# Some classical formulæ for curves and surfaces

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**Abstract.** These notes have been taken on the occasion of the seminar *Degenerazioni e enumerazione di curve su una superficie* run at Roma Tor Vergata 2015–2017. THIS IS ONLY A PRELIMINARY VERSION, still largely to be completed; in particular many references shall be added.

# Contents

### Introduction

While only the topics in Section 2 have been discussed during the seminar, I have decided to include much more material in these notes for further reference.

### Notation

The polar of a hypersurface $X = V(F)$ with respect to a point $P$ is denoted by $\mathrm{D}_P X = V(\mathrm{D}_P F)$.

We write $p_a$ and $p_g$ respectively for the arithmetic and geometric genera of a projective variety.

For a plane curve $C$, we write $\delta$ and $\kappa$ respectively of the number of its ordinary double points and ordinary cuspidal double points, i.e. its $A_1$ and $A_2$ singularities. Notation $\check{\delta}$, $\check{\kappa}$, etc.

Ordinary singularities are intended from the point of view of projective duality.

# 1 – Background material

## 1.1 – Projective duality

In this text we are chiefly concerned with the enumeration of hyperplanes with a prescribed tangency pattern with a smooth surface in $\mathbf{P}^3$. To get started, let us recall some general results in the framework of projective duality that we will use repeatedly in the course of the text. For a more complete introduction, see [28, Chap. 1].

**(1.1) Definition.** *Let $X$ be a variety in $\mathbf{P}^N$. The* dual variety *of $X$ is the Zariski closure $X^\vee$ in the dual projective space $\check{\mathbf{P}}^N$ of the set of points $\varpi \in \check{\mathbf{P}}^N$ such that the hyperplane $\varpi^\perp \subset \mathbf{P}^N$ is tangent to $X$ at one of its smooth points.*

A central result is the following *reflexivity theorem.* See [17, Example 16.20] for an algebraic proof, or [15, Chap. 1] for a proof using symplectic geometry.

**(1.2) Theorem.** *Let $X$ be a variety in $\mathbf{P}^N$. One has $(X^\vee)^\vee = X$. More precisely, let $p$ and $\varpi$ be smooth points of $X$ and $X^\vee$ respectively: The hyperplane $\varpi^\perp \subset \mathbf{P}^N$ is tangent to $X$ at $p$ if and only if the hyperplane $p^\perp \subset \check{\mathbf{P}}^N$ is tangent to $X^\vee$ at $\varpi$.*

This may be refined in the following way. For a proof, see [21], or [29, §14.1].

**(1.3) Theorem.** *Suppose $X$ is smooth and its dual is a hypersurface. A point $\varpi \in X^\vee$ is smooth if and only if the hyperplane $\varpi \subset \mathbf{P}^N$ is tangent to $X$ in a unique point $p$, and $\varpi^\perp \cap X$ has an ordinary double point at $p$.*

The general philosophy is that the singularity of the dual in a point $\varpi$ reflects the tangency scheme of the hyperplane $\varpi^\perp$ with $X$. I shall loosely refer to any result incarnating this philosophy as a *biduality statement.* For instance, when $X$ itself is a smooth hypersurface one has the following result, see [9, Prop. 11.24].

**(1.4) Theorem.** *Let $X$ be a smooth hypersurface in $\mathbf{P}^N$, and consider $\varpi \in X^\vee$ such that the singular locus of $\varpi^\perp \cap X$ consists of finitely many points $p_1, \ldots, p_r$. Then the tangent cone to $X^\vee$ at $\varpi$ is the union of all hyperplanes $p_1^\perp, \ldots, p_r^\perp$, each counted with multiplicity $\mu(X, \varpi, p_i)$, the Milnor number of the singularity of $\varpi^\perp \cap X$ at $p_i$.*

Let us state one more general result before we move on to the more specific study of duality for hypersurfaces. For the proof, see [28, Tm. 1.21]. Put simply it says that linear projections and sections are operations dual to each other. Recall that by elementary linear algebra, if $W$ is a linear subspace of a vector space $V$, then the transpose of the projection $V \to V/W$ canonically identifies with the injection $W^\perp \hookrightarrow V^\vee$. We shall use the notation $\mathbf{P}(V)/\mathbf{P}(W)$ for $\mathbf{P}(V/W)$.

**(1.5) Proposition.** *Let $X$ be a variety in $\mathbf{P}^N$, and $\Lambda$ be a general linear subspace of codimension $m+1$ in $\mathbf{P}^N$. We consider the projection from $\Lambda$,*

$$\pi_\Lambda : \mathbf{P}^N \dashrightarrow \mathbf{P}^N/\Lambda \cong \mathbf{P}^m.$$

i) *If $m > \dim(X)$ and $\pi_\Lambda|_X$ is an isomorphism on its image, then $\pi_\Lambda(X)^\vee = \Lambda^\perp \cap X^\vee$.*
ii) *If $m = \dim(X)$, let $B \subset \mathbf{P}^m$ be the branch divisor of $\pi_\Lambda|_X$. One has $B^\vee = \Lambda^\perp \cap X^\vee$.*
iii) *If $m < \dim(X)$, consider $\tilde{X} \to X$ the blow-up of $X$ at $\Lambda \cap X$, and $\tilde{\pi}_\Lambda : \tilde{X} \to \mathbf{P}^m$ the morphism induced by $\pi_\Lambda$. Let $\Delta \subset \mathbf{P}^m$ be the discriminant of $\tilde{\pi}_\Lambda$. One has $\Delta^\vee = \Lambda^\perp \cap X^\vee$.*

## 1.2 – Projective duality for hypersurfaces

When $X$ is a hypersurface, much information on $X^\vee$ may be obtained by studying the defining equation of $X$ using polynomial calculus. We gather here a few facts relevant for us, and refer to [10, Chap. 1] for a much more complete treatment. A key technical tool is polarity; the main results are included in Appendix A.

In this subsection we always assume $X$ to be a hypersurface in $\mathbf{P}^N$, defined by the degree $n$ homogeneous polynomial $f$ in a system of homogeneous coordinates $(x_0 : \ldots : x_N)$.

**(1.6)** Let $p = (a_0 : \ldots : a_N)$ be a smooth point of $X$. By Euler's formula, the equation of the tangent hyperplane to $X$ at $p$ is

$$\partial_0 f(a) \cdot x_0 + \cdots + \partial_N f(a) \cdot x_N = \mathrm{D}_{(x_0:\ldots:x_N)} f(a) = 0.$$

In other words, the tangent hyperplane $\mathbf{T}_p X$ is the hyperplane of those points $q \in \mathbf{P}^N$ such that $\mathrm{D}_q f(p) = 0$.

**(1.7) Gauss map and degree of the dual.** Since there is a unique tangent hyperplane at each smooth point of $X$, there is a dominant rational map

$$\gamma_X : X \dashrightarrow X^\vee$$

called the *Gauss map*. If $X^\vee$ is a hypersurface, it follows from (1.3) that the Gauss map is birational.

It follows from (1.6) that the Gauss map of $X$ is the restriction to $X$ of the map defined on $\mathbf{P}^N$ by the $N$-dimensional linear system of all first polars of $X$. The latter is a subsystem of $|\mathcal{O}_{\mathbf{P}^N}(n-1)|$, and it is base point free if and only if $X$ is smooth.

In particular this gives a way to compute the degree of the dual of $X$, which is also called the *class* of $X$. If $X$ is smooth, then the degree of $X^\vee$ equals $n(n-1)^{N-1}$. If $X$ is singular, this has to be corrected by taking into account the base locus of the linear system of first polars.

**(1.8)** Suppose $X$ smooth for simplicity. Another consequence of (1.6) is that if $p \in \mathbf{P}^N$ is a general point, then $\mathrm{D}_p X \cap X$ is the locus of those points $q \in X$ such that $p$ sits on the tangent hyperplane $\mathbf{T}_q X$.

Therefore $\mathrm{D}_p X \cap X$ is the ramification divisor of the projection $\pi_p : X \to \mathbf{P}^{N-1}$. It is called the *apparent boundary* of $X$ as seen from the point $p$. The cone projecting $\mathrm{D}_p X \cap X$ from $p$ on the other hand is called the *circumscribed cone.*

**(1.9)** Still assuming that $X$ is smooth, the Hessian hypersurface $\mathrm{Hess}(X)$ cuts out on $X$ the locus of points $p \in X$ such that the tangent section $\mathbf{T}_p X \cap X$ has a singularity at $p$ worse than an ordinary double point. The intersection $X \cap \mathrm{Hess}(X)$ is the ramification divisor of the Gauss map $\gamma_X : X \to X^\vee$.

## 1.3 – Plücker formulae for plane curves

As a first and fundamental application of the previous results, let us explain the classical Plücker formulae for plane curves.

**(1.10)** Let $C \subset \mathbf{P}^2$ be a plane curve. It has a finite number of bitangent lines (i.e., lines tangent to $C$ in two distinct points) and inflection points, which give rise respectively to nodes and cusp on the dual curve $C^\vee \subset \check{\mathbf{P}}^2$: this can be seen for instance with Theorem (1.4). In general there are no lines with a deeper tangent scheme with $C$, so that nodes and cusps are the only singularities of $C^\vee$.

The reflexivity theorem (1.2) suggests that one could look for formulae linking numerical characters of $C$ and $C^\vee$ in a symmetric fashion. In this perspective one should allow $C$ to have nodes and cusps.

**(1.11) Plückerian characters of a plane curve.** We consider a plane curve $C \subset \mathbf{P}^2$ such that both $C$ and its dual $C^\vee \subset \check{\mathbf{P}}^2$ have only ordinary double points (nodes) and ordinary cusps as singularities. We call:
— $n$ the degree of $C$;
— $\check{n}$ the degree of $C^\vee$;
— $\beta$ the number of bitangents of $C$, which is also the number of nodes of $C^\vee$;
— $\iota$ the number of flexes of $C$, which is also the number of cusps of $C^\vee$;
— $\delta$ the number of nodes of $C$, which is also the number of bitangents of $C^\vee$;
— $\kappa$ the number of cusps of $C$, which is also the number of flexes of $C^\vee$.

**(1.12) Plücker formulae for plane curves.** One has the following relations for a curve $C$ as in (1.11).

$$\begin{aligned} \check{n} &= n(n-1) - 2\delta - 3\kappa; & \qquad n &= \check{n}(\check{n}-1) - 2\beta - 3\iota; \\ \iota &= 3n(n-2) - 6\delta - 8\kappa; & \qquad \kappa &= 3\check{n}(\check{n}-2) - 6\beta - 8\iota, \end{aligned}$$

We sketch the proof, as it is emblematic of our approach in this text, and refer to [14] for a detailed and careful implementation.

One first computes the class of a plane curve with nodes and cusps, following (1.7); this gives the two relations on the first line by applying it to both $C$ and $C^\vee$. Let $p \in \mathbf{P}^2$ be a general point. It follows from Theorem (A.8) that the first polar $\mathrm{D}_p C$ passes through all nodes and cusps of $C$, is smooth at these points, and has general tangent line at the nodes and tangent line equal to the reduced tangent cone of $C$ at the cusps. Hence $\mathrm{D}_p C$ intersects $C$ with multiplicity

$\geqslant 2$ at the nodes and $\geqslant 3$ at the cusps, and in fact these are the actual intersection multiplicities. The relation follows.

Then one computes the number of flexes of $C$ by considering its intersection with its Hessian (see (1.9)); this will give the two relations on the second row by applying it to $C^\vee$ as well. Again, one has to substract the contribution of $C \cap \mathrm{Hess}(C)$ corresponding to the singularities of $C$ and not to actual inflection points. If $q$ is a node of $C$, then the Hessian curve also has an ordinary double point at $q$, with tangent cone equal to that of $C$. If $q$ is a cusp of $C$, then the Hessian curve has a triple point with two local branches at $q$, one smooth with general tangent, and one cuspidal with tangent cone equal to that of $C$. It follows that $C$ intersects its Hessian with multiplicity $\geqslant 6$ at the nodes and $\geqslant 8$ at the cusps, and again these are the actual intersection multiplicities, hence the formula.

**(1.13) Smooth plane curves.** When the plane curve $C$ is smooth, one has $\delta = \kappa = 0$ and all the Plückerian characters of $C$ are immediately readable from the formulae in (1.12) except for $\beta$; for the latter they give

$$\beta = \tfrac{1}{2} n(n-2)(n-3)(n+3).$$

# 2 – Double curves of the dual to a surface in $\mathbf{P}^3$

## 2.1 – Local geometry of a surface in $\mathbf{P}^3$ and its dual

We now start our study of surfaces in $\mathbf{P}^3$. Here we give a first description of the dual to a smooth surface in $\mathbf{P}^3$, that will be refined later in Section 3.

We first take the occasion to introduce some necessary notions on the local geometry of a surface in $\mathbf{P}^3$.

**(2.1) Asymptotic tangent lines.** Tangent cone of the tangent section, but also two lines intersecting with multiplicity 3, hence $\mathbf{T}_p S \cap \mathrm{D}^2 S(p)$.

*(2.1.1) Flex tangent lines.* On peut le voir en coordonnées locales. Localement,

$$S \subset \mathbf{P}^3 : \; w^{d-1} z + w^{d-2}\big(xy + z\tilde{f}(x,y,z)\big) + \cdots = 0, \tag{2.1.2}$$

au point $(0:0:0:1)$, le plan tangent est $(z=0)$, on suppose qu'on n'est pas en un point parabolique[1], cône tangent de la section par $(z=0)$ est $(xy=0)$. On regarde un plan $\Pi$ contenant une des droites de ce cône tangent, disons $y=z=0$, donc

$$\Pi : \; ay + bz = 0 \;\rightsquigarrow z = \alpha y$$

($\alpha \neq 0$ si on ne veut pas le plan tangent lui-même). En substituant dans (2.1.2), il vient (je fais $w=1$)

$$\alpha y + xy + \alpha^2 y \tilde{f}(x,y,y) + \cdots = 0.$$

En intersectant avec la droite tangente $(y=0)$, on trouve que le premier terme non nul est de degré $\geqslant 3$, comme il fallait démontrer. $\square$

*(2.1.3) Biduality statement for asymptotic tangents.*

**(2.2) Parabolic points.**

[1] c'est l'hypothèse ubiquitaire chez les géomètres projectifs différentiels, typiquement non satisfaite pour les surfaces développables.

**(2.3) Codimension 1 singular loci of the dual.**

## 2.2 – Degrees of the double curves

*Formules apprises dans Gallarati* Elles sont démontrées dans [20, VII], avec rigueur et des conditions précises de validité. Je ne suis pas certain qu'à présent ma preuve soit beaucoup moins rigoureuse.

Let $S$ be a smooth surface of degree $n$ in $\mathbf{P}^3$. For any point $P \in \mathbf{P}^3$, we let $\Gamma_P$ be the apparent boundary of $S$ from $P$ (viz. the curve $S \cap \mathrm{D}_P S$), and consider the plane curve $\gamma_P \subset \mathbf{P}^2$ projecting $\Gamma_P$ from the point $P$. Harmoniser les notations.

**(2.4) Plückerian characters of the branch curve.** We let $N$ be the degree of the branch curve $B \subset \mathbf{P}^2$, $\check{N}$ be its class, and $\delta$, $\kappa$, $\beta$, $\iota$ be its other Plückerian characters as in (1.12). These numbers may all be expressed as a polynomial in the degree of $S$:
— $N = n(n-1)$ ;
— $\check{N} = n(n-1)^2$ ;
— $\delta = \frac{1}{2}n(n-1)(n-2)(n-3)$ ;
— $\kappa = n(n-1)(n-2)$ ;
— $\iota = 4n(n-1)(n-2)$ ;
— $\beta = \frac{1}{2}n(n-1)(n-2)(n^3-n^2+n-12)$.

**(2.5)** A key observation is that on the one hand $B$ is the birational projection from $p$ of the circumscribed curve $\mathcal{C}_p S$, and on the other hand it is the plane dual to the section of $S^\vee$ by the plane $p^\perp \subset \check{\mathbf{P}}^3$.

This observation is enough to prove all formulae in (2.4). Indeed it implies that i) the degree and class of $B$ equal the degrees of $\mathcal{C}_p S$ and $S^\vee$ respectively, which gives the formulae for $N$ and $\check{N}$ (see (2.7) for details), and ii) that the geometric genus of $B$ equals that of $\mathcal{C}_p S$, which is easy to compute since the latter is a complete intersection. We can then derive all remaining characters using the Plücker formulae together with the genus formula for plane curves. This is carried out in a more general context in Subsection 5.3.

Moreover, the relation of $B$ with the dual surface $S^\vee$ provides the following corollary.

**(2.6) Corollary.** *The numbers of binodal and cuspidal curves in a general net of hyperplane sections of $S$ are respectively*

$$\tfrac{1}{2}n(n-1)(n-2)(n^3-n^2+n-12) \quad \textit{and} \quad 4n(n-1)(n-2).$$

*These numbers are the degrees of the ordinary and cuspidal double curves of $S^\vee$ respectively.*

**(2.7)** Besides the quick proof given in (2.5), all characters in (2.4) but $\beta$ (although, see Remark (2.8)) may be computed directly, and it is instructive to do so.

*(2.7.1) Degree of $B$.* We have seen that it equals the degree of the circumscribed curve $\mathcal{C}_p S = S \cap \mathrm{D}_p S$, that is $\deg(S) \cdot \deg(\mathrm{D}_p S)$.

*(2.7.2) Class of $B$.* We have seen that it equals the degree of $S^\vee$. By (1.7), this is $\deg(S) \cdot \deg(\mathrm{D}_p S)^2$.

*(2.7.3) Nodes of $B$.* The points of $\mathcal{C}_pS$ are those points $q \in S$ such that the line $\langle p, q\rangle$ is tangent to $S$. The nodes of $B$ correspond to unordered pairs of distinct points $q, q' \in \mathcal{C}_pS$ such that $p, q, q'$ are aligned, hence to lines passing through $p$ that are bitangent to $S$. This number may be computed in a natural way, using resultants and elimination theory; this is explained in ref avec Laurent [3].

*(2.7.4) Flexes.* Since $B$ is the branch curve of the projection of $S$ from $p$, its flexes correspond to hyperplane sections of $S$ with a cusp.

This may be seen with the following local computation. Let $q$ be a flex point of $B$, and $q' \in \mathcal{C}_pS$ be the double point in its pull-back to $S$. Locally at $q'$, $S$ may be described as the hypersurface $z^2 = f(x, y)$ in affine coordinates $(x, y, z)$, such that the projection from $p$ is $(x, y, z) \mapsto (x, y)$ and $f(x, y) = 0$ is an equation of $B$ locally at $q$. Assume that the tangent to $B$ at $q$ is $y = 0$. Then the section of $S$ by its tangent plane at $q'$ is the hypersurface $z^2 = f(x, 0)$ in affine coordinates $(x, z)$, and since $q$ is a flex of $B$, $f(x, 0)$ vanishes to the third order at 0, hence the curve $z^2 = f(x, 0)$ has a cusp at the origin, as we wanted to show. The converse is proved in the same fashion.

It follows that the number of flexes of $B$ is the number of hyperplanes passing through $p$ and tangent to $S$ at one of its parabolic points. This is the number of intersection points of $\mathcal{C}_pS$ with the parabolic curve, and that is $\deg(S) \cdot \deg(\mathrm{D}_pS) \cdot \deg(\mathrm{Hess}(S))$.

*(2.7.5) Cusps.* The cusps of $B$ correspond to the points $q \in \mathcal{C}_pS$ such that the line $\langle p, q\rangle$ is tangent to $\mathcal{C}_pS$ at $q$.

Again, this may be seen with a local computation. Assume that $\mathcal{C}_pS$ is described parametrically by $t \mapsto (t, t^2, t^3)$ in affine coordinates centered at $q$. Then projecting from a point on the tangent line amounts to forgetting the first coordinate, and one sees that this gives a cuspidal image. The converse is proved in the same fashion.

In turn, by Lemma (2.9) the points $q \in \mathcal{C}_pS$ such that the line $\langle p, q\rangle$ is tangent to $\mathcal{C}_pS$ at $q$ are the intersection points of $\mathcal{C}_pS$ with the second polar $\mathrm{D}_{p^2}S$. Their number is therefore $\deg(S) \cdot \deg(\mathrm{D}_pS) \cdot \deg(\mathrm{D}_{p^2}S)$.

**(2.8) Remark.** In [27, §605–607], Salmon gives the construction of a polynomial defined from the equation of $S$ by rather elaborate elimination theory, and which cuts out on $S$ the locus of tangency points of all bitangent planes of $S$. As a by-product, this gives the number of bitangents of $B$ without resorting to the Plücker formulae. This is studied in details in [3].

**(2.9) Lemma.** *The asymptotic tangent lines passing through $p$ are tangent to the circumscribed curve $S \cap \mathrm{D}_pS$.*

I shall give three different proofs of this result, as I believe that all three of them are interesting with different respects.

*(2.9.1) Proof by a local computation* (cf. [6, Prop. 3.9]). $S \subset \mathbf{P}^3$ projetée sur $\mathbf{P}^2$ depuis $p \notin S$, je dis que la courbe de branchement $B$ a des cusps aux points correspondants aux droites par $p$ asymptotiquement tangentes à $S$.

On peut supposer projeter depuis $p = (0 : 0 : 1 : 0)$ sur $z = 0$, ce qui correspond à $(x, y, z) \mapsto (x, y)$ en affine, une surface passant par l'origine $q = (0 : 0 : 0 : 1)$, soit le point $(0, 0, 0)$ en affine. On suppose que la droite $\langle p, q\rangle = (x = y = 0)$ est asymptotiquement tangente à $S$. On peut supposer librement que le plan tangent à $S$ en $q$ est $(y = 0)$. Alors l'équations locales de $S$ est (en affine $(x, y, z)$)

$$F = y + f_2(x, y, z) + O(3) = 0, \quad \text{avec} \quad f_2(x, 0, z) = x(ax + bz).$$

La polaire par rapport à $p$ est simplement

$$\partial_z F = \partial_z f_2 + O(2),$$

et il s'agit de voir que la droite $(y = z = 0)$ lui est tangente, autrement dit que la partie linéaire de l'équation ci-dessus s'annule le long de $x = y = 0$. C'est bien le cas, puisque

$$f_2(x, y, z) = x(ax + bz) + yg(x, z)$$

et donc

$$\partial_z f_2 = bx + y\partial_z g.$$

$\square$

*(2.9.2) Proof by abstract nonsense using polarity.* Let $q \in S \cap \mathrm{D}_p S$. The line $\langle p, q\rangle$ is an asymptotic tangent to $S$ if and only if $q \in \mathrm{D}_{p^2} S$, equivalently $p \in \mathrm{D}_{q^{d-2}} S$ by polar symmetry. We already know that $\langle p, q\rangle$ is tangent to $S$, i.e., $p \in \mathbf{T}_q S = \mathrm{D}_{q^{d-1}} S$, so it suffices to prove that $p \in \mathbf{T}_q(\mathrm{D}_p S) = \mathrm{D}_{q^{d-2}p} S = \mathrm{D}_p(\mathrm{D}_{q^{d-2}} S)$. This holds by Euler formula, since by assumption $p \in \mathrm{D}_{q^{d-2}} S$. $\square$

*(2.9.3) Proof by biduality.* The condition that $q \in S \cap \mathrm{D}_p S \cap \mathrm{D}_{p^2} S$ is equivalent to the line $\langle p, q\rangle$ having intersection multiplicity 3 at $q$ with $S$. Hence by biduality, also the line $\langle p, q\rangle^\perp$ has intersection multiplicity 3 at $q^\perp$ with $S^\vee$. In other words, the orthogonal of an asymptotic tangent line is an asymptotic tangent line to the dual surface; we provide two direct proofs of this fact, one by considering the differential of the Gauss map in Proposition (4.2), and the other purely algebraic in [3, `p:vanishing-2nd-polar`].

The curve $B_p$ is the plane dual to the section of $S^\vee$ by the hyperplane $p^\perp$. Since the latter contains the asymptotic tangent $\langle p, q\rangle^\perp$, the hyperplane section $S^\vee \cap p^\perp$ has a flex along the line $\langle p, q\rangle^\perp$, see ??, and correspondingly its dual $B_p$ has a cusp at the image of $q$ under projection from $p$. $\square$

# 3 – Zero-dimensional strata of the dual surface

In this section we study the zero-dimensional strata of the dual of a smooth surface $S$ in $\mathbf{P}^3$, and in particular compute the number of planes triply tangent to $S$. The general strategy is to look at the cuspidal edges of the polar cone of $S^\vee$ with respect to a general point of $\check{\mathbf{P}}^3$.

## 3.1 – Local models

In this preliminary subsection, we build explicit local models for the dual of a smooth surface at zero-dimensional strata of its Whitney stratification.

**(3.1) Swallowtails.**
*(3.1.1) Semi-universal deformation of the tacnode.*

$$y^2 = x^4 + ax^2 + bx + x.$$

The equation of the discriminant locus is

$$\mathrm{Disc}(x^4 + ax^2 + bx + x) = 256c^3 - 128a^2c^2 + 144ab^2c - 27b^4 + 16a^4c - 4a^3b^2. \tag{3.1.2}$$

It may alternatively be obtained from the parametric description ($x^4 + ax^2 + bx + x = (x - u)^2(x - v)(x + 2u + v)$)

$$\begin{cases} a = -3u^2 - 2uv - v^2 \\ b = 2u^3 + 4u^2v + 2uv^2 \\ c = -2u^3v - u^2v^2 \end{cases}$$

by eliminating $u$ and $v$.
*(3.1.3) Parabolic curve.* This time the only way I know to get the equation is to use the parametric description $x^4 + ax^2 + bx + x = (x - u)^3(x + 3u)$

$$\begin{cases} a = -6u^2 \\ b = 8u^3 \\ c = -3u^4 \end{cases}$$

It gives the ideal of equations

$$(a^2 + 12c, 8a^3 + 27b^2).$$

*(3.1.4) Node couple curve.* $x^4 + ax^2 + bx + x = (x - u)^2(x + u)^2$

$$\begin{cases} a = -2u^2 \\ b = 0 \\ c = u^4 \end{cases}$$

gives the ideal of equations

$$(b, a^2 - 4c).$$

*(3.1.5) Intersection multiplicity of the two curves.* One finds that the tangent cone of the intersection is defined by $(b, c, a^2)$ , so this multiplicity seems to be 2.

**(3.2) Points of type $\gamma$.** We look at the product of the semi-universal deformations of a node and a cusp respectively, given by

$$y^2 = x^2 + a \qquad \text{and} \qquad y^2 = x^3 + bx + c.$$

*(3.2.1) Discriminant.* The discriminant is merely the product of the two discriminants, i.e.,

$$\mathrm{Disc}(x^2 + a) \cdot \mathrm{Disc}(x^3 + bx + c) = a \cdot (4b^3 + 27c^2).$$

*(3.2.2) Parabolic curve.* This is the curve parametrizing deformations such that the node smoothes and the cusp is maintained, which has equations

$$b = c = 0.$$

*(3.2.3) Node couple curve.* This is the curve parametrizing deformations such that the node is maintained and the cusp deforms to a node, which has equations

$$a = 4b^3 + 27c^2 = 0.$$

## 3.2 – Polar cones and their cuspidal edges

We set up the theory for surfaces in $\mathbf{P}^3$ with ordinary dual singularities. We shall apply it to the case of the dual to a surface in the next subsection.

**(3.3) Definition.** *Let $X \subset \mathbf{P}^3$ be a surface, and $p \in \mathbf{P}^3$ be a point off $X$. The* polar cone $\mathcal{D}_p X$ *of $X$ with respect to $p$ is the cone projecting from $p$ over the curve $X \cap \mathrm{D}_p X$.*

**(3.4) Polar cones of a surface with ordinary dual singularities.** Let us assume the surface $X \subset \mathbf{P}^3$ to have a curve $B$ of ordinary double points, a curve $C$ of simple cuspidal double points, and no other singularities (except from those arising from the singularities of the curves $B$ and $C$).

A local computation shows that the polar hypersurface $\mathrm{D}_p X$ contains simply the curves $B$ and $C$, with tangent cone equal to that of $X$ at the generic point of $C$. It follows that the polar cone splits as the sum of three cones, namely the two cones over $B$ and $C$ with multiplicities 2 and 3 respectively, and the *circumscribed cone* $\mathcal{T}_p X$, i.e., the closure of the set of all lines in $\mathbf{P}^3$ through $p$, and tangent to $X$ at some smooth point. We denote by $A$ the curve $(X \cap \mathcal{T}_p X)_{\mathrm{red}}$, which we call the *circumscribed curve.*

We call $a$ the degree of the circumscribed cone, and $b$ and $c$ the degrees of $B$ and $C$ respectively. The degree of the polar cone $\mathcal{D}_p X$ reads

$$n(n-1) = a + 2b + 3c. \tag{3.4.1}$$

**(3.5) Intersection of the polar cone with the second polar.** Now we list the various kinds of points in the intersection $X \cap \mathrm{D}_p X \cap \mathrm{D}_{p^2} X$. The lines joining $p$ to these points are the cuspidal edges of the polar cone $\mathcal{D}_p X$, since for $q \in X \cap \mathrm{D}_p X \cap \mathrm{D}_{p^2} X$ the line $\langle p, q \rangle$ intersects $X$ with multiplicity 3 at $q$ by (A.6).

We let:

— $\kappa$ the number of cuspidal edges of the circumscribed cone $\mathcal{T}_p X$;
— $\rho$ the number of intersection points between the circumscribed curve $(X \cap \mathcal{T}_p X)_{\mathrm{red}}$ (equivalently, the curve residual to $2B + 3C$ in $X \cap \mathrm{D}_p X$) and the ordinary double curve $B$;
— $\sigma$ the number of intersection points between the circumscribed curve and the cuspidal double curve $C$.
— $\beta$ the number of intersection points between $B$ and $C$ at which $C$ is cuspidal;
— $\gamma$ the number of intersection points between the double curves $B$ and $C$ at which $B$ is cuspidal;
— $t$ the number of triples points of the ordinary double curve $B$ (note that these are also triple points of the surface $X$).

We say that a curve $Y$ is cuspidal in a point $p$ if there is a point over $p$ where the differential of the normalization map $\bar{Y} \to Y$ vanishes; this is called a stationary point in classical terminology.

Some more notation is introduced later in (4.12) which is not relevant for the present considerations.

À noter : — points $\kappa$: ne correspondent pas aux plans par $p_0$ tangents en un point parabolique ; ces derniers donnent des cusps *à la duale* de $\pi_{p_0}(\mathrm{D}_{p_0} U \cap U)$, et donc des “plans tangents stationnaires” au cône circonscrit.

**(3.6)** Salmon gives the following formulae [27, §610], which describe separately the intersections with the second polar $\mathrm{D}_{p^2} X$ of the three components of the polar cone $\mathcal{D}_p X$:

(𓂀 𓏺) $$a(n-2) = \kappa + \rho + 2\sigma \ ;$$
(𓂀 𓏻) $$b(n-2) = \rho + 2\beta + 3\gamma + 3t \ ;$$
(𓂀 𓏼) $$c(n-2) = 2\sigma + 4\beta + \gamma.$$

He comments: « The reader can see without difficulty that the points indicated in these formulæ are included in the intersections of $[\mathrm{D}_{p^2} X$ with the three components of $\mathcal{D}_p X]$; but it is not so easy to see the reason for the numerical multipliers which are used in the formulæ. Although it is probably not impossible to account for these constants by a priori reasoning, I prefer to explain the method by which I was led to them inductively. » The latter method consists in considering the dual of a smooth cubic surface, for which all the quantities may be directly computed, and then to tune the coefficients to make the formulae work.

Following Salmon's example, I shall not give a rigorous proof of these formulae, although I believe that this should be feasible by a careful local computation, using for instance Theorem (A.8). Let me nevertheless give a few hints, in an attempt to render them plausible. Recall that the points in $\mathcal{D}_p X \cap \mathrm{D}_{p^2} X$ are those points $q$ such that the line $\langle p, q \rangle$ intersects $X$ with multiplicity at least 3 in $q$.

*(3.6.1)* The appearance of $\kappa$ in the first formula is tautological. If $q$ is a point of type $\rho$, then $X$ has two transverse local sheets at $q$, and the line $\langle p, q \rangle$ is an honest tangent to one of them for $q$ belongs to $X \cap \mathcal{T}_p X$. If $q$ is a point of type $\sigma$, then $X$ has only one, cuspidal, local sheet at $q$, and the line $\langle p, q \rangle$ is contained in its tangent cone at $q$, for $q$ belongs to $X \cap \mathcal{T}_p X$.

*(3.6.2)* Let us look at the second formula. As above, points of type $\rho$ are smooth points of $B$, which happen to sit on $\mathrm{D}_{p^2} X$; the latter is smooth and transverse to $B$ at such points. Points of type $t$ are triple for $X$, and it follows from Theorem (A.8) that $\mathrm{D}_{p^2} X$ is smooth and transverse to $B$ at such points; the multiplicity 3 in the formula follows from the multiplicity 3 of $B$ at these points.

For the swallowtail singularity (type $\beta$), I found a triple point with tangent cone a triple plane. I deduce that the second polar is smooth at such a point, with tangent cone the tangent cone of $X$. Hence it meets $B$ with multiplicity 2, being $B$ smooth, and $C$ with multiplicity 4, having $C$ a cusp.

For points $\gamma$, one should find that the second polar is transverse to $C$, but tangent (in the sense: contains the tangent cone) to $B$, hence the multiplicity 3.

Maybe could confirm this by the study of the intersection of the curves $B$ and $C$? (although I think this is only in the case of a dual surface; but should be the general case!)

*(3.6.3) Points of type $\rho$.* These are those points $q \in B$ such that the line $\langle p, q \rangle$ is an honest tangent to one of the two smooth local sheets of $X$ at $q$, which we will call $X'_q$. The first polar $\mathrm{D}_p X$ is smooth at $q$ with the same tangent plane as $X'_q$, and its intersection with $X$ locally at $q$ consists of $B$ and the circumscribed curve $\mathcal{C}_p X$, the latter entirely contained in $X'_q$. The second polar $\mathrm{D}_{p^2} X$ is smooth at $q$ with general tangent plane, hence intersects both $B$ and $\mathcal{C}_p X$ transversely at $q$.

This may be justified by the following local computation. Let $q$ be the point $(0:0:0:1)$ in a homogeneous system of coordinates $(x:y:z:w)$, and assume $X$ is given locally at $q$ by the equation

$$f(x, y, z) = xy + f_3(x, y, z) + \cdots + f_n(x, y, z) = 0$$

in the affine coordinates $(x, y, z)$, where each $f_i$ is homogeneous of degree $i$. We may take $p = (1:0:0:0)$. Then

$$\mathrm{D}_p f = y + \partial_x f_3 + \cdots \qquad \text{and} \qquad \mathrm{D}_{p^2} f = \partial_x^2 f_3 + \cdots ,$$

hence the above description holds.

*(3.6.4) Points of type $\sigma$.* These are those points $q \in C$ such that the line $\langle p, q \rangle$ lies in the tangent cone of the unique, cuspidal, local sheet of $X$ at $q$. The first polar $\mathrm{D}_p X$ has a double point at $q$ with tangent cone a double plane, whose reduced intersection with the tangent cone of $X$ is the tangent line to $C$ at $q$. The circumscribed curve $\mathcal{C}_p X$ is therefore tangent to $C$ at $q$. The second polar $\mathrm{D}_{p^2} X$ is smooth at $q$ with tangent plane equal to the reduced tangent cone of $\mathrm{D}_p X$, hence

it is tangent to both curves $C$ and $\mathcal{C}_p X$, which accounts for the multiplicities 2 in (𓂀 I)and (𓂀 III).

In this case the local computation is the following. We keep $q = (0:0:0:1)$ as in (3.6.3) above, but consider the local equation

$$f(x, y, z) = y^2 + x^3 + f_4(x, y, z) + \cdots + f_n(x, y, z) = 0.$$

We may again take $p = (1:0:0:0)$, and then

$$\mathrm{D}_p f = 3x^2 + \cdots \qquad \text{and} \qquad \mathrm{D}_{p^2} f = 6x + \cdots .$$

Note that the tangent cone of the curve $f = \mathrm{D}_p f = 0$ is the quadruple line defined by $x^2 = y^2 = 0$; this curve indeed consists of $C$ with multiplicity 3 and $\mathcal{C}_p X$.

*(3.6.5) Points of type $t$.* These are the points at which $X$ has three transverse smooth local sheets. The point $p$ is general with respect to these points, so we may directly apply Theorem (A.8). Let $q$ be such a point. The first polar $\mathrm{D}_p X$ has an ordinary double point at $q$, and indeed this is necessary since $\mathrm{D}_p X$ contains the curve $B$ which has a space triple point at $q$. The second polar $\mathrm{D}_{p^2} X$ is smooth with general tangent plane, hence intersects $B$ transversely at $q$: this gives the multiplicity 3 as $q$ is a triple point of $B$.

*(3.6.6) Points of type $\beta$.* At these points $X$ has a swallowtail singularity; in particular they are triple points of $X$ (hence lie on $\mathrm{D}_{p^2} X$), at which the tangent cone of $X$ is a triple plane. refer to appropriate statement. The point $p$ is general with respect to these points, so we may directly apply Theorem (A.8). Let $q$ be a point of type $\beta$. The first polar $\mathrm{D}_p X$ has a double point at $q$ with tangent cone a double plane supported on $(\mathrm{TC}_q X)_{\mathrm{red}}$, and the second polar $\mathrm{D}_{p^2} X$ is smooth with tangent plane equal to $(\mathrm{TC}_q X)_{\mathrm{red}}$.

The ordinary double curve $B$ is smooth at points of type $\beta$, and the second polar is tangent to it. This accounts for the multiplicity 2 in (𓂀 I). For the same reason, since the cuspidal double curve $C$ has cusps at points of type $\beta$, it intersects with multiplicity $\geqslant 3$ at $q$ with $\mathrm{D}_{p^2} X$. However, I claim that $(\mathrm{TC}_q X)_{\mathrm{red}}$ is in fact the osculating plane of $C$ at $q$, so that the intersection multiplicity of $C$ with $\mathrm{D}_{p^2} X$ is indeed 4.

To compute this intersection multiplicity (and hence prove the claim), we work on the normalization of $X$. By biduality, we can think of this normalization as being given locally at $q$ by the Gauss map $S \to S^\vee = X$ of a smooth surface $S \subset \mathbf{P}^3$ at a point $q'$ such that $S \cap \mathbf{T}_{q'} S$ has a tacnode at $q'$. The preimage of $C$ is the parabolic curve of $S$, and we want to compute its intersection multiplicity with the tangent plane of $S$. The latter cuts out on $S$ a curve with a tacnode at $q'$ with tangent cone at $q'$ supported on the double flex tangent line of $S$ at $q'$, and the latter is by Lemma (4.4) the tangent line of the parabolic curve; this proves that $S \cap \mathbf{T}_{q'} S$ and $S \cap \mathrm{Hess}(S)$ intersect with multiplicity 4 at $q'$, as required.

*(3.6.7) Points of type $\gamma$.* These are triple points $q$ of $X$, at which $X$ has two transverse local sheets, one smooth: $X_q^{\mathrm{s}}$, and the other cuspidal: $X_q^{\mathrm{c}}$. The first polar $\mathrm{D}_p X$ has a double point at $q$, with tangent cone the sum of $(\mathrm{TC}_q X_q^{\mathrm{c}})_{\mathrm{red}}$ and a general plane containing the line $(\mathrm{TC}_q B)_{\mathrm{red}}$. The second polar $\mathrm{D}_{p^2} X$ is smooth at $q$, with tangent plane a general plane containing the line $(\mathrm{TC}_q B)_{\mathrm{red}}$. Therefore it intersects $B$, which is cuspidal at $q$, with multiplicity 3, and it is transverse to $C$, which is smooth at $q$. This amounts for the multiplicities 3 and 1 in (𓂀 \\) and (𓂀 III) respectively.

To clarify the situation, let us carry out the local computations proving the above assertions. We let $q = (0:0:0:1)$ and consider the two sheets $X_q^s$ defined respectively by the equations

$$z + f_2(x, y, z) + \cdots = 0 \qquad \text{and} \qquad y^2 + x^3 + f_4(x, y, z) + \cdots = 0.$$

The curve $B$ is the intersection of these two sheets, it has a cusp at $q$ with tangent cone defined by $z = y^2 = 0$. The curve $C$ is the singular locus of $X_q^{\mathrm{c}}$, it is smooth at $q$ with tangent line defined by $y = x = 0$. We may take $p = (u_0 : u_1 : u_2 : 0)$. Then, by Theorem (A.8) the tangent cones of $\mathrm{D}_p X$ and $\mathrm{D}_{p^2} X$ are defined respectively by

$$\mathrm{D}_p(zy^2) = y(2u_1 z + u_2 y) \qquad \text{and} \qquad \mathrm{D}_p(zy^2) = 2u_1(u_1 z + 2u_2 y),$$

which proves our claims.

## 3.3 – Application to the case $X = S^\vee$

We now let $S$ be a smooth, degree $n$, surface in $\mathbf{P}^3$, with dual a hypersurface $X \subset \check{\mathbf{P}}^3$. We explore in this particular case the geometric landscape described above, using decoration with a duality symbol to record that $X = S^\vee$. For instance, we let:
— $\check{n} = n(n-1)^2$, the degree of $X$.

**(3.7)** The double curves $\check{B}$ and $\check{C}$ are the curves in $S^\vee$ parametrizing respectively the bitangent and parabolic tangent planes to $S$. The Guassian map of $S$ induces a $2:1$ map $\mathrm{Kn}_S \to \check{B}$ from the node-couple curve $\mathrm{Kn}_S \subset S$, and a birational map $\mathrm{He}_S \to \check{C}$ from the Hessian curve $\mathrm{He}_S \subset S$. The degrees of the two double curves are (see):
— $\check{b} = \frac{1}{2} n(n-1)(n-2)(n^3 - n^2 + n - 12)$ ;
— $\check{c} = 4n(n-1)(n-2)$.

**(3.8)** The point $\check{p} \in \check{\mathbf{P}}^3$ corresponds to a hyperplane $H_{\check{p}} = (\check{p})^\perp$ in $\mathbf{P}^3$. The circumscribed cone $\mathcal{T}_{\check{p}}(S^\vee)$ is the dual in $\check{\mathbf{P}}^3$ to the plane curve $H_p \cap S \subset \mathbf{P}^3$. This gives:
— $\check{a} = n(n-1)$.
The values of $\check{a}, \check{b}, \check{c}$ fit with formula (3.4.1).

It also follows from the above description of $\mathcal{T}_{\check{p}}(S^\vee)$ that the cuspidal edges of type $\check{\kappa}$ of the polar cone $\mathcal{D}_{\check{p}} S^\vee$ are the lines orthogonals to the osculating tangent lines of the plane curve $H_{\check{p}} \cap S$ (i.e., the orthogonals of the flex tangents of $H_{\check{p}} \cap S$). Thus
— $\check{\kappa} = 3n(n-2)$.

**(3.9)** The circumscribed curve $(S^\vee \cap \mathcal{T}_{\check{p}}(S^\vee))_{\mathrm{red}}$ parametrizes hyperplanes in $\mathbf{P}^3$ tangent to $S$ at some point lying on $H_{\check{p}}$. It follows that the points of type $\check{\rho}$ (resp. $\check{\sigma}$) correspond to hyperplanes that are tangent to $S$ at some point on $H_{\check{p}}$ and at some other point (resp. at some point on $H_{\check{p}}$ which is parabolic). Hence $\check{\rho}$ (resp. $\check{\sigma}$) is the degree of the intersection of $H_{\check{p}}$ with the couple-nodal curve $\mathrm{Kn}_S$ (resp. with the Hessian curve $\mathrm{He}_S$), and thus:
— $\check{\rho} = n(n-2)(n^3 - n^2 + n - 12)$ ;
— $\check{\sigma} = 4n(n-2)$.

**(3.10)** The points of type $\check{\gamma}$ correspond to planes $H$ tangent to $S$ at two points, one of which is parabolic; correspondingly the curve $H \cap S$ has one cusp and one node. Points of type $\check{\gamma}$ are honest points on $\check{C}$, which accounts for the cusp. On the other hand, as a general point of $\check{B}$ moves towards a point of type $\check{\gamma}$, one sees a family of hyperplane sections of $S$ with two nodes, one of which degenerates to a cusp; this accounts for the fact that points of type $\check{\gamma}$ are cusps of the ordinary double curve $\check{B}$. The number $\check{\gamma}$ shall be derived indirectly in (3.11) below.

The points of type $\check{\beta}$ correspond to planes $H$ tangent to $S$ at one point, such that the section $H \cap S$ has a tacnode there. Indeed, points of type $\check{\beta}$ are cusps of $\check{C}$, so that as a general point of $\check{C}$ moves towards a point of type $\check{\beta}$, one must see a family of hyperplane sections of $S$ with a cusp which degenerates to a tacnode. Going to a point of type $\check{\beta}$ along $\check{B}$, one merely sees two nodes tending to a common limit point; this is harmless for $\check{B}$, and indeed points $\check{\beta}$ are smooth points of $\check{B}$.

Those points $q \in S$ such that $\mathbf{T}_q S \cap S$ has a tacnode at $q$ are the intersection points of the parabolic curve $S \cap \mathrm{Hess}(S)$ with the flecnodal locus. The latter is cut out on $S$ by a homogeneous polynomial of degree $11n - 24$, Référence avec Laurent [3]. but one has to take into account the famous fact that the two surfaces defined by the Hessian and the flecnodal polynomials are everywhere tangent. Sadly, I abandoned the idea of including a proof of this fact here. Give reference however! Eventually, one thus finds:
— $\check{\beta} = 2n(n-2)(11n-24)$.

Eventually, the points of type $\check{t}$ correspond to the famous hyplerplanes tangent to $S$ at three points. Their number we shall derive using formula (𓂀 𓏭).

**(3.11) Intersection of the two double curves.** We now analyze the intersection of the two curves $\check{B}$ and $\check{C}$ on $X = S^{\vee}$ in terms of the intersection of the node-couple $\mathrm{Kn}_S$ and Hessian $\mathrm{He}_S$ curves on $S$. Recall that we have two maps $\mathrm{Kn}_S \to \check{B}$ and $\mathrm{He}_S \to \check{C}$, the former a double cover, and the latter birational.

To a point of type $\check{\gamma}$ there corresponds a section of $S$ with one node and one cusp, which gives two points on $\mathrm{Kn}_S$, only one of which lies on $\mathrm{He}_S$.

To a point of type $\check{\beta}$ there corresponds a point $q \in S$ such that $\mathbf{T}_q S \cap S$ has a tacnode at $q$. On the one hand, $q$ belongs to $K_S$ and is a ramification point of $K_S \to \check{B}$, as $\mathbf{T}_q S \cap S$ may be seen as a bi-nodal section of $S$ with the two nodes coalesced. This implies that $T_q \mathrm{Kn}_S$ is the kernel of the differential of the Gauss map of $S$ at $q$. On the other hand $q$ is a ramification point of $\mathrm{He}_S \to \check{C}$ as well, since $\check{C}$ has a cusp at $(\mathbf{T}_q S)^{\perp}$, and this implies that $T_q \mathrm{He}_S$ is the kernel of the differential of the Gauss map of $S$ at $q$ (see also Lemma (4.4)). The upshot is that the two curves $\mathrm{Kn}_S$ and $\mathrm{He}_S$ are tangent at $q$.

The intersection points of $\mathrm{Kn}_S$ and $\mathrm{He}_S$ all correspond to points of $\check{B} \cap \check{C}$ either of type $\check{\beta}$ or of type $\check{\gamma}$. We thus find that

$$(\mathrm{Kn}_S \cdot \mathrm{He}_S) = (n-2)(n^3 - n^2 + n - 12) \cdot 4(n-2) \cdot n = 2\beta + \gamma.$$

*(3.11.1)* The above relation together with the formula for $\check{\beta}$ in (3.10) yields:
— $\check{\gamma} = 4n(n-2)(n-3)(n^3 + 3n - 16)$.

**(3.12) Formulae (𓂀) and the number of tritangent planes.** First of all let me mention that with the quantities we have been able to find so far, formulae (𓂀 𓏤) and (𓂀 𓏤𓏤𓏤) are indeed verified (beware that also $n$ should be replaced by $\check{n} = n(n-1)^2$).

Formula (𓂀 𓏭) on the other hands yields the number of tritangent planes to a smooth surface of degree $n$ in $\mathbf{P}^3$:
— $t = \frac{1}{6} n(n-2)(n^7 - 4n^6 + 7n^5 - 45n^4 + 114n^3 - 111n^2 + 548n - 960)$.

```
a:=n*(n-1);
b:=1/2*n*(n-1)*kn;
c:=he*(n-1);
nn:=n*(n-1)^2;

kappa:=3*n*(n-2);
rho:=n*(n-2)*(n^3-n^2+n-12);
sigma:=4*n*(n-2);

a*(nn-2)-kappa-rho-2*sigma;
expand(%);
```

```
                                   0

beta:=2*n*(n-2)*(11*n-24);
gamm:=(n-2)*(n^3-n^2+n-12)*4*n*(n-2)-2*b;
                                                        3
                 gamm := 4 n (n - 2) (n - 3) (n  + 3 n - 16)

c*(nn-2)-2*sigma-4*beta-gamm;
expand(%);
                                   0

t:=1/3*(b*(nn-2)-rho-2*beta-3*gamm);
factor(expand(t));
               7      6      5       4       3       2
     n (n - 2) (n  - 4 n  + 7 n  - 45 n  + 114 n  - 111 n  + 548 n - 960)
     ------------------------------------------------------------------
                                  6
```

## 4 – Hessian and node-couple developables

Given a curve $Y$ traced on a surface $S \subset \mathbf{P}^3$, one may consider the developable surface $\Sigma_Y \subset \mathbf{P}^3$ corresponding to the family of planes tangent to $S$ at points of $Y$ (some describe it as the family of planes touching $S$ at points of $Y$). Let $\gamma_S : S \dashrightarrow S^\vee$ be the Gauss map of $S$. We are considering the family of planes parametrized by the curve $\gamma_S(Y) \subset \check{\mathbf{P}}^3$; the developable surface $\Sigma_Y$ is then its dual $(\gamma_S(Y))^\vee \subset \mathbf{P}^3$. In classical terms, the developable $\Sigma_Y$ is the surface enveloped by the family of tangent planes to $S$ at points of $Y$.

In this section we study this developable surface in the two cases when $Y$ is either the Hessian (a.k.a. parabolic) curve $H_S = S \cap \mathrm{Hess}(S)$ or the node-couple curve $K_S$. An important step in the computation of the numerical characters of the latter will be the consideration of the ordinary double edges of a general polar cone to $S^\vee$.

**(4.1) The involution on $\mathbf{P}(T_pS)$.** Let $p$ be a smooth, non-parabolic, point of $S$. There is a self-map $\iota_\gamma$ of the projective line $\mathbf{P}(T_pS)$, which is defined as follows. Consider a tangent direction $u \in \mathbf{P}(T_pS)$ (if one prefers, a point of $S$ infinitely near to $p$). The differential of the Gauss map sends $u$ to a direction $\gamma_S(u)$ in $\check{\mathbf{P}}^3$ tangent to $S^\vee$ at the point $(\mathbf{T}_pS)^\perp$, which spans a line $\langle\gamma_S(u)\rangle$ contained in $\mathbf{T}_{(\mathbf{T}_pS)^\perp}S^\vee$. Its orthogonal is a line contained in $\mathbf{T}_pS \subset \mathbf{P}^3$, hence defines a point $v \in \mathbf{P}(T_pS)$ which we take as the image of $u$ by $\iota_\gamma$.

This may be rephrased as follows. The tangent direction $\gamma_S(u)$ gives via the Gauss map a tangent plane to $S$ infinitely near to $\mathbf{T}_pS$. Then $v = \iota_\gamma(u)$ corresponds to the line obtained as the intersection of $\mathbf{T}_pS$ with this infinitely near tangent plane, see (B.5).

**(4.2) Proposition.** *The map $\iota_\gamma$ is an involution of $\mathbf{P}(T_pS)$, with fixed points the directions of the two flex tangents of $S$ at $p$.*

This may be seen as a consequence of biduality, which tells us that the two Gauss maps $\gamma_S$ and $\gamma_{S^\vee}$ give the identity when composed one with another.

*Proof.* We consider homogeneous coordinates $(x : y : z : w)$ in $\mathbf{P}^3$, such that $p = (0 : 0 : 0 : 1)$ and $\mathbf{T}_pS$ is the plane $z = 0$. We write the equation of $S$ in the form

$$f(x,y,z,w) = w^{n-1}z + w^{n-2}(ax^2 + 2rxy + by^2 + 2sxz + 2tyz + cz^2) + f_{\geqslant 3}(x,y,z,w) = 0,$$

where each monomial of $f_{\geqslant 3}$ has degree at most $n-3$ in $w$.

The Gauss map of $S$ sends the point $(x:y:z:w)$ to the transpose of

$$\begin{pmatrix} \partial_x f \\ \partial_y f \\ \partial_z f \\ \partial_w f \end{pmatrix} = \begin{pmatrix} w^{n-2}(2ax+2ry+2sz)+\cdots \\ w^{n-2}(2rx+2by+2tz)+\cdots \\ w^{n-1}+w^{n-2}(2sx+2ty+2cz)+\cdots \\ (n-1)w^{n-2}z+\cdots \end{pmatrix}.$$

We write it in the affine chart $w=1$; its differential at the origin (i.e., at $p$) is then

$$\begin{pmatrix} 2a\,dx+2r\,dy+2s\,dz \\ 2r\,dx+2b\,dy+2t\,dz \\ 2s\,dx+2t\,dy+2c\,dz \\ (n-1)\,dz \end{pmatrix}. \tag{4.2.1}$$

On the other hand, still in the affine chart $w=1$, $S$ is given by an implicit function $z=\varphi(x,y)$ with vanishing differential at the origin. We want to compose the differential at the origin of $(x,y)\mapsto(x,y,\varphi(x,y))$ with that of the Gauss map. This amounts to set $dz=0$ in (4.2.1).

The upshot is that the tangent plane to $S$ at the point infinitely near to $p$ in the direction $(u_0:u_1)$ is given by the equation

$$(2au_0+2ru_1)x+(2ru_0+2bu_1)y+(2su_0+2tu_1)z=0.$$

Therefore its intersection with $\mathbf{T}_pS$ is given by

$$(2au_0+2ru_1)x+(2ru_0+2bu_1)y=0,$$

hence $\iota_\gamma$ is the involution

$$\begin{pmatrix} -r & -b \\ a & r \end{pmatrix} \in \mathrm{PGL}_2(\mathbf{C}); \tag{4.2.2}$$

note that the determinant of the latter matrix is up to sign that of the quadratic form $ax^2+2rxy+by^2$ which defines the tangent cone at $p$ of $S\cap\mathbf{T}_pS$, and thus is invertible because $p$ is not parabolic.

Eventually, a direct computation shows that for

$$ax^2+2rxy+by^2=(\alpha x+\beta y)(\alpha' x+\beta' y),$$

${}^{\mathsf{T}}(-\beta,\alpha)$ and ${}^{\mathsf{T}}(-\beta',\alpha')$ are eigenvectors of the matrix in (4.2.2). □

To wit:

$$\begin{pmatrix} a & r \\ r & b \end{pmatrix} = \begin{pmatrix} \alpha\alpha' & \frac12(\alpha\beta'+\beta\alpha') \\ \frac12(\alpha\beta'+\beta\alpha') & \beta\beta' \end{pmatrix}$$

hence

$$\begin{aligned} \begin{pmatrix} -r & -b \\ a & r \end{pmatrix}\begin{pmatrix} -\beta \\ \alpha \end{pmatrix} &= \begin{pmatrix} -\frac12(\alpha\beta'+\beta\alpha') & -\beta\beta' \\ \alpha\alpha' & \frac12(\alpha\beta'+\beta\alpha') \end{pmatrix}\begin{pmatrix} -\beta \\ \alpha \end{pmatrix} \\ &= \begin{pmatrix} \frac12\alpha\beta\beta'+\frac12\beta^2\alpha'-\alpha\beta\beta' \\ -\alpha\beta\alpha'+\frac12\alpha^2\beta'+\frac12\alpha\beta\alpha' \end{pmatrix} \\ &= \begin{pmatrix} -\frac12\alpha\beta\beta'+\frac12\beta^2\alpha' \\ -\frac12\alpha\beta\alpha'+\frac12\alpha^2\beta' \end{pmatrix} = \tfrac12(\alpha\beta'-\beta\alpha')\begin{pmatrix} -\beta \\ \alpha \end{pmatrix}. \end{aligned}$$

## 4.1 – The Hessian developable surface

We study the family of planes in $\mathbf{P}^3$ tangent to $S$ at its parabolic points, and the associated developable surface in $\mathbf{P}^3$. A crucial point is to adapt the observations of (4.1) and (4.2) to the case when $p$ is a parabolic point.

**(4.3) Proposition.** *If $p$ is a parabolic point of $S$, the construction of $\iota_\gamma$ in (4.1) defines a contraction of $\mathbf{P}(T_pS)$ to its point corresponding to the double flex tangent line at $p$.*

By definition the point $p$ is parabolic if the Hessian of $S$ at $p$ is degenerate; it has rank 1 for general $p$ (unless $S^\vee$ is not a hypersurface) and we call *double flex tangent line* its isotropic cone.

The content of the statement is that the image of the differential of the Gauss map at a parabolic point is the orthogonal of the double flex tangent line. In addition, we shall see that the kernel of this differential is the double flex tangent line itself.

This is a biduality statement, the analog of (2.1.3) for parabolic points. The flex tangent is the line of maximum contact with $S$, and intersects $S$ with multiplicity 3. Its orthogonal is the line of maximum contact with $S^\vee$, namely the tangent line of the cuspidal edge, and intersects $S^\vee$ with multiplicity 4 (intersection multiplicity is 2 for a general line, 3 for a general line in the tangent cone, and 4 for the tangent of the cuspidal edge). I am surprised to see that the intersection multiplicity is not the same on both sides.

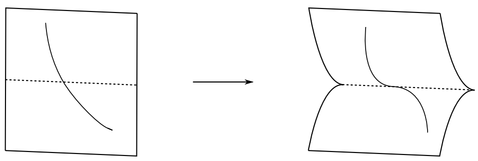

Figure 1: The Gauss map around the parabolic curve

*Proof.* This follows from the computations carried out in the proof of Proposition (4.2) noting that, in the notation introduced there, we may assume $a = r = 0$ since $p$ parabolic. The double flex tangent line is then defined by $u_1 = 0$. □

**(4.4) Lemma.** *Let $p$ be a parabolic point of $S$. The tangent line at $p$ to the curve $S \cap \mathrm{Hess}(S)$ coincides with the double flex tangent line of $S$ at $p$ if and only if the tangent section $S \cap \mathbf{T}_pS$ has a tacnode at $p$ (or a worse singularity).*

This tells us in particular that the cuspidal double curve of $S^\vee$, which is the image of $S \cap \mathrm{Hess}(S)$ by the Gauss map, has cusps at points corresponding to tangent planes cutting out a tacnodal curve on $S$. Moreover, for such points $p$ the plane $\mathbf{T}_pS$ (resp. $p^\perp = (\mathrm{TC}_{(\mathbf{T}_pS)^\perp} S^\vee)_{\mathrm{red}}$) is the osculating plane of the curve $S \cap \mathrm{Hess}(S)$ (resp. of the cuspidal double curve of $S^\vee$), as we have already observed in (3.6.7).

?! In general these points are the intersection points of $S \cap \mathrm{Hess}(S)$ with the *flecnodal* surface of $S$ [reference]. The latter cuts out on $S$ the locus of points $p$ such that there is a line intersecting $S$ with multiplicity 4 at $p$. For a general $p$ on the flecnodal locus, the tangent section of $S$ is a nodal curve, with one of its local branches wich has a flex; for a parabolic $p$ on the flecnodal locus, this degenerates to a tacnode.

*Proof of Lemma (4.4).* Let us consider the equation of $S$ given in the local form at $p$

$$f(x,y,z,w) = w^{n-1}z + w^{n-2}\left(\tfrac{1}{2}y^2 + axz + byz + \tfrac{1}{2}cz^2\right) + w^{n-3}f_3(x,y,z) + f_{\geqslant 4}(x,y,z,w) = 0,$$

in a system of homogeneous coordinates $(x:y:z:w)$ such that $p=(0:0:0:1)$ ($n$ is the degree of $S$, and each monomial of $f_{\geqslant 3}$ has degree at most $n-4$ in $w$). The tangent at $p$ to the Hessian curve is obtained by intersecting $\mathbf{T}_pS$, that is the plane $z=0$, with the tangent plane to the Hessian, which we find with formula (A.9.1).

In the affine chart $w=1$, we have

$$\begin{aligned} \partial_x f &= az + \partial_x f_3 + \cdots \\ \partial_y f &= y + bz + \partial_y f_3 + \cdots , \\ \partial_z f &= 1 + ax + by + cz + \partial_z f_3 + \cdots \end{aligned}$$

so that up to a multiplicative constant $\mathrm{Hess}(f)$ is

$$\begin{vmatrix} \frac{n}{n-1}(z+\frac{1}{2}y^2+\cdots) & az+\partial_x f_3+\cdots & y+bz+\partial_y f_3+\cdots & 1+ax+by+cz+\partial_z f_3+\cdots \\ az+\partial_x f_3+\cdots & \partial_x^2 f_3+\cdots & \partial_x\partial_y f_3+\cdots & a+\partial_x\partial_z f_3+\cdots \\ y+bz+\partial_y f_3+\cdots & \partial_x\partial_y f_3+\cdots & 1+\partial_y^2 f_3+\cdots & b+\partial_y\partial_z f_3+\cdots \\ 1+ax+by+cz+\partial_z f_3+\cdots & a+\partial_x\partial_z f_3+\cdots & b+\partial_y\partial_z f_3+\cdots & c+\partial_z^2 f_3+\cdots \end{vmatrix}.$$

Plugging in $z=0$ and keeping for each entry only the terms of order at most 1, we obtain

$$\begin{vmatrix} 0 & 0 & y & 1+ax+by \\ 0 & \partial_x^2 f_3 & \partial_x\partial_y f_3 & a+\partial_x\partial_z f_3 \\ y & \partial_x\partial_y f_3 & 1+\partial_y^2 f_3 & b+\partial_y\partial_z f_3 \\ 1+ax+by & a+\partial_x\partial_z f_3 & b+\partial_y\partial_z f_3 & c+\partial_z^2 f_3 \end{vmatrix}.$$

The only linear term in this determinant is $\partial_x^2 f_3$, so the equations of the tangent line to $S\cap \mathrm{Hess}(S)$ are

$$z=\partial_x^2 f_3 = 0. \tag{4.4.1}$$

It coincides with the double flex tangent line if and only if there is no term in $x^3$ in $f_3$, which is equivalent to the tangent section of $S$ at $p$ having an equation of the form

$$\frac{1}{2}y^2 + y(a_{21}x^2 + a_{12}xy + a_{03}y^2) + \cdots = 0,$$

which is the local form of a tacnodal singularity. □

**(4.5) Corollary.** *The generators of the Hessian developable surface are the double flex tangent lines at parabolic points of $S$. Its stationary planes are the tangent planes to $S$ at points where the tangent section has a tacnode.*

*(4.5.1)* Let us warn the reader that the Hessian developable surface is not the tangential surface of the Hessian curve on $S$. The only points where the tangent to the Hessian curve coincides with a generator of the Hessian developable are those parabolic points lying on the flecnodal locus.

I don't know of an explicit description of the regression edge of the Hessian developable.

We are now ready to compute the various numerical characters of the Hessian developable.

**(4.6) Lemma.** *Let $S,T$ be two surfaces in $\mathbf{P}^3$ of degrees $s,t$ respectively. The degree of the ruled surface generated by the flex tangents of $S$ at the points of $S\cap T$ equals $st(3s-4)$.*

Note that there is no reason that this ruled surface be developable in general, although it is indeed when $T=\mathrm{Hess}(S)$.

*Proof.* Let $f, g$ be homogeneous equations of $S$ and $T$ respectively. The ruled surface we are interested in is the projection on the first factor of the incidence graph

$$\{(p, p') : p' \in S \cap T \text{ and } p \text{ sits on a flex tangent to } S \text{ at } p'\}$$

which is the complete intersection in $\mathbf{P}^3 \times \mathbf{P}^3$ defined by the four bihomogeneous polynomials $f(p')$, $g(p')$, $\mathrm{D}_p f(p')$, $\mathrm{D}_{p^2} f(p')$, of bidegrees $(0, s)$, $(0, t)$, $(1, s-1)$, and $(2, s-2)$ in the variables $(p, p')$ respectively.

It is therefore the hypersurface in $\mathbf{P}^3$ defined by the homogeneous equation

$$\mathrm{Res}_{s,t,s-1,s-2}(f, g, \mathrm{D}_p f, \mathrm{D}_{p^2} f),$$

taken as a resultant of homogeneous polynomials in the variable $p'$, with coefficients homogeneous polynomials in the variable $p$. By standard homogeneity properties of the resultant référence au texte avec Laurent [3] it is a homogeneous polynomial of degree

$$0 \times t(s-1)(s-2) + 0 \times s(s-1)(s-2) + 1 \times st(s-2) + 2 \times st(s-1) \;=\; st(3s-4).$$

□

**(4.7) Corollary.** *The degree of the Hessian developable surface equals $2n(n-2)(3n-4)$.*

This quantity is called the *rank* of the Hessian system in the classical terminology, cf. Appendix B.

*Proof.* We apply the above lemma with $T = \mathrm{Hess}(S)$. This gives the degree $4m(m-2)(3m-4)$, which gets however divided by two, as the obtained equation is double in this case, being the two flex tangents coincident at parabolic points. □

**(4.8)** The class of the Hessian developable is the number of planes tangent to $S$ at a parabolic point, and passing through a fixed general point in $\mathbf{P}^3$. It is therefore the intersection number of $S$, $\mathrm{Hess}(S)$, and $\mathrm{D}_{p'} S$ for a general $p' \in \mathbf{P}^3$, which equals $4n(n-1)(n-2)$.

**(4.9)** The number of stationary planes of the Hessian developable is by Corollary (4.5) the number of points in the intersection of the parabolic curve $S \cap \mathrm{Hess}(S)$ with the flecnodal locus. The latter is cut out on $S$ by a homogeneous polynomial of degree $11n - 24$, Référence avec Laurent [3]. but one has to take into account the famous fact that the two surfaces defined by the Hessian and the flecnodal polynomials are everywhere tangent. Sadly, I abandoned the idea of including a proof of this fact here. Give reference however! Eventually, one thus finds the number $2n(n-2)(11n-24)$.

**(4.10)** We may now find the remaining characteristic numbers using the formulae of Appendix B. We follow the notation of (B.4) and (B.14), with an additional decoration 'He' to avoid confusion; see also the warning below. We have found so far:
— $r_{\mathrm{He}} = 2n(n-2)(3n-4)$ ;
— $n_{\mathrm{He}} = 4n(n-1)(n-2)$ ;
— $\alpha_{\mathrm{He}} = 2n(n-2)(11n-24)$.
Knowing three quantities is enough to derive all the others by a direct applications of the formulae in (B.16) and (B.17). For instance, one finds as in [27, §608]:
— $m_{\mathrm{He}} = 4n(n-2)(7n-15)$ ;
— $\beta_{\mathrm{He}} = 10n(n-2)(7n-16)$ ;
— $g_{\mathrm{He}} = 2n(n-2)(4n^4 - 16n^3 + 20n^2 - 27n + 39)$ ;
— $h_{\mathrm{He}} = 2n(n-2)(196n^4 - 1232n^3 + 2580n^2 - 1861n + 137)$.

*(4.10.1) Warning.* For ease of reference I have done my best to stick to Salmon's notation, as other have done before me, e.g., Piene [24], Ronga [26]. It sets however various little traps one should be aware of, due to the exchanges performed by duality. For instance, $\alpha_{\mathrm{He}} = \check{\beta}$, and $g_{\mathrm{He}} = \check{h}$. $\alpha_{\mathrm{He}}$ and $g_{\mathrm{He}}$ appear in [27, §608], while $\check{\beta}$ and $\check{h}$ appear in [27, §612] and [27, §613] respectively.

```
r := 2*n*(n-2)*(3*n-4);
nH := 4*n*(n-1)*(n-2);
alpha := 2*n*(n-2)*(11*n-24);

m:=
alpha-3*(nH-r);
                              4 n (n - 2) (7 n - 15)

beta:=
nH-3*(r-m);
                              10 n (n - 2) (7 n - 16)

g:=
1/2*(nH*(nH-1)-3*alpha-r);
                                  4        3        2
                  4 n (n - 2) (4 n  - 16 n  + 20 n  - 27 n + 39)

h:=
1/2*(m*(m-1)-3*beta-r);
                                    4          3          2
         dh := 4 n (n - 2) (196 n  - 1232 n  + 2580 n  - 1861 n + 137)
```

## 4.2 – The node-couple developable surface

**(4.11) Proposition.** *The generators of the node-couple developable surface are the lines joining pairs of conjugated points of the node-couple curve, in other words the lines $\langle p, q\rangle$ such that there exists a plane tangent to $S$ at both $p$ and $q$.*

*Proof.* The node-couple developable is the dual of the curve $\gamma_S(K_S) \subset \check{\mathbf{P}}^3$, so its generators are the orthogonals to the tangents lines of $\gamma_S(K_S)$ (see subsection B.1). Let $\varpi \in \gamma_S(K_S)$ be a general point. There are two points $p, q \in S$ such that the plane $H = \varpi^\perp \subset \mathbf{P}^3$ is tangent to $S$ at both $p$ and $q$. The tangent line $\mathbf{T}_\varpi(\gamma_S(K_S))$ is the intersection of the tangent planes to the two local sheets of $S^\vee$ at $\varpi$, respectively $p^\perp$ and $q^\perp$. Therefore,

$$\mathbf{T}_\varpi(\gamma_S(K_S))^\perp = (p^\perp \cap q^\perp)^\perp = \langle p, q\rangle.$$

□

To determine the numerical characters of the node-couple developable (in subsection 4.4) we need the results of the following subsection 4.3. For the moment let us only mention the interesting [7] in which the node-couple developable of a very general complex quartic surface is carefuly studied.

## 4.3 – Ordinary double edges of the polar cone

**(4.12)** We adopt once again the set-up of (3.4); the polar cone $\mathcal{D}_pX$ of $X$ with respect to $p \in \mathbf{P}^3$ is the cone projecting from $p$ over the curve $X \cap \mathrm{D}_pX$. Its ordinary double edges are the lines intersecting $X$ with multiplicity 2 in two distinct points, and we have seen in 2.2 that their contact points with $X$ are in the number $d(d-1)(d-2)(d-3)$, all on the reducible curve $X \cap \mathrm{D}_pX$. We now list the various kinds of such contact points.

**(4.13) Notation.** We keep the notation from (3.5), and let in addition:
- $\delta$ the number of ordinary double edges of the circumscribed cone $\mathcal{T}_pX$ (these are the honest bitangent lines to $X$);
- $k$ the number of apparent double points of the ordinary double curve $B$ of $X$;
- $h$ the number of apparent double points of the cuspidal double curve $C$ of $X$;

Recall that the number of apparent double points of $B$ (resp. $C$) is the number of double points of its projection from the point $p$.

**(4.14)** For $Y$ and $Y'$ two distinct curves among $A, B, C$, we let $[Y, Y']^\circ$ be their number of *apparent only double points*, which we define as the number of intersection points of their projections from $p$, $\pi_p(Y)$ and $\pi_p(Y')$, that do not come from an actual intersection point of $Y$ and $Y'$.

Of course the total number of intersection points of $\pi_p(Y)$ and $\pi_p(Y')$ is $\deg(Y) \cdot \deg(Y')$, and we shall discuss below the quantities to be subtracted to obtain $[Y, Y']^\circ$.

**(4.15)** Salmon gives the formulae

(𓏠𓂧𓇋 𓏺) $\quad a(n-2)(n-3) = 2\delta + 2[A \cdot B]^\circ + 3[A \cdot C]^\circ \;;$

(𓏠𓂧𓇋 𓏭) $\quad b(n-2)(n-3) = 4k + [A \cdot B]^\circ + 3[B \cdot C]^\circ \;;$

(𓏠𓂧𓇋 𓏼) $\quad c(n-2)(n-3) = 6h + [A \cdot C]^\circ + 2[B \cdot C]^\circ$

without any comment.

My understanding of these formulae is as follows. The number $n(n-1)(n-2)(n-3) = (a+2b+3c)(n-2)(n-3)$ counts the contact points of the ordinary double edges of the polar cone $\mathcal{D}_pX$ with the curve $X \cap \mathrm{D}_pX = A + 2B + 3C$. Let $Y, Y'$ be two curves (possibly the same) among $A, B, C$, and $m_Y, m_{Y'}$ their multiplicities in the cycle $X \cap \mathrm{D}_pX$. An intersection point between $\pi_p(Y)$ and $\pi_p(Y')$ gives a double edge of the polar cone with multiplicity $m_Y \cdot m_{Y'}$, hence two contact points $y$ and $y'$ with $X \cap \mathrm{D}_pX$ with the same multiplicity. Taking into account the multplicities of $Y$ and $Y'$ in $X \cap \mathrm{D}_pX$, one finds that the polynomial of degree $(n-2)(n-3)$ cutting out the contact points should intersect $Y$ with multiplicity $m_{Y'}$ at $y$ and $Y'$ with multiplicity $m_Y$ at $y'$. This explains all the coefficients in the formulae. The actual intersection points of $A, B, C$ should not be counted, as they give rise to cuspidal double edges of the polar cone, and not to ordinary double edges.

The polynomial of degree $(n-2)(n-3)$ cutting out the contact points is constructed in ref.w.Laurent [3]. I have not been able to derive a more rigorous proof of the above formulae.

**(4.16) Number of apparent only intersection points.** We now discuss the various contributions to be removed from the total number of apparent double points to obtain the number of apparent only intersection points. We use one more notation:
- $i$ the number of intersection points of $B$ and $C$ smooth for both curves (when $X = S^\vee$ for a smooth $S$, this number vanishes).

The formulae are the following:

(𓎆𓏺) $[A \cdot B]^\circ = ab - 2\rho$ ;

(𓎆𓏻) $[A \cdot C]^\circ = ac - 3\sigma$ ;

(𓎆𓏼) $[B \cdot C]^\circ = bc - 3\beta - 2\gamma - i$.

Once more, Salmon takes them as self-evident.

In our setup, every actual intersection point of two curves among $A, B, C$ is of one of the types $\rho, \sigma, \beta, \gamma, i$, so only the coefficients are to be explained. They are the intersection multiplicities at the corresponding points of the projections of the two curves under consideration. This is straightforward for points of type $i$.

*(4.16.1) Points of type $\rho$.* These are the points $q \in A \cap B$. At such a point, the surface $X$ has two smooth local sheets $X'_q$ and $X''_q$, $B = X'_q \cap X''_q$ and the circumscribed curve $A$ is contained in one of them, say $X'_q$, and transverse to $B$. The curve $A$ is the ramification curve of the projection of $X'_q$, hence the projections of $A$ and $B$ are tangent at the projection of $q$, and intersect with multiplicity 2.

*(4.16.2) Points of type $\sigma$.* These are the points $q \in A \cap C$. At such a point, the surface $X$ has one cuspidal local sheet $X_q$, and $C = \mathrm{Sing}(X_q)$. The curve $A$ is the ramification curve of the projection of $X_q$ lifted to its normalization $\bar{X}_q$. We have seen in (3.6.4) that $A$ and $C$ are smooth and tangent at $q$. The claim is that after the projection they intersect with multiplicity 3. I have not been able to produce a definitive direct proof of this fact. However the number $[A \cdot C]^\circ$ occurs only in (𓍢𓆼𓏺) and (𓍢𓆼𓏼) which we may verify otherwise since we know all the other numbers involved. This gives an indirect proof of Formula (𓎆𓏻).

*(4.16.3) Points of type $\beta$.* These are the swallowtail singularities of $X$. At these points $B$ is smooth and $C$ is cuspidal, and we can see on the local model in (3.1) that they have the same reduced tangent cone. It follows that after projection they intersect with multiplicity 3 at these points.

*(4.16.4) Points of type $\gamma$.* These are the points at which $X$ has two transverse local sheets, one smooth: $X^{\mathrm{s}}_q$, and the other cuspidal: $X^{\mathrm{c}}_q$. There $C$ is smooth and $B$ is cuspidal, and these two curves have transverse reduced tangent cones. It follows that after projection they intersect with multiplicity 2 at these points.

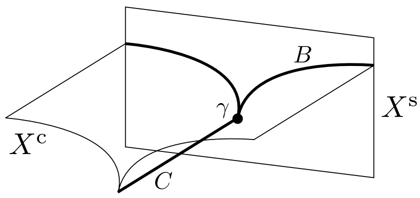


Figure 2: Local picture of $X$ at a point of type $\gamma$

**(4.17) Corollary.** Putting together formulas (𓍢𓆼) and (𓎆) one obtains:

(𓍢𓆼𓆼𓏺) $a(n-2)(n-3) = 2\delta + 2ab + 3ac - 4\rho - 9\sigma$ ;

(𓍢𓆼𓆼𓏻) $b(n-2)(n-3) = 4k + ab + 3bc - 9\beta - 6\gamma - 3i - 2\rho$ ;

(𓍢𓆼𓆼𓏼) $c(n-2)(n-3) = 6h + ac + 2bc - 6\beta - 4\gamma - 2i - 3\sigma$.

## 4.4 – Application to the case $X = S^\vee$

We now proceed as in Subsection 3.3, and maintain the setup introduced there, to apply the formulae of Subsection 4.3 above to the case when $X$ is the dual of a smooth, degree $n$, surface in $\mathbf{P}^3$.

**(4.18)** As in (3.8), the circumscribed cone $\mathcal{T}_{\check{p}}(S^\vee)$ is the dual in $\check{\mathbf{P}}^3$ of the plane curve $\check{p}^\perp \cap S$. The number $\check{\delta}$ of ordinary edges of this cone is therefore the number of bitangents of the plane curve $\check{p}^\perp \cap S$, which by (1.13) is:
— $\check{\delta} = \frac{1}{2}n(n-2)(n-3)(n+3)$.

**(4.19)** The numbers $\check{h}$ and $\check{k}$ are the invariants $g$ (introduced in (B.14)) of the Hessian and node-couple developables respectively. The former has been computed in (4.10) and the latter we don't know yet:
— $\check{h} = g_{\mathrm{He}} = 2n(n-2)(4n^4 - 16n^3 + 20n^2 - 27n + 39)$ ;
— $\check{k} = g_{\mathrm{Kn}}$.

**(4.20)** The numbers $\check{n}, \check{a}, \check{b}, \check{c}$ have been given in Subsection 3.3, and the numbers $\check{\rho}, \check{\sigma}, \check{\beta}, \check{\gamma}$ have already been computed. As we mentioned earlier, the number $\check{\imath}$ vanishes.

One may thus check by a direct computation that the first and third identities of (4.17) are indeed verified. The second one on the other hand gives the formula:
— $\check{k} = \frac{1}{8}n(n-2)(n^{10}-6n^9+16n^8-54n^7+164n^6-288n^5+547n^4-1058n^3+1068n^2-1214n+1464)$.

```
                             2
               nn := n (n - 1)

                a := n (n - 1)

                        3    2
      n (n - 1) (n - 2) (n  - n  + n - 12)
  b:= ------------------------------------
                     2

            c := 4 n (n - 2) (n - 1)

               n (n - 2) (n - 3) (n + 3)
       delta := -------------------------
                          2

                        3    2
       rho := n (n - 2) (n  - n  + n - 12)

              sigma := 4 n (n - 2)

a*(nn-2)*(nn-3)-2*delta-2*a*b-3*a*c+4*rho+9*sigma;
expand(%);
                       0

         beta := 2 n (n - 2) (11 n - 24)

                                  3
   gamm := 4 n (n - 2) (n - 3) (n  + 3 n - 16)
```

```
                                      4        3        2
              h := 2 n (n - 2) (4 n  - 16 n  + 20 n  - 27 n + 39)

c*(nn-2)*(nn-3)-6*h-a*c-2*b*c+6*beta+4*gamm+3*sigma;
factor(expand(%));
                                          0

1/4*(b*(nn-2)*(nn-3)-a*b-3*b*c+9*beta+6*gamm+2*rho);
factor(expand(%));
           10       9        8        7         6         5         4         3
n (n - 2) (n   - 6 n  + 16 n  - 54 n  + 164 n  - 288 n  + 547 n  - 1058 n

             2
     + 1068 n  - 1214 n + 1464)/8
```

**(4.21) Rank of the node-couple developable.** We shall compute $r_{\mathrm{Kn}}$ with the formula $r = n(n-1) - 2g - 3\alpha$ from (B.16); the latter has to be adapted however, due to the presence of triple points on the curve $\check{B}$, which plays the role of $C$ in the context of (B.16). This may be taken care of in various ways, but the idea is always the same, namely that an ordinary triple point counts exactly as three nodes. The upshot is the following:

$$\begin{aligned} r_{\mathrm{Kn}} &= n_{\mathrm{Kn}}(n_{\mathrm{Kn}} - 1) - 2\big(g_{\mathrm{Kn}} + 3\#(\text{triple points})\big) - 3\alpha_{\mathrm{Kn}} \\ &= \check{b}(\check{b} - 1) - 2\check{k} - 6\check{t} - 3\check{\gamma}, \end{aligned}$$

which gives
— $r_{\mathrm{Kn}} = n(n-2)(n-3)(n^2+2n-4)$.

**(4.22) Numerical characters of the node-couple developable.** We have found the three numerical characters $n_{\mathrm{Kn}} = \check{b}$, $g_{\mathrm{Kn}} = \check{k}$ and $\alpha_{\mathrm{Kn}} = \check{\gamma}$, so we are able to derive all the others using the formula of Subsection B.3, suitably modified in order to take into account the presence of $\check{t}$ ordinary triple points on the curve $C_{\mathrm{Kn}} = \check{B}$ as in (4.21) above. We leave this as an exercise to the not-tired-yet reader.

## 5 – Formulae for surfaces projected in $\mathbf{P}^3$

In this section, we provide enumerative formulae for the number of curves with particular features in a possibly incomplete linear system of dimension $\leqslant 3$ on a smooth surface.

**(5.1) Setup.** We consider a smooth surface $S$ and a possibly incomplete linear system $|V|$ of dimension $r \leqslant 3$ on $S$, with smooth irreducible general member. Up to adding a fixed part to $|V|$, we may assume that it is a linear subsystem of a very ample linear system $|V^\sharp|$. We make the further assumption that $|V|$ is a *general* subsystem of $|V^\sharp|$, so that we may apply the general projection theorem. Reference? If $r > 1$, this implies in particular that $|V|$ is base point free.

We call:
— $C$ the linear equivalence class of divisors in $|V|$ (and sometimes a member of $|V|$ by abuse of notation) ;
— $n$ the degree of $|V|$, that is $n = C^2$ ;
— $\pi$ the common arithmetic genus of all members of $|V|$.

We shall frequently use the identity

$$K_S \cdot C = 2\pi - 2 - n \tag{5.1.1}$$

given by the adjunction formula.

**(5.2) Warm up: the 1-dimensional case.** If $r = 1$, the generality assumption implies that $|V|$ has $n$ simple base points, and its singular members are all 1-nodal curves. In this case, a topological formula discussed in details at various other places in this volume gives the class $\check{n}$ of $|V|$, that is the number of its singular members:
— $\check{n} = e(S) + n - e(C) \cdot e(\mathbf{P}^1) = c_2(S) + n + 4\pi - 4$.

Next we treat somewhat unexpectedly the case $r = 3$ before the case $r = 2$: this is before we shall use the geometric description of the former situation to comprehend the latter.

## 5.1 – The 3-dimensional case: surfaces with ordinary singularities

**(5.3)** If $r = 3$, the generality assumption implies that $|V|$ defines a birational morphism from $S$ to a degree $n$ surface $S^\flat \subset \mathbf{P}^3$ with ordinary singularities, that is $S^\flat$ has an ordinary double curve $D$ with $\tau$ pinch points and $t$ triple points: locally at a general point of $D$, $S^\flat$ looks like the hypersurface $xy = 0$ in the affine space with coordinates $x, y, z$; locally at a pinch point it has a Whitney umbrella singularity, that is it looks like the hypersurface $x^2 - zy^2 = 0$; locally at a triple point it looks like the hypersurface $xyz = 0$, so these points are non-planar triple points for $D$. We assume for simplicity that $D$ is irreducible.

The morphism $\nu : S \to S^\flat$ is the normalization map. We let $\Delta \subset S$ be the pre-image of $D \subset S^\flat$; there is a double cover $\Delta \to D$, which ramifies exactly over the $\tau$ pinch points. By adjunction theory (see, e.g., [18, Exer. II.8.5]), one has $\omega_S \cong \mathcal{O}_S(-\Delta) \otimes \nu^* \omega_{S^\flat}$, and $\nu_* \omega_S = \mathcal{I}_D|_{S^\flat}$, where $\mathcal{I}_D$ is the sheaf of ideals defining $D$ in $\mathbf{P}^3$. Being $S^\flat$ a divisor in $\mathbf{P}^3$, we have $\omega_{S^\flat} \cong \omega_{\mathbf{P}^3}(S^\flat)\big|_{S^\flat} = \mathcal{O}_{S^\flat}(n-4)$.

We use the following notation, some of which has already been introduced:
— $\tau$ is the number of pinch points of $S^\flat$ ;
— $t$ is the number of triple points of $S^\flat$ ;
— $d$ is the degree of the double curve $D \subset \mathbf{P}^3$ ;
— $\rho$ is the geometric genus of $D$ ;
— $\Pi$ is the geometric genus of $\Delta \subset S$.

**(5.4)** The curves $C$ are represented in $\mathbf{P}^3$ as hyperplane sections of $S^\flat$, hence as degree $n$ plane curves with $d$ nodes. Therefore one has

$$\pi = \binom{n-1}{2} - d. \tag{5.4.1}$$

**(5.5) Effective degree of first polars.** We compute the class $\check{n}$ of $S^\flat$, that is the number of singular curbes in a general subpencil of $|V|$, following the method explained in (1.7). For a general point $p \in \mathbf{P}^3$, the first polar $\mathrm{D}_p S^\flat$ is a degree $n-1$ hypersurface passing simply through $D$; moreover, the intersection $\mathrm{D}_p S^\flat \cap S^\flat$ residual to $D$ is a curve passing through all pinch points. This follows from a local computation, which at this point we may leave to the reader.

Therefore, for a general pair of points $p, q \in \mathbf{P}^3$ the intersection of the two curves $(\mathrm{D}_p S^\flat \cap S^\flat) - D$ and $(\mathrm{D}_q S^\flat \cap S^\flat) - D$ consists of the $\tau$ pinch points of $S^\flat$ plus the $\check{n}$ points $a$ such that $p, q \in \mathbf{T}_a S$. To compute this intersection number $\tau + \check{n}$, we pull-back the curves to the normalization $S$. They are sections of

$$\mathcal{I}_D \otimes \mathcal{O}_{S^\flat}(n-1) \cong \mathcal{I}_D \otimes \omega_{S^\flat} \otimes \mathcal{O}_{S^\flat}(3),$$

hence pull-back to curves in the class $K_S + 3C$. We thus find the identity

$$\check{n} + \tau = (K_S + 3C)^2 = K_S^2 + 3n + 12\pi - 12, \tag{5.5.1}$$

using (5.1.1) to substitute the intersection number $K_S \cdot C$.

$$\begin{aligned} (K_S + 3C)^2 &= K^2 + 9n + 6K \cdot C \\ &= K^2 + 9n + 6(2\pi - 2 - n) \\ &= K^2 + 3n + 12\pi - 12 \\ &= p^{(1)} + 3n + 12\pi - 13. \end{aligned}$$

**(5.6) Genus of the curve of neutral pairs of $|V|$.** One first observes that the curve $\Delta$ has 3 nodes above each triple point of $S^\flat$, as is best understood looking at the picture below.

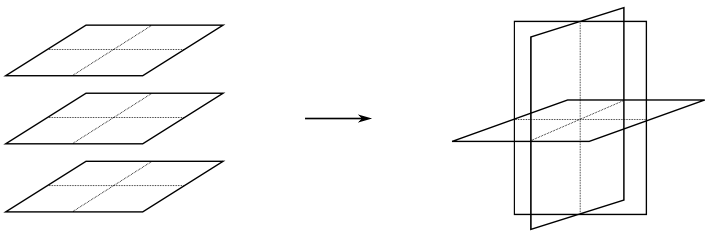

Figure 3: Normalization of $S^\flat$ at a triple point

Since it may be assumed to be otherwise smooth, one finds its arithmetic genus to be $p_a(\Delta) = p_g(\Delta) + 3t = \Pi + 3t$. The latter may also be computed using the adjunction formula. We have already seen the identity $K_S = (n-4)C - \Delta$. One thus finds

$$2(\Pi + 3t) - 2 = (K_S + \Delta) \cdot \Delta = (n-4)C \cdot \Delta = 2(n-4)d, \tag{5.6.1}$$

equivalently

$$p_a(\Delta) = (n-4)d + 1. \tag{5.6.2}$$

On the other hand, the map $\Delta \to D$ lifts to a $2:1$ covering $\bar{\Delta} \to \bar{D}$ between the normalizations of $\Delta$ and $D$. It is ramified at $\tau$ points, so the Riemann–Hurwitz formula gives

$$2\Pi - 2 = 2(2\rho - 2) + \tau. \tag{5.6.3}$$

**(5.7) Intersections of circumscribed curves with the double curve.** Let $p \in \mathbf{P}^3$ be a general point. We consider the circumscribed curve $\mathcal{C}_p S^\flat := (\mathrm{D}_p S^\flat \cap S^\flat) - D$ and its intersection points with the curve $D$. To see clearly the situation, it is best to work on the normalization $S$. One has

$$\nu^* \mathcal{C}_p S^\flat \cdot \Delta = (K_S + 3C) \cdot \big((n-4)C - K_S\big) = -K^2 + 2\pi(n-7) + 2n^2 - 7n + 14.$$

Geometrically this intersection amounts to the $\tau$ pinch points plus the points $q \in D$ such that the tangent plane to one of the two local sheets of $S^\flat$ at $q$ passes through $p$. Let $a$ be the number of the latter points. It may be computed by considering the intersection of the second polar $\mathrm{D}_{p^2} S^\flat$ with $D$. It follows from Theorem (A.6) that this intersection consists of the $a$ points we are interested in plus the $t$ triples points, and since those are triple for $D$ one finds

$$a + 3t = (n-2)d.$$

Eventually we end up with

$$\tau + \big((n-2)d - 3t\big) = -K^2 + 2\pi(n-7) + 2n^2 - 7n + 14. \tag{5.7.1}$$[2]

$$\begin{aligned}(K_S + 3C) \cdot \big((n-4)C - K_S\big) &= -K^2 + 3n(n-4) + (n-7)KC \\ &= -K^2 + 3n(n-4) + (n-7)(2\pi - 2 - n) \\ &= -K^2 + 2\pi(n-7) + 3n^2 - 12n - (n^2 - 5n - 14) \\ &= -K^2 + 2\pi(n-7) + 2n^2 - 7n + 14 \\ &= -p^{(1)} + 1 + 2\pi(n-7) + 2n^2 - 7n + 14.\end{aligned}$$

**(5.8) Postulation formula.** We shall also use the following formula for the arithmetic genus of $S$, which we will discuss in the dedicated subsection 5.2.

$$p_a(S) = \binom{n-1}{3} - (n-4)d + \rho + 2t - 1. \tag{5.8.1}$$

**(5.9) Conclusion.** We may now express the quantities $\check{n}$, $\tau$, $t$, $d$, $\rho$, $\Pi$ in terms of the four invariants $n$, $\pi$, $p_a(S)$, $K_S^2$. The formula for $d$ is readily given by (5.4.1); we use it so substitute for $d$ in the five equations (5.5.1), (5.6.1), (5.6.3), (5.7.1), (5.8.1), thus obtaining

$$\begin{pmatrix} 1 & & & & 1 \\ & 2 & 6 & & \\ & 2 & 0 & -4 & -1 \\ & & -3 & 0 & 1 \\ & & -2 & -1 & 0 \end{pmatrix} \begin{pmatrix} \check{n} \\ \Pi \\ t \\ \rho \\ \tau \end{pmatrix} = \begin{pmatrix} 3n + 12\pi + K_S^2 - 12 \\ (n-4)(n-1)(n-2) - 2(n-4)\pi + 2 \\ -2 \\ -\frac{1}{2}n^3 + \frac{9}{2}n^2 - 11n + 16 - K_S^2 + \pi(3n-16) \\ \frac{1}{6}(n-1)(n-2)(-2n+9) + \pi(n-4) - p_a(S) - 1 \end{pmatrix}.$$

This eventually gives:
— $\check{n} = n + 4\pi + 12p_a(S) - K_S^2 + 8$ ;
— $d = \frac{1}{2}(n-1)(n-2) - \pi$ ;
— $\rho = \frac{1}{2}(n^2 - 7n) + \pi(n-12) + 9p_a(S) - 2K_S^2 + 22$ ;
— $\Pi = n^2 - 6n + 2\pi(n-10) + 12p_a(S) - 3K_S^2 + 33$ ;
— $t = \frac{1}{6}(n^3 - 9n^2 + 26n) - \pi(n-8) - 4p_a(S) + K_S^2 - 12$ ;
— $\tau = 2n + 8\pi - 12p_a(S) + 2K_S^2 - 20$.

[2] There is a typo in [13, Formula (6), p. 175], but the final formulas on p. 176 agree with ours in (5.9).

Start with

$$\begin{cases} d = \binom{n-1}{2} - \pi \\ \check{n} + \tau = K_S^2 + 3n + 12\pi - 12 \\ 2(\Pi + 3t) - 2 = 2(n-4)d \\ 2\Pi - 2 = 2(2\rho - 2) + \tau \\ \tau + \big((n-2)d - 3t\big) = -K^2 + 2\pi(n-7) + 2n^2 - 7n + 14 \\ p_a(S) = \binom{n-1}{3} - (n-4)d + \rho + 2t - 1. \end{cases}$$

Next substitute $d$, then leave the first equation aside, and rearrange the equations:

$$\begin{cases} \check{n} + \tau = K_S^2 + 3n + 12\pi - 12 \\ 2\Pi + 6t = (n-4)(n-1)(n-2) - 2(n-4)\pi + 2 \\ 2\Pi - 4\rho - \tau = -2 \\ \tau - 3t = -K^2 + 2\pi(n-7) + 2n^2 - 7n + 14 \\ \qquad - (n-2)\dfrac{(n-1)(n-2)}{2} + (n-2)\pi \\ -\rho - 2t = -p_a(S) + \dfrac{1}{6}(n-1)(n-2)(-2n+9) + \pi(n-4) - 1. \end{cases}$$

```
with(LinearAlgebra);

A:=Matrix(5,5,[
1 ,0, 0, 0, 1,
0, 2, 6, 0, 0,
0, 2, 0, -4, -1,
0, 0, -3, 0, 1,
0, 0, -2, -1, 0]);

invA:=MatrixInverse(A);


2*n^2-7*n+14-1/2*(n-1)*(n-2)^2;
factor(expand(%));
                                      2                     3
                          9/2 n  - 11 n + 16 - 1/2 n

B:=Matrix(5,1,[
K+3*n+12*pi-12,
(n-4)*(n-1)*(n-2)-2*(n-4)*pi+2,
-2,
-K+pi*(3*n-16)+9/2*n^2-11*n+16-1/2*n^3,
-p+1/6*(n-1)*(n-2)*(-2*n+9)+pi*(n-4)-1
]);

X:=MatrixMatrixMultiply(invA,B);

ncheck
collect(X[1],n);
                            [n - K + 12 p + 4 pi + 8]
```

```
Pi
eval(X[2],n=0);
                            [33 - 20 pi - 3 K + 12 p]
eval(X[2],[p=0,K=0]);collect(%,pi);
                                (2 n - 20) pi + ...
eval(X[2],[p=0,K=0,pi=0]);value(%[1]);
%-n^2+6*n-33;
expand(%);
                                        0

t
value(X[3]);
                         K - 4 p + (-n + 8) pi +...
eval(%,[K=0,p=0,pi=0]);value(%[1]);expand(6*%);
                               3      2
                              n  - 9 n  + 26 n - 72

rho
(n - 12) pi + 9 p - 2 K ...
eval(X[4,1],[K=0,p=0,pi=0]);expand(2*%);
                                   2
                                  n  - 7 n + 44

tau
factor(expand(X[5,1]));
                         2 K + 2 n - 12 p + 8 pi - 20
```

**(5.10) Choice of four independent invariants.** I have chosen to follow Enriques' notation for ease of reference. I have however substituted the old-fashioned *linear genus* $p^{(1)}(S)$ with $K_S^2$. Indeed, $p^{(1)}(S)$ is defined as the *virtual* genus of canonical sections of $S$, that is the genus of the curves in $|K_S|$ obtained from the adjunction formula, even though $|K_S|$ may in fact be empty. In other words $p^{(1)}$ is defined by the formula

$$2p^{(1)} - 2 = (K_S + K_S) \cdot K_S,$$

hence $p^{(1)} = K_S^2 + 1$.

Nowadays the custom is to use $C^2$, $K_S \cdot C$, $K_S^2$, $c_2(S)$ rather than $n$, $\pi$, $p_a(S)$, $K_S^2$. To obtain the above formulas in this form, simply use

(5.10.1) $$C^2 = n; \qquad K_S \cdot C + C^2 = 2\pi - 2; \qquad 1 + p_a(S) = \frac{1}{12}\big(K_S^2 + c_2(S)\big).$$

The former two relations we have already encountered, and the latter is the well-known Noether formula (see [18, Example A.4.1.2] for instance).

## 5.2 – The postulation formula

This subsection is a digression about the postulation formula (5.8.1), which gives an expression for the arithmetic genus $p_a(S)$ in terms of the projection $S^\flat$ in $\mathbf{P}^3$. A modern proof may be found in [25]; here instead we explain the classical definition of the arithmetic genus, how it gives Formula (5.8.1), and why it coincides with the modern definition. This may seem a bit off-topic, I included it here nevertheless by lack of a suitable reference. The interested reader may also wish to consult [30, Chap. III] in addition to [13, Cap. IV].

**(5.11)** Recall that by definition, the arithmetic genus of a variety $X$ is

$$p_a(X) := (-1)^{\dim X}\big(\chi(\mathcal{O}_X) - 1\big).$$

If $X$ is a degree $d$ hypersurface in $\mathbf{P}^N$, one finds $p_a(X) = h^0(\mathcal{O}_{\mathbf{P}^N}(d-N-1)) = \binom{d-1}{N}$. When $X$ is smooth, its geometric genus is by definition

$$p_g(X) := h^0(X, \omega_X).$$

**(5.12)** For a smooth curve there is no difference between the arithmetic and geometric genera; it is an important point that the same does not hold in higher dimension. One defines the geometric genus of an arbitrary curve to be the genus of its normalization.

Let $X^1$ be a curve. One usually relates its arithmetic and geometric genera with an exact sequence

$$0 \to \mathcal{O}_{X^1} \to \nu_*\mathcal{O}_{\bar{X}_1} \to \textstyle\bigoplus_{p\in \mathrm{Sing}(X^1)} (\mathbf{C}_p)^{\oplus\delta(p)} \to 0, \tag{5.12.1}$$

where $\nu : \bar{X}_1 \to X_1$ is the normalization map, and the third term is a skyscraper sheaf supported on the singular locus of $X^1$, with $\delta(p)$ an integer depending on the type of singularity. Taking Euler characteristics in (5.12.1), one obtains

$$p_g(X^1) = p_a(X^1) - \textstyle\sum_{p\in\mathrm{Sing}(X^1)} \delta(p).$$

*(5.12.2)* Locally at a space triple point, the exact sequence (5.12.1) looks like

$$\begin{array}{ccccc} 0 \to \mathbf{C}[x,y,z]/(xy,xz,yz) \to & \mathbf{C}[x]\oplus\mathbf{C}[y]\oplus\mathbf{C}[z] & \to & \mathbf{C}^2 & \to 0 \\ & (P,Q,R) & \mapsto & \big(P(0)-Q(0), Q(0)-R(0)\big) & \end{array}$$

so that a space triple point make the genus drop by 2.

**(5.13)** The arithmetic genus of a smooth surface $S$ may be computed in a similar fashion in terms of a generic projection $S^\flat$ in $\mathbf{P}^3$ (we keep the notation from the previous Subsection 5.1). There is indeed an exact sequence

$$\begin{array}{ccccc} 0 \to \mathcal{O}_{S^\flat} \to \nu_*\mathcal{O}_S \to & \nu_*\mathcal{O}_\Delta & \to & \mathcal{O}_D & \to 0, \\ & (f_1,f_2) & \mapsto & f_1 - f_2 & \end{array}$$

which gives, taking Euler characteristics:

$$p_a(S) = p_a(S^\flat) - p_a(\Delta) + p_a(D).$$

Now, the arithmetic genus of $S^\flat$ is that of a degree $n$ hypersurface in $\mathbf{P}^3$, see (5.11), that of $\Delta$ is given in (5.6.2), and that of $D$ is $\rho + 2t$ by (5.12.2), so we have obtained a first proof of Formula (5.8.1).

To explain the classical definition of the arithmetic genus of a surface, we shall need the following lemma. It uses in an essential way the relations between the canonical line bundles of $S$, $S^\flat$, and $\mathbf{P}^3$, cf. (5.3).

**(5.14) Lemma.** *For all $i = 0,1,2$, one has $H^i(S,\omega_S) \cong H^i(S^\flat, \mathcal{O}_{S^\flat}(n-4)\otimes\mathcal{I}_D)$.*

*Proof.* By the projection formula, we have

$$\nu_*\omega_S = \nu_*\big(\nu^*\mathcal{O}_{S^\flat}(n-4)\otimes\mathcal{O}_S(-\Delta)\big) = \mathcal{O}_{S^\flat}(n-4)\otimes\mathcal{I}_D. \tag{5.14.1}$$

Moreover one has $R^i\nu_*(\omega_S) = 0$ for all $i = 1,2$, because $\nu$ is finite, so $H^i(S,\omega_S) \cong H^i(S^\flat, \nu_*\omega_S)$ for all $i = 0,1,2$ by Leray's spectral sequence (or merely [18, Exer. III.8.1]), and the result follows. $\square$

**(5.15)** As a corollary of Lemma (5.14), one has the following formula for the geometric genus of $S$:

$$p_g(S) = h^0(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(n-4) \otimes \mathcal{I}_D), \tag{5.15.1}$$

i.e., the geometric genus of $S$ is the number of independent degree $n-4$ hypersurfaces containing $D$, in more classical words of degree $n-4$ hypersurfaces adjoint to $S^\flat$.

To compute this number, one may consider the restriction exact sequence

$$0 \to \mathcal{O}_{\mathbf{P}^3}(n-4) \otimes \mathcal{I}_D \to \mathcal{O}_{\mathbf{P}^3}(n-4) \to \mathcal{O}_D(n-4) \to 0. \tag{5.15.2}$$

If the linear series cut out on $D$ by $|\mathcal{O}_{\mathbf{P}^3}(n-4)|$ is complete and non-special, one finds $h^0(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(n-4) \otimes \mathcal{I}_D)$ to equal

$$h^0\big(\mathcal{O}_{\mathbf{P}^3}(n-4)\big) - h^0\big(\mathcal{O}_D(n-4)\big) = \tbinom{n-3}{3} - \big(1 - p_a(D) + \deg \mathcal{O}_D(n-4)\big) \tag{5.15.3}$$

$$= \tbinom{n-3}{3} - 1 + (\rho + 2t) + (n-4)d \tag{5.15.4}$$

(the first equality is given by the Riemann–Roch theorem, and the second one by the computation of $p_a(D)$ in (5.13)), which is the right-hand-side term in (5.8.1).

Classically, the arithmetic genus of $S$ is defined as the *virtual number* of independent degree $n-4$ hypersurfaces adjoint to $S^\flat$, that is one *defines* $p_a(S)$ by the value (5.15.4) of $p_g(S)$ under the postulation that the linear series cut out on $D$ by $|\mathcal{O}_{\mathbf{P}^3}(n-4)|$ is complete and non-special (hence the name "postulation formula").

Cohomology provides a better understanding of this definition: in fact the exact sequence (5.15.2) gives

$$\chi\big(\mathcal{O}_{\mathbf{P}^3}(n-4) \otimes \mathcal{I}_D\big) = \chi\big(\mathcal{O}_{\mathbf{P}^3}(n-4)\big) - \chi\big(\mathcal{O}_D(n-4)\big) \tag{5.15.5}$$

$$= \tbinom{n-3}{3} - 1 + \rho + 2t + (n-4)d \tag{5.15.6}$$

regardless of any assumption on the linear series cut out on $D$ by $|\mathcal{O}_{\mathbf{P}^3}(n-4)|$. This leads to the cohomological definition of the arithmetic genus, as I claim that

$$\chi\big(\mathcal{O}_{\mathbf{P}^3}(n-4) \otimes \mathcal{I}_D\big) = \chi(\mathcal{O}_S) - 1. \tag{5.15.7}$$

To prove this claim, consider the exact sequence

$$0 \to \omega_{\mathbf{P}^3} \cong \mathcal{O}_{\mathbf{P}^3}(n-4) \otimes \mathcal{I}_{S^\flat} \to \mathcal{O}_{\mathbf{P}^3}(n-4) \otimes \mathcal{I}_D \to \mathcal{O}_{S^\flat}(n-4) \otimes \mathcal{I}_D \to 0,$$

with $\mathcal{I}_{S^\flat}$ the sheaf of ideals defining $S^\flat$ in $\mathbf{P}^3$. It implies

$$\chi\big(\mathcal{O}_{\mathbf{P}^3}(n-4) \otimes \mathcal{I}_D\big) = -1 + \chi\big(\mathcal{O}_S(n-4) \otimes \mathcal{I}_D\big).$$

Next, one has $\chi(S^\flat, \mathcal{O}_{S^\flat}(n-4) \otimes \mathcal{I}_D) = \chi(S, \omega_S)$ by Lemma (5.14), and eventually $\chi(S, \omega_S) = \chi(S, \mathcal{O}_S)$ by Serre duality. This proves the claim, and makes the link between the classical and modern definitions of the arithmetic genus of a surface.

## 5.3 – The 2-dimensional case: multiple planes

**(5.16)** For any $r \leqslant 3$, given the way we have set up the situation in (5.1) we may assume that $|V|$ is a linear subspace of the space of hyperplane sections of a surface $S^\flat \subset \mathbf{P}^3$ as in (5.3). Equivalently, the morphism $S \to \mathbf{P}^r$ given by $|V|$ is the projection of $S^\flat$ from a $(3-r)$-dimensional linear subspace of $\mathbf{P}^3$. We shall thus deduce our formulas from the study carried out in Subsection 5.1.

**(5.17)** As a warm up, we note that the formula given in (5.2) for the class of a 1-dimensional linear system may be recovered with the formula for the class in (5.9). Indeed, the discriminant of the 1-dimensional $|V|$ is merely the section of that of the 3-dimensional $|V|$ by a line in $\check{\mathbf{P}}^3$. The identity

$$c_2(S) + n + 4\pi - 4 = n + 4\pi + 12p_a(S) - K_S^2 + 8$$

is equivalent to the Noether formula stated in (5.10.1).

**(5.18)** We now concentrate on the $r = 2$ case. The linear system defines an $n : 1$ covering $S \to \mathbf{P}^2$ with branch curve $B \subset \mathbf{P}^2$, which we assume to be the projection of $S^\flat \subset \mathbf{P}^3$ from a point $p$. We consider:
— $N$ the degree of the branch curve $B \subset \mathbf{P}^2$ ;
— $P$ the geometric genus of $B$ ;
— $\kappa$ the number of points $q \in S$ such that there exist $C, C' \in |V|$ intersecting with multiplicity $\geqslant 3$ at $q$ ;
— $\iota$ the number of cuspidal members of $|V|$ ;
— $\delta$ the number of unordered pairs $q, q' \in S$ such that there exist $C, C' \in |V|$ tangent at both $q$ and $q'$ ;
— $\beta$ the number of 2-nodal members of $|V|$.
Of course the dual of $B$ is the discriminant of $|V|$, so the class of $B$ is $\check{n}$, which always assume the same value as in (5.2) and (5.9), see (5.17). Moreover, $\delta$, $\kappa$, $\beta$, $\iota$ are respectively the numbers of nodes, cusps, bitangents, flexes of $B$. Note that $B$ and its dual have no other singularities by our generality assumption.

**(5.19) Characters of the circumscribed curve of $S^\flat$.** The branch curve $B$ is the image of the circumscribed curve $\mathcal{C}_p S^\flat = (\mathrm{D}_p S^\flat \cap S^\flat) - D$ by the map $S^\flat \to \mathbf{P}^2$, and in fact $\mathcal{C}_p S^\flat$ is the normalization of $B$.

This implies that $N$ and $P$ are respectively the degree and genus of $\mathcal{C}_p S^\flat$, and since we have computed that its class in $S$ is $K_S + 3C$ one finds:
— $N = (K_S + 3C) \cdot C = 2n + 2\pi - 2$ ;
— $P = 9\pi + K_S^2 - 8$.

$$\begin{aligned} 2P - 2 = (2K + 3C) \cdot (K + 3C) &= 2K^2 + 9C^2 + 9K \cdot C \\ &= 2K^2 + 9n + 9(2\pi - 2 - n) \\ &= 2(K^2 + 9\pi - 9) \end{aligned}$$

**(5.20) Conclusion.** Since we know in addition the class $\check{n}$ of $B$, we may derive all remaining numerical characters by applying the Plücker formulae to $B$:

$$\begin{aligned} 2\delta + 3\kappa &= N(N-1) - \check{n}; & 2\beta + 3\iota &= \check{n}(\check{n}-1) - N; \\ 2\delta + 2\kappa &= (N-1)(N-2) - 2P; & 2\beta + 2\iota &= (\check{n}-1)(\check{n}-2) - 2P; \end{aligned}$$

(while the identities on the first line are genuinely Plücker relations, those on the second line come from the formula for the genus of a plane curve with nodes and cusps, written for $B$ and its dual which both have the same geometric genus $P$). One thus finds:
— $\delta = 2\big[(n+\pi)^2 - 5n - 17\pi - 2K_S^2 + 6p_a(S) + 22\big]$ ;
— $\kappa = 3(n + 6\pi + K_S^2 - 4p_a(S) - 10)$ ;
— $\beta = \frac{1}{2}\big[(n + 4\pi - K_S^2 + 12p_a(S) - 1)^2 + 15n - 6\pi - 17K_S^2 + 132p_a(S) + 57\big]$ ;

— $\iota = 24(\pi + p_a(S))$.

$$\kappa = 2N + 2P - \check{n} - 2; \qquad \iota = 2\check{n} + 2P - N - 2;$$
$$2\delta = (N-1)(N-6) + 2\check{n} - 6P; \qquad 2\beta = (\check{n}-1)(\check{n}-6) + 2N - 6P.$$

```
nn:=n+4*pi+12*p-K+8;
N:=2*n+2*pi-2;
P:=9*pi+K-8;

kappa:=2*N+2*P-nn-2;
                              3 (n + 6 pi - 10 + K - 4 p)

delta:=1/2*((N-1)*(N-6)+2*nn-6*P);
1/2*%-(n+pi)^2;
                          -2 K - 5 n + 6 p - 17 pi + 22

iota:=2*nn+2*P-N-2;
                                   24 pi + 24 p

beta:=1/2*((nn-1)*(nn-6)+2*N-6*P);
%-(n+4*pi+12*p-K-1)^2;
                         -17 K + 15 n + 132 p - 6 pi + 57
```

**(5.21) Number of cuspidal curves in a net.** It is possible to derive directly the quantity $\iota$ by projective methods. We end this section by a sketch on how this goes.

The parabolic curve of $S^\flat$ and its circumscribed curve with respect to $p$ intersect at the pinch points, and at those points $q$ on the parabolic curve such that the tangent plane at $q$ contains $p$; the number of the latter points is the quantity $\iota$.

To find the intersection number of these two curves we compute the class of the pull-back of the parabolic curve on $S$. A local computation shows that the Hessian hypersurface of $S^\flat$ contains the double curve $D$ with multiplicity 4. Since the degree of the Hessian is $4(n-2)$, one finds as in (5.5) the class $4K_S + 8C$.

On the other hand, we claim that the two curves intersect with multiplicity 2 at each pinch point. Indeed if $q$ is a pinch point, with preimage $q' \in S$, the members of $|V|$ passing through $q'$ form a pencil of curves all singular at $q'$. The condition to be cuspidal in such a pencil is that the determinant of the $2 \times 2$ matrix of second derivatives vanishes, hence this is a degree 2 condition. We thus find 2 cuspidal curves at $q'$ in $|V|$, hence the claim.

Eventually, we have

$$(K_S + 3C) \cdot (4K_S + 8C) = \iota + 2\tau, \tag{5.21.1}$$

which indeed gives $\iota = 24(\pi + p_a)$ with the expression for $\tau$ in (5.9).

# A – Polarity

We give here a brief recap on polarity, so that the reader unfamiliar with this may conveniently consult the relevant material. We refer the reader to [10, Chapter 1] and [2, § 5.4 and 5.6] for more details.

**(A.1) The polar pairing.** Let $f$ be a complex homogeneous degree $d$ polynomial in $n+1$ variables $(x_0,\dots,x_n)$. For $\mathbf{a}=(a_0,\dots,a_n)\in\mathbf{C}^{n+1}$, we define

(A.1.1)
$$\mathrm{D}_a f = a_0\partial_0 f + \cdots a_n\partial_n f,$$

and for $\mathbf{a}_1,\dots,\mathbf{a}_k\in\mathbf{C}^{n+1}$, we let

(A.1.2)
$$\mathrm{D}_{\mathbf{a}_1\cdots\mathbf{a}_k} f = (\mathrm{D}_{\mathbf{a}_1}\circ\cdots\circ\mathrm{D}_{\mathbf{a}_k})(f).$$

Extending (A.1.2) by $\mathbf{C}$-linearity, one obtains a perfect bilinear pairing

$$\begin{aligned}\mathrm{D}:\mathrm{Sym}^k(\mathbf{C}^{n+1})\otimes\mathrm{Sym}^d(\check{\mathbf{C}}^{n+1}) &\longrightarrow \mathrm{Sym}^{d-k}(\check{\mathbf{C}}^{n+1})\\ \xi\otimes f &\longmapsto \mathrm{D}_\xi f\end{aligned}$$

extending the natural pairing between $\mathbf{C}^{n+1}$ and $\check{\mathbf{C}}^{n+1}$.

**(A.2) Polar hypersurfaces.** Consider the hypersurface $X=V(f)\subset\mathbf{P}^n$. For $\hat{a}=(a_0,\dots,a_n)\in\mathbf{C}^{n+1}$, and $k\in\mathbf{N}$,

$$\mathrm{D}_{\hat{a}^k} f = \big(a_0\partial_0+\cdots+a_n\partial_n\big)^k f,$$

may be viewed as a bihomogeneous polynomial of bidegree $(k,d-k)$ in the variables $(a_0,\dots,a_n)$ and $(x_0,\dots,x_n)$. The hypersurface $V(\mathrm{D}_{\hat{a}^k}f)$ depends only on $X$ and the point $a=(a_0:\ldots:a_n)\in\mathbf{P}^n$, see [2, Thm. 5.4.2]; we call it the *$k$-th polar of $X$ with respect to $a$*, and denote it by $\mathrm{D}_{a^k}X$. We will often abuse notation and consider $\hat{a}$ and $a$ without distinction.

We shall also use the following useful notation: $\mathrm{D}^kX(a)$, referred to as the *polar $k$-ic of $X$ at $a$*, is the hypersurface defined by the degree $k$ polynomial in the variables $(x_0,\dots,x_n)$,

$$\mathrm{D}^k f(\hat{a}) = \big(x_0\partial_0+\cdots x_n\partial_n\big)^k f(a_0,\dots,a_n).$$

It follows from Euler formula that if $a\in X$, then $a\in\mathrm{D}_{a^k}X$ for all $k\geqslant 1$. Also the converse holds, as follows.

**(A.3) Proposition.** *Assume $d=\deg(f)>1$, and let $a\in\mathbf{P}^n$. Then,*

$$a\in X \iff \exists k\in[\![1,d-1]\!] : a\in\mathrm{D}_{a^k}X.$$

*Proof.* One has $\mathrm{D}_{a^k}f(a)=d(d-1)\cdots(r-k+1)f(a)$ by Euler formula. □

**(A.4) Proposition** (polar symmetry)**.** *Consider $X\subset\mathbf{P}^n$ a degree $d$ hypersurface. Let $a,b\in\mathbf{P}^n$, and $k\in[\![1,d-1]\!]$. One has the equivalence:*

$$a\in\mathrm{D}_{b^k}X \iff b\in\mathrm{D}_{a^{d-k}}X.$$

This says that $\mathrm{D}^kX(a)=\mathrm{D}_{a^{d-k}}X$. In case $a$ is a singular point of $\mathrm{D}_{b^k}X$, we have the following. Let $k,l$ be positive integers such that $k+l<d$. If $a$ is a point of multiplicity $\geqslant d-k-l+1$ of $\mathrm{D}_{b^l}X$, then $b$ is a point of multiplicity $\geqslant d-k-l+1$ of $\mathrm{D}_{a^k}X$. The proof is mere polynomial calculus.

**(A.5)** If $b\in X=V(f)$ is a smooth point, then $\mathrm{D}^1f(b)$ is “the” linear homogeneous polynomial defining the tangent hyperplane to $X$ at $b$. Therefore, for all $a\in\mathbf{P}^n$,

$$X\cap\mathrm{D}_aX=\{x\in X : \mathbf{T}_xX\ni a\}.$$

This generalizes to the following fundamental property.

If $X\subset\mathbf{P}^n$ is a hypersurface and $\ell\subset\mathbf{P}^n$ a line, for all $p\in\ell$ we let $i(X,\ell)_p$ be the multiplicity with which $p$ appears in $X\cap\ell$.

**(A.6) Theorem.** *Let $X$ be a degree $d$ hypersurface, $a \in X$, and $b \in \mathbf{P}^n$. For all integer $s \geqslant 0$, one has*

$$\begin{aligned} i\big(X, \langle a, b\rangle\big)_a \geqslant s+1 \qquad &\Longleftrightarrow \qquad \forall k \leqslant s, \quad b \in D_{a^{d-k}} X \\ &\Longleftrightarrow \qquad \forall k \leqslant s, \quad a \in D_{b^k} X. \end{aligned}$$

It turns out that for $a \in X$, all polars of $X$ with respect to $a$ (equivalently, all $\mathrm{D}^k X(a)$) are tangent at $a$. Actually, $X$ and its polar $k$-ic at $a$, $\mathrm{D}^k X(a)$ have the same polar $s$-ics at $a$ for all $s \leqslant k$, as the following identities show:

$$\mathrm{D}^s\big(\mathrm{D}^k X(a)\big)(a) = \mathrm{D}_{a^{k-s}}\big(\mathrm{D}^k X(a)\big) = \mathrm{D}_{a^{k-s}}\big(\mathrm{D}_{a^{d-k}} X\big) = \mathrm{D}_{a^{d-s}} X = \mathrm{D}^s X(a).$$

This has the following remarkable consequence.

**(A.7) Corollary.** *Let $X$ be a hypersurface in $\mathbf{P}^n$, and $a$ a point of $X$. The $n$ hypersurfaces $\mathrm{D}^1 X(a), \ldots, \mathrm{D}^n X(a)$ intersect with multiplicity at least $n!$ at $a$.*

*Proof.* We consider only the case in which all intersections are complete, otherwise the result is trivial. Then the intersection $\mathrm{D}^1 X(a), \ldots, \mathrm{D}^{n-1} X(a)$ consists of $(n-1)!$ lines each intersecting $X$ with multiplicity at least $n$ at $a$, as follows from Theorem (A.6). Since the polar $k$-ics at $a$ of $\mathrm{D}^n X(a)$ are the same as those of $X$ as indicated above, each of these lines also intersect $\mathrm{D}^n X(a)$ with multiplicity at least $n$ at $a$, and the result follows. $\square$

The polar hyperplane $\mathrm{D}^1 X(a)$ is well-defined only if $a$ is a smooth point of $X$ (otherwise its equation is 0, and the more reasonable thing to do is to set $\mathrm{D}^1 X(a) = \mathbf{P}^n$). When $a$ is singular, the following holds.

**(A.8) Theorem.** *Let $X$ be a degree $d$ hypersurface, and $a \in X$ a point of multiplicity $m$. We consider an integer $r \leqslant d - m$.*
*(A.8.1) The polar hypersurface $\mathrm{D}_{a^r} X$ has multiplicity $m$ at $a$, and it has the same tangent cone at $a$ as $X$: $\mathrm{TC}_a(\mathrm{D}_{a^r} X) = \mathrm{TC}_a(X)$.*
*(A.8.2) Let $b \in \mathbf{P}^n - \{a\}$. The polar hypersurface $\mathrm{D}_{b^r} X$ has multiplicity $\geqslant m - r$ at $a$; this multiplicity is exactly $m - r$ for general $b \in \mathbf{P}^n$, and in this case the tangent cone of $\mathrm{D}_{b^r} X$ at $a$ equals the $r$-th polar with respect to $b$ of the tangent cone $\mathrm{TC}_a(X)$:*

$$\mathrm{TC}_a\big(\mathrm{D}_{b^r} X\big) = \mathrm{D}_{b^r}\big(\mathrm{TC}_a(X)\big).$$

In particular, (A.8.2) tells us that $\mathrm{D}_{b^r} X$ contains $a$ if $r \leqslant m-1$, and is singular at $a$ if $r \leqslant m-2$.

We end this section by recalling the following definition.

**(A.9) Hessian of a hypersurface.** Let $X$ be a hypersurface in $\mathbf{P}^n$, defined by the homogeneous polynomial $f \in \mathbf{C}[x_0, \ldots, x_n]$. The *Hessian hypersurface* of $X$ is the hypersurface $\mathrm{Hess}(X)$ defined by the homogeneous polynomial

$$\mathrm{Hess}(f) = \det\big(\partial_i \partial_j f\big)_{0 \leqslant i,j \leqslant n}.$$

Using the Euler formula and the standard properties of determinants, one may obtain the alternative expression

$$\text{(A.9.1)} \qquad \operatorname{Hess}(f) = \frac{(d-1)^2}{x_0^2} \begin{vmatrix} \frac{d}{d-1} \cdot f & \partial_1 f & \cdots & \partial_n f \\ \partial_1 f & \partial_1^2 f & \cdots & \partial_1 \partial_n f \\ \vdots & \vdots & \ddots & \vdots \\ \partial_n f & \partial_1 \partial_n f & \cdots & \partial_n^2 f \end{vmatrix}.$$

It is useful to study the intersection of $X$ and $\operatorname{Hess}(X)$.

# B – Developable surfaces

In this apendix we review the basic theory of developable surfaces in the projective space of dimension 3. following the beautiful [11, Chap. V].

## B.1 – Synthetic description

**(B.1) Definition.** A *developable surface* $S \subset \mathbf{P}^3$ is a ruled surface such that the tangent plane to $S$ is constant along the lines of the rulings. In other words, for all line $\Lambda$ of the ruling, and for all $p, p' \in \Lambda$, one has $\mathbf{T}_p S = \mathbf{T}_{p'} S$. An important aspect of developable surfaces is that they are the surfaces that can be unfolded into the plane with no distortion; we will however not consider this idea any farther in this text.

In classical terminology, if $S$ is a ruled surface, the lines of the rulings are called *generators*, and a generator $\Lambda$ is said to be *torsal* if the tangent plane to $S$ is constant along $\Lambda$. Thus developable surfaces are those ruled surfaces for which all generators are torsal.

**(B.2)** Developable surfaces in $\mathbf{P}^3$ are exactly those surfaces having their dual which is not a hypersurface, see [28, Thm. 1.18]. It therefore follows from biduality that any non-degenerate developable surface is projective dual to a curve $C$ in $\check{\mathbf{P}}^3$. In particular it has an ordinary double curve, corresponding to bitangent planes to $C$, and a cuspidal double curve, corresponding to osculating planes to $C$ (unless $C$ is degenerate, in which case $S$ is a cone).

(We say that a plane $H$ is bitangent [resp. osculating] to $C$ if the intersection cycle $H \cap C$ contains a subcycle of the form $2p + 2q$ [resp. $3p$], $p, q \in C$).

This description may be refined as follows.

**(B.3)** The following fact refines the description of (B.2); a detailed proof can be found in [2, 5.8.14]. Also [19, §30] is very enlightening. Let $S$ be a developable surface in $\mathbf{P}^3$. On every generator there is a singular point, at which $S$ does not have a well-defined tangent plane. If $S$ is not a cone, the locus of the singular points of the various generators is a curve $\Gamma$, called the *regression edge*, which is none other than the cuspidal double curve of $C^\vee$ mentioned in (B.2). At a general point $p \in \Gamma$, the tangent line $\mathbf{T}_p\Gamma$ is the generator of $S$, and the osculating plane $\operatorname{Osc}_p\Gamma$ is the plane tangent to $S$ along the generator $\mathbf{T}_p\Gamma$.

This tells us that $S$ is the *tangential surface* of $\Gamma$, namely

$$S = \operatorname{Tan}(\Gamma) := \overline{\bigcup_{p \in \Gamma_{\mathrm{sm}}} \mathbf{T}_p \Gamma}.$$

Moreover, the curve $C = S^\vee \subset \check{\mathbf{P}}^3$ is what I call the *oscdual* curve of $\Gamma \subset \mathbf{P}^3$, namely

$$C = \overline{\bigcup_{p \in \Gamma^\circ} (\operatorname{Osc}_p \Gamma)^\perp},$$

where $\Gamma^\circ$ denotes the open subset of $\Gamma$ of points $p$ such that $\mathrm{Osc}_p\Gamma$ is a plane, i.e., $\Gamma$ and $\mathbf{T}_p\Gamma$ intersect with multiplicity exactly 2 at $p$.

The dual of $\Gamma$ is a developable surface in $\check{\mathbf{P}}^3$, the generators of which are the lines $\mathbf{T}_p\Gamma^\perp$ for all smooth points $p \in \Gamma$. It follows from biduality and the above description that this is the tangential surface of the curve $C$. In particular, $\Gamma$ is the oscdual curve of $C$: this is a reflexivity statement for oscduality.

## B.2 – Analysis of contacts

We pursue our analysis of developable surfaces, building upon the description given in the previous subsection (in particular, we keep the same notation), and following the general philosophy of biduality. We leave aside for the moment the peculiar behaviour that the curve $\Gamma \subset \mathbf{P}^3$ may have at isolated points in some special situations.

**(B.4) Characteristic numbers, I.** We let
— $m = \deg \Gamma$ ;
— $n = \deg C$ ;
— $r = \deg\big(\mathrm{Tan}(\Gamma)\big)$.

Classically, developable surfaces were considered as systems packaging together three 1-dimensional families in $\mathbf{P}^3$ of planes, points, and lines respectively; these are the planes parametrized by points of $C$, the points of $\Gamma$, and the tangent lines of $\Gamma$. In this perspective, $m$ is the number of points of the system contained in a general plane of $\mathbf{P}^3$, $n$ is the number of planes of the system passing through a general point of $\mathbf{P}^3$, and $r$ is the number of lines of the system meeting a general line of $\mathbf{P}^3$. The integers $m, n, r$ are called respectively the order, class, and rank of the system.

**(B.5)** Knowing the family of planes given by $C \subset \check{\mathbf{P}}^3$, one may find the generators of the developable surface $\mathrm{Tan}(\Gamma) \subset \mathbf{P}^3$ by taking the intersections in $\mathbf{P}^3$ of pairs of infinitely near members of $C$.

Indeed, the generators of $\mathrm{Tan}(\Gamma)$ are the orthogonals of the tangent lines of $C$. Now for $p, q \in C$, the orthogonal of the line $\langle p, q\rangle \subset \check{\mathbf{P}}^3$ is the intersection of the two planes $p^\perp$ and $q^\perp$ in $\mathbf{P}^3$. Letting $q$ tend to $p$ yields the asserted statement.

**(B.6)** Let $o$ and $H$ be a general point and a general plane in $\mathbf{P}^3$. The duality between projections and hyperplanes sections (see [28, Thm. 1.21]) tells us on the one hand that the projection of $\Gamma$ from $o$ is dual as a plane curve to the hyperplane section of $\Gamma^\vee = \mathrm{Tan}(C)$ by $o^\perp$, and on the other hand that the section of $S = \mathrm{Tan}(\Gamma)$ by $H$ is dual to the projection of $C$ by $H^\perp$. This may be established using the following elementary statement, see Proposition (B.8).

**(B.7) Lemma.** *Let $\varpi \in \check{\mathbf{P}}^3$ be a point off $C$, and consider the projection $\pi_\varpi : C \to \check{\mathbf{P}}^3/\varpi$ from $\varpi$.*
(i) *An ordinary (resp. osculating) tangent plane $H$ at $p \in C$ gives rise to an ordinary tangent (resp. a flex tangent) line $\pi_\varpi(H)$ at $\pi_\varpi(p)$ to the plane curve $\pi_o(C)$.*
(ii) *A point $p \in C$ such that the line $\langle \varpi, p\rangle$ is tangent to $C$ gives rise to a cusp of $\pi_\varpi(C)$ at $\pi_\varpi(p)$.*

*Proof.* This is a local computation that i will eventually include. □

*(B.7.1)* Part (ii) of the lemma tells us that if we project from $C$ a point $\varpi$ lying on $\mathrm{Tan}(C)$, then $\pi_\varpi(C)$ acquires one more cusp than if $\varpi$ were general. Accordingly the number of flex points of $\pi_\varpi(C)$ drops by 2: indeed, the number of flexes of a plane curve is given by its intersection number with its Hessian minus $6\delta + 8\kappa$ (in the usual notation), and acquiring one more cusp means passing from $(\delta, \kappa)$ to $(\delta - 1, \kappa + 1)$ (note that the geometric genus of $\pi_\varpi(C)$ remains unchanged, so "acquiring one more cusp" actually means that a node has become a cusp).

**(B.8) Proposition.** *The section of* $\mathrm{Tan}(\Gamma)$ *by a plane* $H$ *is the dual curve of the plane curve* $\pi_{H^\perp}(C)$.

*Proof.* Each point of $\mathrm{Tan}(\Gamma) \cap H$ is a point $y = H \cap \mathbf{T}_x\Gamma$ for some $x \in \Gamma$. By biduality, the orthogonal line $(\mathbf{T}_x\Gamma)^\perp$ is the tangent to $C$ at the point $(\mathrm{Osc}_x C)^\perp$. It follows that the plane $y^\perp$ is tangent to $C$, as it contains $(\mathbf{T}_x\Gamma)^\perp$; as it passes through $H^\perp$, its projection is a tangent line of $\pi_{H^\perp}(C)$. Conversely, all tangent lines to $\pi_{H^\perp}(C)$ are obtained in this way. □

*(B.8.1)* We now observe that everything fits well together. The plane curve $H \cap \mathrm{Tan}(\Gamma)$ has cusps at the points where $H$ meets the cuspidal edge of $\mathrm{Tan}(\Gamma)$, which is $\Gamma$ itself. Its dual $\pi_{H^\perp}(C)$ correspondingly has flex points given by the osculating planes to $C$ passing through $H^\perp$, i.e., the planes $x^\perp$ for $x \in H \cap \Gamma$.

**(B.9) Proposition.** *The two tangential surfaces* $\mathrm{Tan}(\Gamma)$ *and* $\mathrm{Tan}(C)$ *have the same degree.*

*Proof.* Let $p \in \mathbf{P}^3$ be a point off $\mathrm{Tan}(\Gamma)$. The projected curve $\pi_p(\Gamma)$ is the plane dual of the hyperplane section $p^\perp \cap \Gamma^\vee = \mathrm{Tan}(C)$, so the degree of $\mathrm{Tan}(C)$ is the number of lines tangent to $\pi_p(\Gamma)$ passing through a fixed point $q \in \mathbf{P}^3/p$, equivalently the number of planes tangent to $\Gamma$ containing the line $\langle p, q\rangle \subset \mathbf{P}^3$. Now the planes tangent to $\Gamma$ are those containing a generator of $\mathrm{Tan}(\Gamma)$. Hence the number of planes tangent to $\Gamma$ and containing $\langle p, q\rangle$ equals the number of generators of $\mathrm{Tan}(\Gamma)$ meeting $\langle p, q\rangle$, which is none other than the degree of $\mathrm{Tan}(\Gamma)$. □

*(B.9.1) Remark.* Proposition (B.9) is a twisted version of the fact that the class of a (non-developable) ruled surface equals its degree.

To compare the two situations, let's recall the proof. Let $S \subset \mathbf{P}^3$ be a ruled surface, and assume $\check{S}$ is a surface. The class of $S$ is the number of planes tangent to $S$ containing an arbitrary prescribed line $\Lambda \subset \mathbf{P}^3$. Since $\check{S}$ is a surface, the planes tangent to $S$ are exactly the planes that contain a ruling of $S$. It follows that the class of $S$ is the number of rulings of $S$ that meet $\Lambda$. But the latter number also equals the degree of $S$.

If $S$ is developable, it is no longer true that its tangent planes are those containing a ruling (indeed there is only one tangent plane per ruling), and the number of planes tangent to $S$ containing a line $\Lambda$ is zero, as $S^\vee$ actually has codimension 2 in $\check{\mathbf{P}}^3$.

It is useful to analyze the sections of the developable surface by planes containing a generator.

**(B.10) Lemma.** *Let the hyperplane* $H \subset \mathbf{P}^3$ *contain the line* $\mathbf{T}_x\Gamma$ *for some* $x \in \Gamma$*, and write* $H \cap \mathrm{Tan}(\Gamma)$ *as* $\mathbf{T}_x\Gamma + R$. *In general, the residual section* $R$ *has one more flex and two less cusps than the general plane section of* $\mathrm{Tan}(\Gamma)$. *The extra flex point is* $x$*, and* $\mathbf{T}_x R = \mathbf{T}_x\Gamma$.

*Proof.* The point $H^\perp$ sits on the line $(\mathbf{T}_x\Gamma)^\perp$ which is tangent to $C$ at the point $(\mathrm{Osc}_x\Gamma)^\perp$, so the curve $\pi_{H^\perp}(C)$ has one more cusp and two less flexes than in the general situation, by Lemma (B.7) (ii), and the comment after it. Dually, this gives the first assertion about $R$. Note one may also verify with similar considerations that the class of $\pi_{H^\perp}(C)$ drops by one with respect to the general projection.

The tangent cone to $\pi_{H^\perp}(C)$ at its extra cusp is the projection from $H^\perp$ of the plane $\mathrm{Osc}_{(\mathrm{Osc}_x\Gamma)^\perp}C$; its orthogonal is the extra flex of $R$, and it is the point $x$. By biduality, the tangent to $R$ at $x$ is the orthogonal of the cuspidal point of $\pi_{H^\perp}(C)$; the latter is the projection from $H^\perp$ of the line $(\mathbf{T}_x\Gamma)^\perp$, hence $\mathbf{T}_xR$ equals $\mathbf{T}_x\Gamma$. □

**(B.11) Corollary.** *The degree of the ordinary double curve of* $\mathrm{Tan}(\Gamma)$ *equals* $r-4$.

*Proof.* Consider the section of $\mathrm{Tan}(\Gamma)$ by a plane $H$ containing a generator $\mathbf{T}_x\Gamma$ as in Lemma (B.10): it consists of the line line $\mathbf{T}_x\Gamma$ and a residual curve $R$ of degree $r-1$. They meet with multiplicity 3 at $x$, and at $r-4$ further points $x_1,\dots,x_{r-4}$. □

Being $\mathrm{Tan}(\Gamma)$ ruled, its double points lie on two generators: if they are distinct then the double point is on the ordinary double curve, otherwise it is on the cuspidal edge. The points $p_1,\dots,p_{r-4}$ from the above proof are the intersection points of $\mathbf{T}_p\Gamma$ and the ordinary double curve of $\mathrm{Tan}(\Gamma)$ (in particular, they do not move as $H$ varies in the pencil of planes containing $\mathbf{T}_p\Gamma$). For all $i=1,\dots,r-4$, there is a point $q_i\in\Gamma$ such that $p_i=\mathbf{T}_p\Gamma\cap\mathbf{T}_{q_i}\Gamma$. It follows that the plane $p_i^\perp\subset\check{\mathbf{P}}^3$ is tangent to $C$ at the two points $(\mathrm{Osc}_p\Gamma)^\perp$ and $(\mathrm{Osc}_{q_i}\Gamma)^\perp$, which explains why it should be an ordinary double point of the surface $\mathrm{Tan}(\Gamma)=C^\vee$ according to the general biduality principles.

**(B.12) Lemma.** *The section of* $\mathrm{Tan}(\Gamma)$ *by one of its tangent planes decomposes as* $2\mathbf{T}_x\Gamma+R$ *for some* $x\in\Gamma$*, and the curve* $R$ *is tangent to* $\mathbf{T}_x$ *at* $x$.

*Proof.* Consider the plane $H$ tangent to $\mathrm{Tan}(\Gamma)$ along the generator $\mathbf{T}_x\Gamma$. Then $H=\mathrm{Osc}_x\Gamma$, and the point $H^\perp\in\check{\mathbf{P}}^3$ sits on the curve $C$. Necessarily $H\cap\mathrm{Tan}(\Gamma)=2\mathbf{T}_x\Gamma+R$, and $R$ is the plane dual of the projected curve $\pi_{H^\perp}(C)$. The latter in general is smooth at the point image of $H^\perp$, namely $\pi_{H^\perp}(\mathbf{T}_{H^\perp}C)$, with tangent line $\pi_{H^\perp}(\mathrm{Osc}_{H^\perp}C)$. This gives $x=(\mathrm{Osc}_{H^\perp}C)^\perp\in R$, and $\mathbf{T}_xR=(\mathbf{T}_{H^\perp}C)^\perp=\mathbf{T}_x\Gamma$. □

*(B.12.1) Moreover, the curve $R$ passes through all the ordinary double points $p_1,\dots,p_{r-4}$ of $Tan(\Gamma)$ along its generator $\mathbf{T}_p\Gamma$.*

Indeed, recalling that $p_i=\mathbf{T}_p\Gamma\cap\mathbf{T}_{q_i}\Gamma$, we have seen that the plane $p_i^\perp$ passes through $H^\perp=(\mathrm{Osc}_p\Gamma)^\perp$ and is tangent to $C$ at $(\mathrm{Osc}_{q_i}\Gamma)^\perp$. Thus $\pi_{H^\perp}(p_i^\perp)$ is the tangent line to $\pi_{H^\perp}(C)$ at $\pi_{H^\perp}((\mathrm{Osc}_{q_i}\Gamma)^\perp)$, hence $p_i\in R$ since the latter is the plane dual of $\pi_{H^\perp}(C)$. □

*(B.12.2) Remark.* It is interesting to convince oneself that the degree of $R$ is indeed $r-2$. Let $g$ be the genus of $C$. When it is projected from a general point of $\mathbf{P}^3$, it gives a nodal degree $n$ plane curve of genus $g$, which thus has $\delta=\frac{1}{2}(n-1)(n-2)-g$ nodes, hence class $r=n(n-1)-2\delta$. If however it gets projected from a general point on itself, it still gives a genus $g$ nodal curve, but with degree $n-1$, so that the number of nodes becomes $\delta'=\frac{1}{2}(n-2)(n-3)-g=\delta-(n-2)$. The class of this curve is thus $r-2$ as we want.

**(B.13) Remark.** Consider a generator $\Lambda=\mathbf{T}_p\Gamma$. Then the pencil of curves cut out on $\mathrm{Tan}(\Gamma)$ by planes containing $\Lambda$ is a family of plane curves with prescribed contact $3p+p_1+\cdots+p_{r-4}$ with $\Lambda$, which degenerates to a curve containing $\Lambda$ itself. This is exactly the situation considered by Caporaso and Harris in [4] and studied in the first half of this volume.

## B.3 – Plücker formulae

We now explain how the Plücker formulae for plane curves may be used to derive relations between the numerical characters of a developable surface, following an original idea of Cayley [5]. As was already the case for plane curves, it turns out that the appropriate setup is to let the curves $\Gamma$ and $C$ have various kinds of singularities. We thus consider a slightly more general situation than what we did in Subsection B.2, by allowing the stationary behaviours indicated below.

**(B.14) Characteristic numbers, II.** In addition to the quantities introduced in (B.4), we let
— $\alpha$ the number of ordinary cusps of $C$, i.e., points at which $C$ has a local parametrization of the form $t \mapsto (t^2, t^3, t^4)$ ;
— $\beta$ the number of ordinary cusps of $\Gamma$ ;
— $x$ the degree of the ordinary double curve of $\mathrm{Tan}(\Gamma)$;
— $y$ the degree of the ordinary double curve of $\mathrm{Tan}(C)$;
— $g$ the number of apparent double points of $C$;
— $h$ the number of apparent double points of $\Gamma$.
The quantities $\alpha$ and $\beta$ are classically referred to respectively as the number of stationary planes and points of the system. The quantity $x$ is the number of points in a general plane $H \subset \mathbf{P}^3$ that lie on two distinct tangent lines of $\Gamma$, see (B.11); dually, $y$ is the number of planes passing through a general point, and containing two distinct tangent lines of $\Gamma$. The quantity $g$ is the number of lines lying in a general plane $H \subset \mathbf{P}^3$ that are contained in two osculating planes of $\Gamma$, and $h$ is merely the number of lines passing through a general point of $\mathbf{P}^3$ and meeting $\Gamma$ in two points.

**(B.15) Lemma.** *Consider the osculatory Gauss map $\Gamma \to C$, mapping a general point $p \in \Gamma$ to the point $(\mathrm{Osc}_p\Gamma)^\perp \in \check{\mathbf{P}}^3$. It sends a point where $\Gamma$ has a local parametrization of the form $t \mapsto (t, t^2, t^4)$ (resp. $t \mapsto (t^2, t^3, t^4)$) to a point where $C$ has a local parametrization of the form $t \mapsto (t^2, t^3, t^4)$ (resp. $t \mapsto (t, t^2, t^4)$).*

The upshot is that the osculatory Gauss map turns points of types $\alpha$ and $\beta$ of $\Gamma$ respectively to points of types $\beta$ and $\alpha$ of $C$.

*Proof.* This is a local computation: to a local branch of $\Gamma$ parametrized by $t \in \mathbf{C} \mapsto f(t) \in \mathbf{C}^4$, the osculatory Gauss maps associates the local branch of $C$ parametrized by $t \in \mathbf{C} \mapsto \big(f \wedge \partial_t f \wedge \partial_t^2 f\big)(t) \in \bigwedge^3 \mathbf{C}^4 \cong \check{\mathbf{C}}^4$. In the former case, the result follows from the identity

$$\begin{pmatrix} 1 \\ t \\ t^2 \\ t^4 \end{pmatrix} \wedge \begin{pmatrix} 0 \\ 1 \\ 2t \\ 4t^3 \end{pmatrix} \wedge \begin{pmatrix} 0 \\ 0 \\ 2 \\ 12t^2 \end{pmatrix} = \begin{pmatrix} -6t^4 & 16t^3 & -12t^2 & 2 \end{pmatrix}.$$

We leave the computation in the latter case to the reader. □

**(B.16)** Let us apply the Plücker formulae to a general hyperplane section $H \cap \mathrm{Tan}(\Gamma)$, which is a plane curve dual to $\pi_{H^\perp}(C)$. The former is a curve of degree $r$, with $x$ nodes and $m$ cusps, and the latter has degree $n$ and $g$ nodes; for general $H$, its only cusps come from cusps of $C$, which are in the number $\alpha$. We thus get the following identities:

$$\begin{aligned} n &= r(r-1) - 2x - 3m; & r &= n(n-1) - 2g - 3\alpha; \\ \alpha &= 3r(r-2) - 6x - 8m; & m &= 3n(n-2) - 6g - 8\alpha, \end{aligned}$$

hence

$$\alpha - m = 3(n-r); \qquad 2(g-x) = (n-r)(n+r-9).$$

**(B.17)** Dually, one may apply the Plücker formulae to the pair of dual plane curves $\pi_o(\Gamma)$ and $o^\perp \cap \mathrm{Tan}(C)$ for a general point $o \in \mathbf{P}^3$. This gives:

$$\begin{aligned} r &= m(m-1) - 2h - 3\beta; \qquad & m &= r(r-1) - 2y - 3n; \\ n &= 3m(m-2) - 6h - 8\beta; \qquad & \beta &= 3r(r-2) - 6y - 8n, \end{aligned}$$

hence

$$n - \beta = 3(r-m) \qquad 2(y-h) = (r-m)(r+m-9).$$

**(B.18)** Eventually, one may combine the identities from (B.16) and (B.17) to obtain:

$$\alpha - \beta = 2(n-m), \qquad x - y = n - m,$$
$$2(g-h) = (n-m)(n+m-7).$$

**(B.19)** It seems natural to allow the system to have stationary lines as well, i.e., the curve $\Gamma$ to have points at which it is locally parametrized by $t \mapsto (t, t^3, t^4)$. We do this only in a second time, for leaving this feature aside allows a more straightforward application of the Plücker formulae. A local computation as in the proof of (B.15) shows that the osculatory Gauss map sends a point at which $\Gamma$ has a stationary tangent line to a point at which $C$ has a stationary tangent line.

The essential modification induced by the presence of stationary lines is the content of the following statement. One may take this into account to modify in a suitable way the previous applications of the Plücker formulae; we leave this to the reader.

*(B.19.1) The stationary lines of the system give cuspidal lines to both tangential surfaces* $\mathrm{Tan}(\Gamma)$ *and* $\mathrm{Tan}(C)$.

*Proof.* Let $p \in \Gamma$ be a point at which $\Gamma$ is locally parametrized by $t \mapsto (t, t^3, t^4)$, and $\Lambda := \mathbf{T}_p\Gamma$ be the corresponding (stationary) generator. We shall prove that for a general plane $H \subset \mathbf{P}^3$, the section $H \cap \mathrm{Tan}(\Gamma)$ has a cusp at the point $H \cap \Lambda$, or equivalently the curve $\pi_{H^\perp}(C)$ has a flex at $\pi_{H^\perp}\big((\mathrm{Osc}_p\Gamma)^\perp\big)$ along the line $\pi_{H^\perp}(\Lambda^\perp)$.

As we have seen, the curve $C$ has a local parametrization of the form $t \mapsto (t, t^3, t^4)$ at the point $(\mathrm{Osc}_p\Gamma)^\perp$. Thus after projection from the general point $H^\perp$, the curve $\pi_{H^\perp}(C)$ has a local parametrization of the form $t \mapsto (t, t^3)$. The result follows since $\Lambda^\perp$ is the tangent line to $C$ at $(\mathrm{Osc}_p\Gamma)^\perp$. □

## B.4 – Generalized Plücker formulae

In this last subsection I outline the generalization of the previous formulae to curves in a projective space of arbitrary dimension, which are due to Piene. This is not used in the course of the text, but it would be a pity not to mention this.

**(B.20)** So far we have only considered extrinsic invariants of the curve $\Gamma \subset \mathbf{P}^3$. Adding its geometric genus $p_g$ in the picture, one finds:

$$r = 2m + 2p_g - 2 - \beta. \tag{B.20.1}$$

*Proof.* Let $\Lambda \subset \mathbf{P}^3$ be a general line. We want to compute the number of tangent lines to $\Gamma$ meeting $\Lambda$, or equivalently the number of planes tangent to $\Gamma$ and containing $\Lambda$. To this end we consider the projection from $\Lambda$ and the degree $n$ covering $\bar{\Gamma} \to \mathbf{P}^1$ it induces, where $\bar{\Gamma}$ is the normalization of $\Gamma$. By the Riemann–Hurwitz formula, it has $2n + 2g - 2$ ramification points, of which $\alpha$ come from the ramification points of the normalization map of $C$. The other ones correspond to planes tangent to $\Gamma$ and containing $\Lambda$. $\square$

*(B.20.2) Remark.* The same reasoning may be applied to the curve $C$; one gets in this way the identity

$$r = 2n + 2p_g - 2 - \alpha.$$

Indeed, the osculatory Gauss map gives a birationality between $\Gamma$ and $C$, so these two curves have the same geometric genus, and $r$ is the degree of both $\mathrm{Tan}(C)$ and $\mathrm{Tan}(\Gamma)$ by Prop. (B.9).

Putting the two identities together, one finds the formula $2(n - m) = \alpha - \beta$ which we have already gotten in (B.18).

**(B.21) Question.** What kind of stationary points does a general curve in $\mathbf{P}^3$ have? This question doesn't make any sense if one doesn't first decide what is a genral curve in $\mathbf{P}^3$.
▷ voir "Ordinary Ramification Theorem" [12, Thm. 2].

### Curves in a projective space of higher dimension

**(B.22) New setup.** From now on, we consider a curve $\Gamma \subset \mathbf{P}^N$ of degree $m$ and geometric genus $p_g$. Let $\bar{\Gamma} \to \Gamma$ be its normalization, and $f : \bar{\Gamma} \to \mathbf{P}^N$ its composition with $\Gamma \hookrightarrow \mathbf{P}^N$. For all $p \in \Gamma$ one may choose a local parameter $t$ at $p$ on $\bar{\Gamma}$ and a system of homogeneous coordinates $(x_0 : x_1 : \ldots : x_N)$ on $\mathbf{P}^N$ such that $f : \bar{\Gamma} \to \mathbf{P}^N$ is given locally by

$$t \mapsto \left(1 : t^{1+s_1} : \ldots : t^{i+s_1+\cdots+s_i} : \ldots\right)$$

up to highesr order terms, where $s_1, \ldots, s_N$ are non-negative integers depending only on $p$ (i.e., the $i$-th coordinate of $f$ is a formal power series in $t$ whose leading term is $t^{i+s_1+\cdots+s_i}$). The point $p$ is *ordinary* if $s_1 = \cdots = s_N = 0$, and *stationary* or *hyperosculating* otherwise.

For all $i = 0, \ldots, N-1$, the $i$-th osculating subspace of $\Gamma$ at $p \in \bar{\Gamma}$, denoted $\mathrm{Osc}^i_p(\Gamma)$, is the $(N - i)$-plane in $\mathbf{P}^N$ defined by the equations $x_N = \cdots = x_{N-i+1} = 0$ in a system of homogeneous coordinates as above. We shall consider the $i$-th *osculating variety* of $\Gamma$, denoted $\mathrm{Osc}^i(\Gamma)$, that is the union of the $i$-th osculating subspaces of $\Gamma$ at all the points of $\bar{\Gamma}$. For $i = 0$ this is $\Gamma$ itself, for $i = 1$ this is the tangential surface of $\Gamma$ that we have considered in the previous subsections; for $i = N - 1$ this fills the whole $\mathbf{P}^N$, hence it is best seen as a curve $C \subset \check{\mathbf{P}}^N$ which I call the *osc-dual* curve of $\Gamma$.

**(B.23) The $i$-th ranks.** For all $i = 0, \ldots, N-1$, let $r_i$ be $i$-th rank of $\Gamma$, that is the degree of its $i$-th osculating variety; for $i = N - 1$ this is by definition the degree of the osc-dual curve $C \subset \check{\mathbf{P}}^N$. We define integers $k_i$ by

$$k_i := \sum\nolimits_{p\in\Gamma} s_i(p),$$

where $s_i(p)$ is defined as in (B.22).

A standard computation on the Chern classes of the bundles of principal parts of the embedding of $\bar{\Gamma}$ in $\mathbf{P}^N$, see [10, Prop. 10.4.13], gives the following formulae:

(B.23.1) $$r_i = (i+1)\big(m + i(p_g - 1)\big) - \sum_{j=1}^{i} \big((i-j+1)k_j\big) \qquad \text{for all } i = 0, \dots, N-1$$

(B.23.2) $$\sum_{j=1}^{N} \big((N-j+1)k_j\big) = (N+1)\big(m + N(p_g - 1)\big).$$

Formula (B.23.1) for $i = 0$ is trivial, as $r_0 = \deg(\Gamma) = m$, and formula (B.23.2) has to be understood as the incarnation for $i = N$ of formula (B.23.1), setting $r_N = 0$. For $i = 1$, one gets a formula which is exactly (B.20.1). Eventually, note that formula (B.23.2) may be interpreted as counting the stationary points of $\Gamma$, with the multiplicity

$$s_N + 2s_{N-1} + \cdots + (N-1)s_2 + Ns_1.$$

For $i = 1, 2$, this gives

(B.23.3) $$r_1 = 2(m + p_g - 1) - k_1,$$

(B.23.4) $$r_2 = 3(m + 2p_g - 2) - 2k_1 - k_2.$$

**(B.24) Generalized Plücker formulae.** As a straightforward corollary of the formulas in (B.23), one has

(B.24.1) $$r_{i-1} - 2r_i + r_{i+1} = 2p_g - 2 - k_{i+1}, \qquad \text{for all } i = 0, \dots, N-1,$$

with the convention that $r_N = r_{-1} = 0$.

*(B.24.2) Plane curves.* For $N = 2$, we are considering a plane curve $\Gamma$ of degree $m$. The number $k_1$ (resp. $k_2$) counts, with multiplicities, the cusps (resp. the flexes) of $\Gamma$, so we will call it $\kappa$ (resp. $\iota$) as usual. Formula (B.24.1) gives the two identities

$$-2m + \check{m} = 2p_g - 2 - \kappa \ ; \qquad m - 2\check{m} = 2p_g - 2 - \iota.$$

Assuming $\Gamma$ has only $\delta$ nodes and $\kappa$ cusps as singularities, these may be recovered from the usual formulae

$$\check{m} = m(m-1) - 2\delta - 3\kappa \ ; \qquad 2p_g - 2 = m(m-3) - 2\delta - 2\kappa,$$

where the first one is the computation of the class using polars for a singular curve as in (1.12), and the second one is the adjunction formula for a singular plane curve as in part on Enriques?. For instance, one gets the first one by writing

$$\check{m} - 2m = m(m-1) - 2\delta - 3\kappa - 2m = m(m-3) - 2\delta - 3\kappa = 2g - 2 - \kappa.$$

*(B.24.3) Space curves.* For $N = 3$ we are considering a space curve $\Gamma \subset \mathbf{P}^3$ as in the previous subsections. In the notation therein, the ranks $r_0, r_1, r_2$ are respectively the degree $m$, the rank $r$, and the class $n$. On the other hand, $k_1, k_2, k_3$ are respectively the number $\beta$ of cusps of $\Gamma$, the number of stationary lines (let us call it $\gamma$), and the number $\alpha$ of cusps of $C$. Formula (B.24.1) gives the three identities

$$-2m + r = 2p_g - 2 - \beta \ ; \qquad r - 2n = 2p_g - 2 - \alpha.$$
$$m - 2r + n = 2p_g - 2 - \gamma \ ;$$

**(B.25) Piene duality** [23, Thm. 5.1]**.** *Let $\Gamma \subset \mathbf{P}^N$ be a curve, and $C \subset \check{\mathbf{P}}^N$ its osc-dual curve. The osc-dual curve of $C$ is $\Gamma$, and the following relations hold:*

$$r_i(C) = r_{N-i-1}(\Gamma) \tag{B.25.1}$$

$$k_i(C) = k_{N-i-1}(\Gamma). \tag{B.25.2}$$

Formula (B.25.1) generalizes the fact that for $N = 3$ the two tangential surfaces $\mathrm{Tan}(\Gamma)$ and $\mathrm{Tan}(C)$ have the same degree, see Prop. (B.9), and Formula (B.25.2) the fact that the osc-Gauss map turns stationary planes, lines and points of $\Gamma$ into stationary points, lines, and planes of $C$ respectively, see Lemma (B.15).

**(B.26) Interpretation as strata of the dual hypersurface.** Underlying the merely numerical relations of (B.25) is the following geometric duality, generalizing (B.3).

For all $i > 0$, we say that a hyperplane $H \subset \mathbf{P}^N$ is *$i$-osculating* to $\Gamma$ at $p \in \bar{\Gamma}$ if it contains $\mathrm{Osc}^i_p(\Gamma)$. If $p \in \bar{\Gamma}$ is an ordinary point of $\Gamma$, this is equivalent to the divisor $f^*H - (i+1)p$ on $\bar{\Gamma}$ being effective. We let $\Gamma^\vee_i \subset \check{\mathbf{P}}^N$ be the closed subset parametrizing hyperplanes $i$-osculating to $\Gamma$ at some point on $\bar{\Gamma}$; thus $\Gamma^\vee_1$ is the plain dual hypersurface $\Gamma^\vee$, and $\Gamma^\vee_{N-1}$ is the osc-dual curve $C$.

One has $\Gamma^\vee_i = \mathrm{Osc}^{N-1-i}(C)$ for all $i = 1, \ldots, N-1$, as follows from biduality see statement = Dimca?. Moreover, one has $(\mathrm{Osc}^i(\Gamma))^\vee = \Gamma^\vee_{i+1}$. In fact, for $p \in \bar{\Gamma}$ the linear space $\mathrm{Osc}^i_p(\Gamma)^\perp$ is the $(N-1-i)$-th osculating space of $C$ at $p$ (note that the map $p \in \bar{\Gamma} \mapsto [\mathrm{Osc}^{n-1}_p(\Gamma)] \in \check{\mathbf{P}}^N$ is the normalization of $C$).

Thus if we let $f_i : \bar{\Gamma} \to \mathbf{G}(\mathbf{P}^N, i)$ (resp. $f^\vee_{N-1-i} : \bar{\Gamma} \to \mathbf{G}(\check{\mathbf{P}}^N, N-1-i)$) be the map to sending $p \in \bar{\Gamma}$ to the $i$-th osculating space to $\Gamma$ at $p$ (resp. to the $(N-1-i)$-th osculating space of $C$ at $p$) seen as a point of the Grassmannian, then $f^\vee_{N-1-i}$ equals the composition of $f_i$ with the orthogonality map $\Lambda \in \mathbf{G}(\mathbf{P}^N, i) \mapsto \Lambda^\perp \in \mathbf{G}(\check{\mathbf{P}}^N, N-1-i)$. It follows that the two curves $f_i(\bar{\Gamma})$ and $f^\vee_{N-1-i}(\bar{\Gamma})$ have the same degree under the Plücker embeddings of $\mathbf{G}(\mathbf{P}^N, i)$ and $\mathbf{G}(\check{\mathbf{P}}^N, N-1-i)$ respectively, and these are $r_i(\Gamma)$ and $r_{N-1-i}(C)$ respectively (see, e.g., [17, Example 19.11]).

## B.5 – De Jonquières' formula

The next result is even more general than the previous ones. See [1, Chap. VIII] for a proof and pointers to the previous literature.

**(B.27) Theorem** (de Jonquières' formula)**.** *Let $m$ be a positive integer, and $m_1, \ldots, m_a$ non-negative integers such that $m = \sum_{1 \leqslant s \leqslant a} s \cdot m_s$. The virtual number of divisors having $m_s$ points of multiplicity $s$ for all $s = 1, \ldots, a$ in a given linear series of degree $m$ and dimension $i = \sum_{1 \leqslant s \leqslant a} (s-1) \cdot m_s$ on a smooth, genus $g$ curve is the coefficient of the $t_1^{m_1} \cdots t_a^{m_a}$ term in the formal power series*

$$\left(1 + 1^2 t_1 + 2^2 t_2 + \cdots + a^2 t_a\right)^g \cdot \left(1 + 1\, t_1 + 2t_2 + \cdots + a t_a\right)^{m-i-g}.$$

**(B.28) Link with the $i$-th ranks.** We now show how the de Jonquières' formula may be used to derive the $i$-th ranks of a curve $\Gamma \subset \mathbf{P}^N$ of degree $m$ and geometric genus $g$ as in (B.22), thus recovering formula (B.23.1).

The key observation is that by (B.25) and (B.26),

$$r_i(\Gamma) = r_{N-1-i}(C) = \deg\bigl(\mathrm{Osc}^{N-1-i}(C)\bigr) = \deg\bigl(\Gamma_i^\vee\bigr),$$

hence $r_i(\Gamma)$ is the number of hyperplanes $i$-osculating to $\Gamma$ in a general $i$-dimensional linear subsystem of $|\mathcal{O}_\Gamma(1)|$.

By de Jonquières' formula, the number of hyperplanes $H$ such that $f^*H-(i+1)p$ is effective for some $p \in \bar{\Gamma}$ in a general $i$-dimensional linear subsystem of $|\mathcal{O}_\Gamma(1)|$ is the coefficient of the monomial $t_1^{m-i-1}t_{i+1}$ in

$$\bigl(1+1^2t_1+2^2t_2+\cdots+(i+1)^2t_{i+1}\bigr)^g \cdot \bigl(1+1\,t_1+2t_2+\cdots+(i+1)t_{i+1}\bigr)^{m-i-g}.$$

This is also the coefficient of $t_1^{m-i-1}t_{i+1}$ in

$$\begin{aligned}\bigl(1+1^2t_1+(i+1)^2t_{i+1}\bigr)^g &\cdot \bigl(1+1\,t_1+(i+1)t_{i+1}\bigr)^{m-i-g}\\ &= \bigl(1+1^2t_1+[(i+1)+i(i+1)]t_{i+1}\bigr)^g \cdot \bigl(1+1\,t_1+(i+1)t_{i+1}\bigr)^{m-i-g}\\ &= \sum_{j=0}^{g}\binom{g}{j}\bigl(1+t_1+(i+1)t_{i+1}\bigr)^{(g-j)+m-i-g}\cdot\bigl(i(i+1)t_{i+1}\bigr)^j,\end{aligned}$$

which amounts to the coefficient of $t_1^{m-i-1}t_{i+1}$ in

$$\bigl(1+t_1+(i+1)t_{i+1}\bigr)^{m-i}+i(i+1)t_{i+1}\cdot g\bigl(1+t_1+(i+1)t_{i+1}\bigr)^{m-i-1},$$

that is

$$(m-i)(i+1)+i(i+1)\cdot g = (i+1)\bigl(m+(g-1)i\bigr). \tag{B.28.1}$$

To obtain $r_i$ from the number (B.28.1) one ought to subtract the excess contributions of the various stationary points of $\Gamma$. For $i=1$ for instance, for all cusp $p$ of $\Gamma$ (i.e., for all $p \in \bar{\Gamma}$ such that $s_1(p)>0$), the whole $p^\perp \subset \check{\mathbf{P}}^N$ gives rise to hyperplanes contributing to (B.28.1), so that the latter number actually is the degree of the reducible hypersurface sum of $\Gamma_1^\vee$ and the various $p^\perp$ for all cusps $p$ of $\Gamma$. One thus finds $r_1$ from (B.28.1) as in (B.20).

For $i=2$, also the points $p \in \bar{\Gamma}$ such that $s_1(p)=0$ and $s_2(p)>0$ contribute to (B.28.1), to the effect that $(\mathrm{Osc}_p^1(\Gamma))^\perp$ occurs together with $\Gamma_2^\vee$. In general, all $p \in \bar{\Gamma}$ such that $s_j(p)>0$ for some $j \leqslant i$ contribute to (B.28.1), and to conclude we need to show that each contributes with the multiplicity

$$s_i + 2\,s_{i-1}+\cdots+i\,s_1$$

in de Jonquières' formula. I have not found this particular computation in the literature, and will leave this as an open exercise to the reader.

# C – Biduality for asymptotic tangents by enumerative means

Let $\Lambda$ be a line (asymptotic) tangent to a surface $S$ at a smooth point $p$. In this appendix, we identify the line $\Lambda^\perp$ as an (asymptotic) tangent to the dual surface $S^\vee$ by computing the intersection multiplicity of $\Lambda^\perp$ and $S^\vee$ at the point $(\mathbf{T}_pS)^\perp$. To do so we consider the pencil of hyperplane sections parametrized by $\Lambda^\perp$ and count its singular members with a topological formula on Euler numbers we have encoutered several times before, see [8, subsec. `s:pencil`]. Then the intersection multiplicity we are looking for is the multiplicity of the tangent section $\mathbf{T}_pS \cap S$ as a singular member of this pencil (indeed the singular members of $\Lambda^\perp$ correspond to the intersection of $\Lambda^\perp$ with $S^\vee$). This gives the following lemma.

**(C.1) Lemma.** *Let $S \subset \mathbf{P}^3$ be a smooth surface, $\Lambda$ be a line, and $H$ be a hyperplane containing $\Lambda$. Consider $\tilde{S} \to S$ the minimal sequence of blow-ups at reduced points such that the proper transform $\tilde{\Lambda}$ of the pencil $\Lambda^\perp$ is base-point-free. Let $\tilde{C}_H$ and $\tilde{C}_{\mathrm{gen}}$ be the members of $\tilde{\Lambda}$ corresponding respectively to $C_H := H \cap S$ and a general member $C_{\mathrm{gen}}$ of $\Lambda^\perp$. Then the intersection multiplicity of $\Lambda^\perp$ and $S^\vee$ at $H^\perp$ equals the difference in Euler numbers $e(\tilde{C}) - e(\tilde{C}_{\mathrm{gen}})$.*

The assumption that $S$ is smooth is only to simplify matters and may be removed without having to change the method. The lemma follows from a direct application of [8, Lemma `l:pencil`] to the morphism $\tilde{S} \to \mathbf{P}^1$ given by $\tilde{\Lambda}$.

With this tool at hand, we shall prove the following biduality statement.

**(C.2) Theorem.** *Let $S \subset \mathbf{P}^3$ be a smooth surface, and $p \in S$. The orthogonal line to*
*(i) a tangent at $p$;*
*(ii) one of the two asymptotic tangents at a non-parabolic $p$;*
*(iii) the asymptotic tangent at a general parabolic point $p$;*
*(iv) the asymptotic tangent at a parabolic point $p$ of type $\beta$;*
*is*
*(i) a tangent at $(\mathbf{T}_pS)^\perp$;*
*(ii) one of the two asymptotic tangents at $(\mathbf{T}_pS)^\perp$;*
*(iii) the tangent at $(\mathbf{T}_pS)^\perp$ of the cuspidal double curve of $S^\vee$;*
*(iv) the common reduced tangent cone at $(\mathbf{T}_pS)^\perp$ of both the ordinary and cuspidal double curves of $S^\vee$.*

*Proof.* We proceed to a case by case analysis, along the lines described at the beginning of the appendix. We use similar notation (and abuse of notation) in all cases; we introduce it in the first case, and use it freely in the others.

(i) Let $\Lambda$ be a general tangent line to $S$, and call $p$ the contact point. Then all members of $\Lambda^\perp$ are tangent at $p$, and we have to blow-up twice at $p$ to resolve the base points, that is we first consider $\varepsilon_1 : S_1 \to S$ the blow-up at $p$, with exceptional divisor $E_1$, then $\varepsilon_2 : S_2 \to S_1$ the blow-up at the point $p_1 \in E_1$ corresponding to the tangent direction at $p \in S$ defined by $\Lambda$, with exceptional divisor $E_2$.

Locally over $p \in S$, the $\tilde{S} \to S$ of Lemma (C.1) is $\varepsilon = \varepsilon_2 \circ \varepsilon_1 : S_2 \to S$. Let $C$ be the hyperplane class on $S$. Forgetting about the base points of $\Lambda^\perp$ away from $p$, the base-point-free pencil $\tilde{\Lambda}$ of Lemma (C.1) is the complete linear system $|\varepsilon^* C - \varepsilon_2^* E_1 - E_2|$. Our task now is to compute the transform of the curve $C_0 = S \cap \mathbf{T}_pS$ in this pencil.

Since $C_0$ has a double point at $p$, we have

$$\varepsilon_1^* C_0 = C_0 + 2E_1,$$

where by abuse of notation $C_0$ also denotes the proper transform of $C_0$ in $S_1$. Next, one has

$$\varepsilon^* C_0 = \varepsilon_2^*(C_0 + 2E_1) = C_0 + 2E_1 + 2E_2,$$

with the same abuse of notation this time for both $C_0$ and $E_1$. Eventually, we find

$$\tilde{C}_0 = \varepsilon^* C_0 - \varepsilon_2^* E_1 - E_2 = \varepsilon^* C_0 - (E_1 + E_2) - E_2 = C_0 + E_1.$$

This is a reduced curve with two ordinary double points, hence it counts with multiplicity 2 as a singular member of $\Lambda^\perp$ (see [8, subsec. `s:pencil`] for the details of the computation). It follows that $\Lambda^\perp$ is tangent to $S^\vee$ at $(\mathbf{T}_pS)^\perp$.

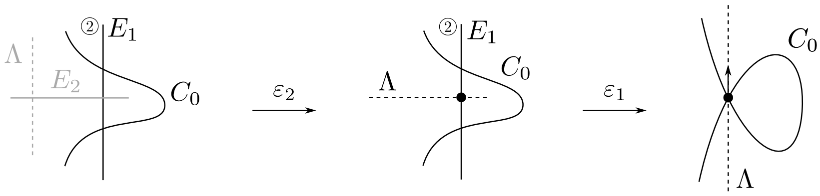


Figure 4: Pencil with base locus an ordinary tangent line $\Lambda$

(ii) Let $\Lambda$ be an asymptotic tangent at a general point $p$. Then the members of $\Lambda^\perp$ are all osculating at the order 3 at $p$ (in fact, we know they all have a flex along $\Lambda$), and we need to blow-up three times to resolve the base points, with exceptional divisors $E_3, E_2, E_1$. The base-point-free pencil is

$$|\varepsilon^* C - \varepsilon_3^* \varepsilon_2^* E_1 - \varepsilon_3^* E_2 - E_3| = |\varepsilon^* C - \varepsilon^* E_1 - \varepsilon^* E_2 - E_3| = |\varepsilon^* C - E_1 - 2E_2 - 3E_3|,$$

where we introduce yet another abuse of notation, to the effect that $\varepsilon$ denotes whatever the appropriate composition of the various $\varepsilon_i$'s is.

Now the curve $C_0 = S \cap \mathbf{T}_p S$ has an ordinary double point at $p$, with one local branch simply tangent to $\Lambda$. We thus have successively

$$\begin{aligned} \varepsilon_1^* C_0 &= C_0 + 2E_1 \\ \varepsilon_2^* \varepsilon_1^* C_0 &= C_0 + 2E_1 + 3E_2 \\ \varepsilon_3^* \varepsilon_2^* \varepsilon_1^* C_0 &= C_0 + 2E_1 + 3E_2 + 3E_3, \end{aligned}$$

so that

$$\tilde{C}_0 = \varepsilon^* C_0 - E_1 - 2E_2 - 3E_3 = C_0 + E_1 + E_2$$

is reduced with 3 nodes, hence counts with multiplicity 3 as a singular member of $\Lambda^\perp$. This shows that $\Lambda^\perp$ is an asymptotic tangent to $S^\vee$.

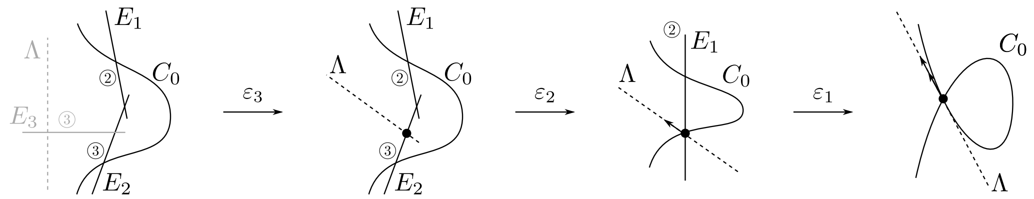


Figure 5: Pencil with base locus an asymptotic tangent line $\Lambda$

(iii) Let $\Lambda$ be an asymptotic tangent at a general parabolic point $p$. The situation is the same as in case (ii), except that the tangent section $C_0 = S \cap \mathbf{T}_p S$ now has a cusp at $p$, with tangent cone supported on $\Lambda$. We have

$$\begin{aligned} \varepsilon_1^* C_0 &= C_0 + 2E_1 \\ \varepsilon_2^* \varepsilon_1^* C_0 &= C_0 + 2E_1 + 3E_2 \\ \varepsilon_3^* \varepsilon_2^* \varepsilon_1^* C_0 &= C_0 + 2E_1 + 3E_2 + 3E_3 \end{aligned}$$

as in case (ii), so that again the relevant curve is

$$\tilde{C}_0 = \varepsilon^* C_0 - E_1 - 2E_2 - 3E_3 = C_0 + E_1 + E_2.$$

But this time it has a triple point, hence counts with multiplicity 4 by [8, Lemma `l:pencil-mult`].

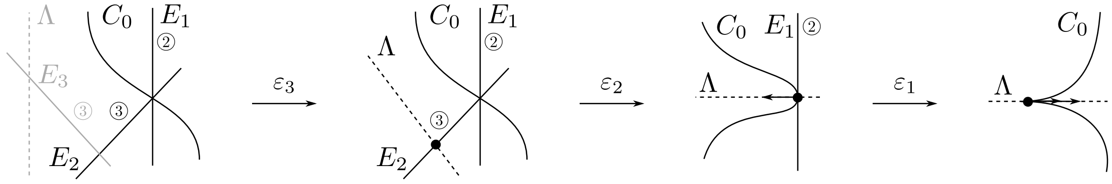


Figure 6: Pencil with base locus the double asymptotic tangent $\Lambda$ at a parabolic point

The point $\Lambda^\perp$ is on the cuspidal double curve of $S^\vee$. A general line through $\Lambda^\perp$ intersects $S^\vee$ with multiplicity 2, a general line in the tangent cone of $S^\vee$ with multiplicity 3, and the tangent line of the cuspidal double curve with multiplicity 4. Hence $\Lambda^\perp$ is the latter line.

(iv) Eventually, we consider a point $p$ such that $C_0 = S \cap \mathbf{T}_p S$ has a tacnode at $p$, and $\Lambda$ the asymptotic tangent of $S$ at $p$, which is also the support of the tangent cone of $C_0$ at $p$. The base-point-free pencil is defined as in cases (ii) and (iii). The computations now give

$$\begin{aligned} \varepsilon_1^* C_0 &= C_0 + 2E_1 \\ \varepsilon_2^*\varepsilon_1^* C_0 &= C_0 + 2E_1 + 4E_2 \\ \varepsilon_3^*\varepsilon_2^*\varepsilon_1^* C_0 &= C_0 + 2E_1 + 4E_2 + 4E_3 \end{aligned}$$

so that

$$\tilde{C}_0 = \varepsilon^* C_0 - E_1 - 2E_2 - 3E_3 = C_0 + E_1 + 2E_2 + E_3.$$

This time the curve is not reduced. However this is not a reason for the topological formula behind Lemma (C.1) to fail, and the latter is still valid. The only caution is that the computation of $e(\tilde{C}_0)$ is no longer liable for [8, Lemma `l:pencil-mult`]; we compute it directly as the Euler number of $(\tilde{C}_0)_{\mathrm{red}}$, as indeed the topological formula doesn't see multiplicities.

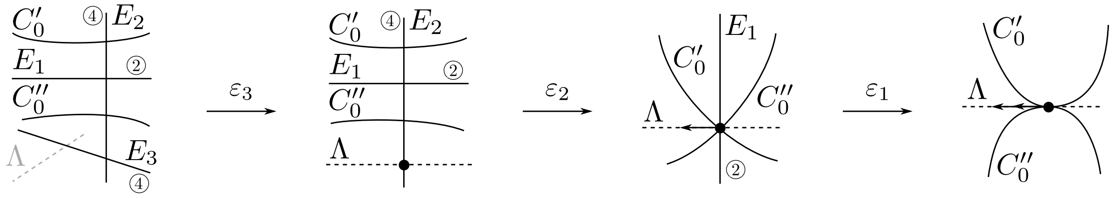


Figure 7: Pencil with base locus the double asymptotic tangent $\Lambda$ at a swallowtail point

Since the curve $C_0 \subset S$ has a tacnode, its proper transform in $\tilde{S}$ has genus $g-2$, where $g$ stands for the genus of a general member of $\Lambda^\perp$. It then follows from the additivity of the Euler number as in the proof of [8, Lemma `l:pencil-mult`], that

$$\begin{aligned} e(C_0 + E_1 + E_2 + E_3) &= e(C_0) + e(E_1) + e(E_2) + e(E_3) - 4 \\ &= \big(2 - 2(g-2)\big) + 2 + 2 + 2 - 4 = e(C_{\mathrm{gen}}) + 6. \end{aligned}$$

One then argues as in case (iii), remembering the local swallowtail model of $S^\vee$ at $(\mathbf{T}_p S)^\perp$ given in (3.1). The point $(\mathbf{T}_p S)^\perp$ is triple for $S^\vee$, and the only line through this point intersecting $S^\vee$ with multiplicity strictly larger than 4 is the support of the tangent cones of the double curves of $S^\vee$. □

*(C.2.1) Remark.* In the swallowtail model of (3.1), the support of the tangent cones of the double curves of $S^\vee$ is the line $b = c = 0$, which is straight out contained in the discriminant hypersurface (3.1.2). Of course it is not the case however that in the notation of (iv), the line $\Lambda^\perp$ is contained $S^\vee$.

## References


[1] E. Arbarello, M. Cornalba, P. A. Griffiths and J. Harris, *Geometry of algebraic curves. Vol. I*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 267, Springer-Verlag, New York, 1985.

[2] M. Beltrametti, E. Carletti, D. Gallarati and G. Monti Bragadin, *Lectures on curves, surfaces and projective varieties – a classical view of algebraic geometry*, EMS Textbooks in Mathematics, European Mathematical Society, 2009, Original version issued in 2003.

[3] L. Busé and T. Dedieu, *Generalized weight properties of resultants and discriminants, and applications to projective enumerative geometry*, Chapter in this volume.

[4] L. Caporaso and J. Harris, *Counting plane curves of any genus*, Invent. Math. **131** (1998), no. 2, 345–392.

[5] A. Cayley, *Mémoire sur les courbes à double courbure et les surfaces développables*, Journal de mathématiques pures et appliquées 1re série **10** (1845), 245–250.

[6] C. Ciliberto and F. Flamini, *On the branch curve of a general projection of a surface to a plane*, Trans. Amer. Math. Soc. **363** (2011), no. 7, 3457–3471.

[7] P. Corvaja and F. Zucconi, *The surface of gauss double points*, prepublication arXiv:2008.02012.

[8] T. Dedieu, *Enumerative geometry of $K3$ surfaces*, in this volume.

[9] A. Dimca, *Topics on real and complex singularities*, Advanced Lectures in Mathematics, Friedr. Vieweg & Sohn, Braunschweig, 1987, An introduction.

[10] I. V. Dolgachev, *Classical algebraic geometry*, Cambridge University Press, Cambridge, 2012, A modern view.

[11] W. L. Edge, *The theory of ruled surfaces*, Cambridge University Press, 1931, `http://www.maths.ed.ac.uk/cheltsov/edge2013/read.html`.

[12] D. Eisenbud and J. Harris, *Divisors on general curves and cuspidal rational curves*, Invent. Math. **74** (1983), no. 3, 371–418.

[13] F. Enriques, *Le superficie algebriche*, N. Zanichelli, Bologna, 1949 (Italian).

[14] G. Fischer, *Plane algebraic curves*, Student Mathematical Library, vol. 15, American Mathematical Society, Providence, RI, 2001, Translated from the 1994 German original by Leslie Kay.

[15] I. M. Gelfand, M. M. Kapranov and A. V. Zelevinsky, *Discriminants, resultants and multidimensional determinants*, Modern Birkhäuser Classics, Birkhäuser Boston, Inc., Boston, MA, 2008, Reprint of the 1994 edition.

[16] L. Gruson and C. Peskine, *On the smooth locus of aligned Hilbert schemes, the $k$-secant lemma and the general projection theorem*, Duke Math. J. **162** (2013), no. 3, 553–578.

[17] J. Harris, *Algebraic geometry*, Graduate Texts in Mathematics, vol. 133, Springer-Verlag, New York, 1995, A first course.

[18] R. Hartshorne, *Algebraic geometry*, Springer-Verlag, New York-Heidelberg, 1977, Graduate Texts in Mathematics, No. 52.

[19] D. Hilbert and S. Cohn-Vossen, *Anschauliche Geometrie.*, vol. 37, Springer, Berlin, 1932 (German), engl. transl. available as "Geometry and the Imagination".

[20] B. Iversen, *Numerical invariants and multiple planes*, Amer. J. Math. **92** (1970), 968–996.

[21] N. M. Katz, *Pinceaux de lefschetz: théorèmes d'existence*, SGA 7 II, Exposé XVII.

[22] V. S. Kulikov and V. S. Kulikov, *Generic coverings of the plane with $A$-$D$-$E$-singularities*, Izv. Ross. Akad. Nauk Ser. Mat. **64** (2000), no. 6, 65–106.

[23] R. Piene, *Numerical characters of a curve in projective $n$-space*, in *Real and complex singularities (Proc. Ninth Nordic Summer School/NAVF Sympos. Math., Oslo, 1976)*, 1977, 475–495.

[24] ———, *Some formulas for a surface in* $\mathbf{P}^3$, in *Algebraic geometry (Proc. Sympos., Univ. Tromsø, Tromsø, 1977)*, Lecture Notes in Math., vol. 687, Springer, Berlin, 1978, 196–235.

[25] ———, *A proof of Noether's formula for the arithmetic genus of an algebraic surface*, Compositio Math. **38** (1979), no. 1, 113–119.

[26] F. Ronga, *On the number of planes triply tangent to a surface in projective space*, in *Proceedings of the 1984 Vancouver conference in algebraic geometry*, CMS Conf. Proc., vol. 6, Amer. Math. Soc., Providence, RI, 1986, 389–395.

[27] G. Salmon, *A Treatise on the Analytic Geometry of Three Dimensions, Vol. II*, Chelsea Publishing Company, New York, 1965, reprint of the fifth (latest) edition.

[28] E. A. Tevelev, *Projective duality and homogeneous spaces*, Encyclopaedia of Mathematical Sciences, vol. 133, Springer-Verlag, Berlin, 2005, Invariant Theory and Algebraic Transformation Groups, IV.

[29] C. Voisin, *Théorie de hodge et géométrie algébrique complexe*, Cours spécialisés, vol. 10, Société Mathématique de France, 2002, Engl. transl./ available.

[30] O. Zariski, *Algebraic surfaces*, Classics in Mathematics, Springer-Verlag, Berlin, 1995, With appendices by S. S. Abhyankar, J. Lipman and D. Mumford, Preface to the appendices by Mumford, Reprint of the second (1971) edition.

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