathbf{T}_j$ for each $j \neq i$, such that $(h, \zeta_1, \ldots, z_i^{-1}, \ldots, \zeta_n) \in \tilde{G}$, and therefore s is conjugated in $\tilde{G}$ to an element of type

$$(g', y_1 d_1^{\ell_1} y_1^{-1}, \ldots, d_i^{\ell_i}, \ldots, y_n d_n^{\ell_n} y_n^{-1}). \tag{3.7}$$

Now we claim that there exists finite sets $A_j \subset \mathbf{T}_j$, $j = 1, \ldots, \widehat{i}, \ldots, n$, such that each element of $\tilde{G}$ as in (3.7) is conjugated in $\tilde{G}$ to some

$$(g'', x_1 d_1^{\ell_1} x_1^{-1}, \ldots, d_i^{\ell_i}, \ldots, x_n d_n^{\ell_n} x_n^{-1}),$$

with $x_j \in A_j$ for each $j \neq i$. Then it is clear that one can build $\mathcal{N}_i$ as required.

To prove our claim, first note that if $(g, z_1 d_1^{\ell_1} z_1^{-1}, \ldots, d_i^{\ell_i}, \ldots, z_n d_n^{\ell_n} z_n^{-1}) \in \tilde{G}$, then an $(n+1)$-uple $(g, \zeta_1 d_1^{\ell_1} \zeta_1^{-1}, \ldots, d_i^{\ell_i}, \ldots, \zeta_n d_n^{\ell_n} \zeta_n^{-1})$ corresponds to an element of the fibered product $\tilde{G}$ if and only if for each $j \neq i$, $\varphi_j(z_j^{-1}\zeta_j)$ belongs to the centralizer $C_{H_j}(\varphi_j(d_j^{\ell_j}))$ of $\varphi_j(d_j^{\ell_j})$ in H_j.

Second, note that if $k_j \in R_j$ for some $j \neq i$, then $(1, 1, \ldots, k_j, \ldots, 1) \in \tilde{G}$, and therefore any element $(g, \ldots, z_j d_j^{\ell_j} z_j^{-1}, \ldots) \in \tilde{G}$ is conjugated to

$$(g, \ldots, (k_j z_j) d_j^{\ell_j} (k_j z_j)^{-1}, \ldots) \in \tilde{G}.$$

Then our claim follows from the fact that for each $j \neq i$, $R_j \trianglelefteq \varphi_j^{-1}\left(C_{H_j}(\varphi_j(d_j^{\ell_j}))\right)$ has finite index.

□

From now on, we let $\mathcal{N}_1, \ldots, \mathcal{N}_n$ be as in Lemma 3.9.

Lemma 3.11. *For $i = 1, \ldots, n$, if*

$$(g, z_1 d_1^{\ell_1} z_1^{-1}, \ldots, d_i^{\ell_i}, \ldots, z_n d_n^{\ell_n} z_n^{-1}) \in \mathcal{N}_i,$$

then for all $k_i \in R_i$, we have

$$(1, 1, \ldots, d_i^{\ell_i} k_i d_i^{-\ell_i} k_i^{-1}, \ldots, 1) \in \tilde{G}'.$$

Proof. Let $k_i \in R_i$. Then $\tilde{k}_i := (1, 1, \ldots, k_i, \ldots, 1) \in \tilde{G}$, and our result follows from the equality

$$(1, 1, \ldots, d_i^{\ell_i} k_i d_i^{-\ell_i} k_i^{-1}, \ldots, 1) =$$
$$(g, z_1 d_1^{\ell_1} z_1^{-1}, \ldots, d_i^{\ell_i}, \ldots, z_n d_n^{\ell_n} z_n^{-1})\, \tilde{k}_i\, (g, z_1 d_1^{\ell_1} z_1^{-1}, \ldots, d_i^{\ell_i}, \ldots, z_n d_n^{\ell_n} z_n^{-1})^{-1}\, \tilde{k}_i^{-1},$$

and the fact that $\langle\langle \mathcal{N}_i \rangle\rangle_{\tilde{G}} = \tilde{G}'$.

□

For $i = 1, \ldots, n$, we let $L_i \subset \Sigma_i$ be the image of $\mathcal{N}_i$ by the projection $\tilde{G} \to \Sigma_i$. The first projection $\psi_i : \Sigma_i \to G$ then induces an epimorphism $\hat{\psi}_i : \hat{\Sigma}_i := \hat{\Sigma}_i(R_i, L_i) \to G$ (see Subsection 3.2 for a definition of $\hat{\Sigma}_i$).

Eventually, we let $\hat{G}$ be the fibered product

$$\hat{\Sigma}_1 \times_G \cdots \times_G \hat{\Sigma}_n \cong G \times_{(G \times \cdots \times G)} (\hat{\Sigma}_1 \times \cdots \times \hat{\Sigma}_n), \tag{3.8}$$

and we define a map $\Phi : \hat{G} \to \tilde{G}/\tilde{G}'$ by the formula

$$\Phi\left([s_1], \ldots, [s_n]\right) = \left[(s_1, \ldots, s_n)\right], \tag{3.9}$$

where $s_i \in \Sigma_i$ for each i (here we see $\hat{G}$ as contained in $\hat{\Sigma}_1 \times \cdots \times \hat{\Sigma}_n$, using its description by the left-hand side of (3.8) rather than by its right-hand side, and similarly we see $\tilde{G}$ as contained in $\Sigma_1 \times \cdots \times \Sigma_n$). It is a consequence of Lemma 3.11 that Φ is well-defined by (3.9). Now we have:

Lemma 3.12. *The morphism Φ is surjective, and has finite kernel.*

Proof. The surjectivity follows at once from (3.9). On the other hand, an element $([s_1], \ldots, [s_n]) \in \hat{G}$ lies in $\ker \Phi$ if and only if $(s_1, \ldots, s_n) \in \tilde{G}'$. This implies for $i = 1, \ldots, n$ that $s_i \in \langle\langle L_i \rangle\rangle_{\Sigma_i}$, because $\tilde{G}' = \langle\langle \mathcal{N}_i \rangle\rangle_{\tilde{G}}$. We thus see that $\ker \Phi$, seen as contained in $\hat{\Sigma}_1 \times \cdots \times \hat{\Sigma}_n$, is contained in $\langle\langle \hat{L}_1 \rangle\rangle_{\hat{\Sigma}_1} \times \cdots \times \langle\langle \hat{L}_n \rangle\rangle_{\hat{\Sigma}_n}$, which is finite by Lemma 3.7.

□

Next, we define a morphism

$$\Theta : \prod_{i=1}^{n} \hat{\Sigma}_i \to \prod_{i=1}^{n} \frac{\Sigma_i}{\langle\langle L_i \rangle\rangle} \xrightarrow{(\bar{q}_1, \ldots, \bar{q}_n)} \prod_{i=1}^{n} \frac{\mathbf{T}_i}{\langle\langle q_i(L_i) \rangle\rangle} \tag{3.10}$$

as in Subsection 3.2: the left-hand side map in (3.10) is the product of the projections

$$\hat{\Sigma}_i \to \hat{\Sigma}_i / \langle\langle \hat{L}_i \rangle\rangle_{\hat{\Sigma}_i} \cong \Sigma_i / \langle\langle L_i \rangle\rangle_{\Sigma_i},$$

and the $\bar{q}_i$'s are induced by the second projections $q_i : \Sigma_i \to \mathbf{T}_i$ as in Lemma 3.8. We have

$$\ker \Phi \subset \langle\langle \hat{L}_1 \rangle\rangle_{\hat{\Sigma}_1} \times \cdots \times \langle\langle \hat{L}_1 \rangle\rangle_{\hat{\Sigma}_n} \subset \ker \Theta, \tag{3.11}$$

and $\ker \Theta$ is finite by both Lemmas 3.7 and 3.8.

Let us set

$$T := \Theta(\hat{G}).$$

Notice that $\hat{G}$ has finite index in $\prod_{i=1}^{n} \hat{\Sigma}_i$ because G is finite, and therefore that T has finite index in $\prod_{i=1}^{n} \mathbf{T}_i / \langle\langle q_i(L_i) \rangle\rangle$. We have a short exact sequence

$$1 \to E_1 \to \hat{G} \xrightarrow{\Theta|_{\hat{G}}} T \to 1. \tag{3.12}$$

Clearly, $\ker\Phi \subset E_1$. Therefore, setting $E := E_1/\ker\Phi$, we obtain the following commutative diagram

$$
\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & & \\
1 \arrow[r] & \ker\,\Phi \arrow[r, "="] \arrow[d] & \ker\,\Phi \arrow[r] \arrow[d] & 1 \arrow[d] & \\
1 \arrow[r] & E_1 \arrow[r] \arrow[d] & \hat{G} \arrow[r, "\Theta|_{\hat{G}}"] \arrow[d, "\Phi"'] & T \arrow[r] \arrow[d, "="'] & 1 \\
1 \arrow[r] & E \arrow[r] \arrow[d] & \tilde{G}/\tilde{G}' \arrow[r, "\theta"] \arrow[d] & T \arrow[r] \arrow[d] & 1 \\
 & 1 & 1 & 1 &
\end{tikzcd}
\tag{3.13}
$$

where θ is the morphism induced by $\Theta|_{\hat{G}}$ which makes the diagram commutative.

We then claim that the lower row of the diagram (3.13) is the short exact sequence we are looking for: exactness follows from an easy diagram chase; the finiteness of E follows from that of E_1; eventually, each $\mathbf{T}_i/\langle\langle q_i(L_i)\rangle\rangle$ is an orbifold surface group, because $q_i(L_i)$ consists of finite order elements (see e.g. [BCGP09, Lem. 4.7]), so that T is a finite index subgroup in a product of orbifold surface groups. This concludes the proof of Proposition 3.5.

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