Geometry of logarithmic Severi varieties at a general point

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Abstract. This is a set of notes based on the results of Caporaso and Harris' [3, §2], taken on the occasion of the seminar *Degenerazioni e enumerazione di curve su una superficie* run at Roma "Tor Vergata" 2015–2017.

Contents

1 – Statement of results

(1.1) Let S be a nonsingular projective connected algebraic surface over the field $\mathbf{C}$ of complex numbers, and fix $R \subset S$ a reduced curve.

By a curve on S, we always mean a closed subscheme of S of pure dimension 1. The *geometric genus* of a reduced curve C is the (arithmetic) genus of its normalization $\bar{C}$, namely

$$1 - \chi(\mathcal{O}_{\bar{C}}) = \sum_{i=1}^{n} g_i - n + 1$$

where $g_1, \ldots, g_n$ are the respective genera of the connected components $C_1, \ldots, C_n$ of $\bar{C}$.

For every $\xi \in \mathrm{NS}(S)$ (i.e. ξ is a homology class on S that can be represented by a divisor) and integer g, we consider $M_g^{\xi,\mathrm{bir}}(S)$ the space parametrizing morphisms

$$\phi : C \to S$$

from a smooth genus g curve C (projective, but possibly disconnected) that are birational on their image, and such that $\phi_*[C] = \xi$.

We also consider $V_g^{\xi}(S)$ the locally closed subscheme of $\mathrm{Curve}(S)$ consisting of those points $[C]$ such that C is reduced and has geometric genus g and homology class ξ, where $\mathrm{Curve}(S)$ denotes the Hilbert scheme of curves on S.

(1.2) We denote by $\underline{\mathbf{N}}$ the set of sequences $\alpha = [\alpha_1, \alpha_2, \ldots]$ of non-negative integers with all but finitely many α_i non-zero. In practice we shall omit the infintely many zeros at the end.

For $\alpha \in \underline{\mathbf{N}}$, we let

$$|\alpha| = \alpha_1 + \alpha_2 + \cdots ;$$
$$I\alpha = \alpha_1 + 2\alpha_2 + \cdots + n\alpha_n + \cdots .$$

For $\alpha, \alpha' \in \underline{\mathbf{N}}$, we say that $\alpha \geqslant \alpha'$ if $\alpha_i \geqslant \alpha'_i$ for all $i \geqslant 1$.

By a set Ω of cardinality $\alpha \in \underline{\mathbf{N}}$, we mean a sequence of sets $\Omega = (\Omega_1, \Omega_2, \ldots)$ such that each Ω_i has cardinality α_i.

(1.3) Definition. *Let $g \in \mathbf{Z}$, $\xi \in \mathrm{NS}(S)$, $\alpha, \beta \in \underline{\mathbf{N}}$ such that*

$$I\alpha + I\beta = \xi \cdot R,$$

and consider a general set $\Omega = \big(\{p_{i,j}\}_{1 \leqslant j \leqslant \alpha_i}\big)_{i \geqslant 1}$ of α points on R. [1]

We define $M_g^\xi(\alpha, \beta)(\Omega)$ as the locally closed subset of $M_g^{\xi, bir}(S)$ consisting of those $[\phi : C \to S]$ such that the intersection $\phi(C) \cap R$ is proper and contained in the smooth locus of R, and there exist α points $q_{i,j} \in C$, $1 \leqslant j \leqslant \alpha_i$, and β points $r_{i,j} \in C$, $1 \leqslant j \leqslant \beta_i$, such that

(1.3.1) $$\forall\, 1 \leqslant j \leqslant \alpha_i : \quad \phi(q_{i,j}) = p_{i,j} \qquad \textit{and}$$

(1.3.2) $$\phi^* R = \sum_{1 \leqslant j \leqslant \alpha_i} i\, q_{i,j} + \sum_{1 \leqslant j \leqslant \beta_i} i\, r_{i,j}.$$

Remark. The above definition is functorial. In other words, $M_g^\xi(\alpha, \beta)(\Omega)$ represents a certain functor, see (2.10).

Correspondingly, we let $\mathring{V}_g^\xi(\alpha, \beta)(\Omega)$ be the locally closed subscheme of $V_g^\xi(S)$ consisting of those points $[C]$ such that the normalisation $\nu : \bar{C} \to C \subset S$ belongs to $M_g^\xi(\alpha, \beta)(\Omega)$. We call *logarithmic Severi variety* of the pair (S, R) its Zariski closure $V_g^\xi(\alpha, \beta)(\Omega)$ scheme of curves on S.

(1.3.1) *Notation, examples, and comments.* In practice we will try to find a balance between rigorous and decipherable notation. For instance we will frequently drop the Ω, and replace ξ with an adequate shorthand.

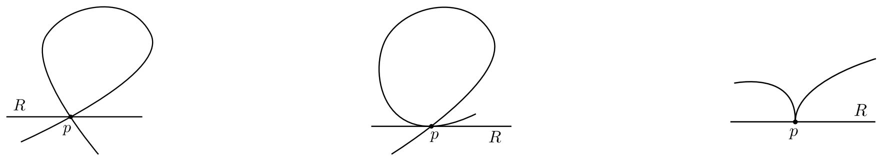
Let us consider the emblematic case when $S = \mathbf{P}^2$ and R is a line. Let $\xi = d[H]$ with $[H]$ the hyperplane class; we shall simply denote ξ by d. For all g, $V_g^d\big([0,1], d-2\big)$ is the family of plane d-ics of genus g and tangent to the line R at some prescribed general point $p \in R$ (in this case, $\Omega = \{p\}$).

Let $\mathrm{p}(d)$ be the arithmetic genus of plane curves of degree d. The "open" Severi variety $\mathring{V}_{\mathrm{p}(d)}^d\big([0,1], d-2\big)$ parametrizes smooth d-ics tangent to R at p; it has codimension 2 in the linear system $|dH|$. The "open" Severi variety $\mathring{V}_{\mathrm{p}(d)-1}^d\big([0,1], d-2\big)$ parametrizes plane d-ics of cogenus 1, tangent to R at p; it has codimension 3 in the linear system $|dH|$ (hence $V_{\mathrm{p}(d)-1}^d\big([0,1], d-2\big)$ is a divisor in $V_{\mathrm{p}(d)}^d\big([0,1], d-2\big)$), and its general member is a curve with one node at a general point of $\mathbf{P}^2$.

On the other hand the family $\mathring{W}$ of plane d-ics with a node at the point p has codimension 3 in $|dH|$ as well, and its closure W is a divisor in the "closed" Severi variety $V_{\mathrm{p}(d)}^d\big([0,1], d -$

1. If R is reducible one should specify the distribution of Ω on the various components of R. This distribution however doesn't change anything in the framework of this text, so we don't dwell on this.

2). However W is not part of $V^d_{\mathrm{p}(d)-1}\big([0,1],d-2\big)$, as Definition (1.3) requires that curves in $\mathring{V}^d_{\mathrm{p}(d)-1}\big([0,1],d-2\big)$ have a local branch tangent to R at p, as illustrated below.



not in $V^d_{p(d)-1}\big([0,1]\big)$ in $V^d_{p(d)-1}\big([0,1]\big)$ in $V^d_{p(d)-1}\big([0,1]\big)$

(1.4) Theorem. *Let $g \in \mathbf{Z}$, $\xi \in \mathrm{NS}(S)$, $\alpha, \beta \in \underline{\mathbf{N}}$, $\Omega = \{p_{i,j}\}_{1 \leqslant j \leqslant \alpha_i} \subset R$ be as in Def. (1.3), and consider an irreducible component V of $V^\xi_g(\alpha,\beta)(\Omega)$. Let $[C]$ be a general member of V, $\phi : \bar{C} \to C \subset S$ its normalization, $q_{i,j}$ $(1 \leqslant j \leqslant \alpha_i)$, $r_{i,j}$ $(1 \leqslant j \leqslant \beta_i)$ points on $\bar{C}$ such that* (1.3.1) *and* (1.3.2) *hold. Set*

$$D = \sum_{1 \leqslant j \leqslant \alpha_i} i\, q_{i,j} + \sum_{1 \leqslant j \leqslant \beta_i} (i-1)\, r_{i,j}.$$

(1.4.0) *If $-K_S \cdot C_i - \deg \phi_* D|_{C_i} \geqslant 1$ for every irreducible component C_i of C, then*

$$\dim V = -(K_S + R) \cdot \xi + g - 1 + |\beta|.$$

(1.4.1) *If $-K_S \cdot C_i - \deg \phi_* D|_{C_i} \geqslant 2$ for every irreducible component C_i of C, then*
- (a$^{\flat\flat}$) *the normalization map ϕ is an immersion, except possibly at the points $r_{i,j}$;*
- (b) *the points $q_{i,j}$ and $r_{i,j}$ of $\bar{C}$ are pairwise distinct ;*
- (c$^\flat$) *none of the points $s_{i,j} := \phi(r_{i,j})$ belongs to Ω ;*
- (d) *for every curve $G \subset S$ and finite set $\Gamma \subset S$ such that $(G \cup \Gamma) \cap \Omega = \emptyset$, if $[C]$ is general with respect to G and Γ then C intersects G transversely and does not intersect Γ.*

(1.4.2) *If $-K_S \cdot C_i - \deg \phi_* D|_{C_i} \geqslant 3$ for every irreducible component C_i of C, then*
- (a$^\flat$) *the normalization map ϕ is an immersion ;*
- (c) *the points $p_{i,j}$ and $s_{i,j} = \phi(r_{i,j})$ on C are pairwise distinct ;*
- (e) *the curve C is smooth at its intersection points with R.*

(1.4.3) *If $-K_S \cdot C_i - \deg \phi_* D|_{C_i} \geqslant 4$ for every irreducible component C_i of C, then*
- (a) *the curve C is nodal.*

Note that if $\beta = 0$, then (b) holds under the weaker condition (1.4.0) for trivial reasons, see (3.4).

2 – Background from deformation theory

2.1 – Deformations of maps with fixed target

(2.1) Let $\phi : C \to S$ be a non-constant morphism from a smooth projective curve C. A *deformation of ϕ with fixed target* over a pointed base $(B, 0)$ is the data of a deformation $\mathcal{C} \xrightarrow{\pi} B$ of C over $(B, 0)$ together with a morphism $\Phi : \mathcal{C} \to S \times B$ of B-schemes, such that the restriction of Φ over 0 equals ϕ.

This defines the *deformation functor* $\mathrm{Def}_{\phi/S}$ of ϕ with fixed target S. It is prorepresented by a complete local $\mathbf{C}$-algebra R_ϕ, see [7, Thm. 3.4.8].

(2.2) The deformations of ϕ with fixed target S are controlled by the *normal sheaf of* ϕ, i.e. the sheaf N_ϕ of $\mathcal{O}_C$-modules defined by the exact sequence on C

$$0 \to T_C \xrightarrow{d\phi} \phi^* T_S \to N_\phi \to 0 \ : \tag{2.2.1}$$

the spaces $H^0(C, N_\phi)$ and $H^1(C, N_\phi)$ are respectively the Zariski tangent space and an obstruction space for the deformations of ϕ with fixed target S. In particular, we have

$$\chi(N_\phi) \leqslant \dim R_\phi \leqslant h^0(N_\phi). \tag{2.2.2}$$

(2.3) The rank 1 sheaf N_ϕ may have torsion. We denote by $\mathcal{H}_\phi$ its torsion part and by $\bar{N}_\phi$ its maximal torsion-free quotient ; they fit in an exact sequence

$$0 \to \mathcal{H}_\phi \to N_\phi \to \bar{N}_\phi \to 0. \tag{2.3.1}$$

The torsion sheaf $\mathcal{H}_\phi$ is supported on the divisor Z of zeroes of the differential $d\phi$, and it is zero if and only if $Z = 0$; in this case, we say that ϕ is an immersion. Moreover, there is an exact sequence of locally free sheaves on C

$$0 \to T_C(Z) \to \phi^* T_S \to \bar{N}_\phi \to 0, \tag{2.3.2}$$

which readily implies the identification of line bundles on C

$$\bar{N}_\phi \cong \omega_C \otimes \phi^* \omega_S^{-1}(-Z). \tag{2.3.3}$$

(2.4) Construction as $\mathrm{Hom}\big(\mathcal{C}_g/\mathcal{M}_g,\, S \times \mathcal{M}_g/\mathcal{M}_g\big)$. Other possible notation : $\mathrm{Mor}_{\mathcal{M}_g}\big(\mathcal{C}_g,\, S \times \mathcal{M}_g\big)$. With proviso.

2.2 – Comparison of the spaces of maps and curves

(2.5) From maps to curves. Consider the universal morphism $\Phi : \mathcal{U}_M \to S \times M_g^{\xi,\mathrm{bir}}$ defined over $M_g^{\xi,\mathrm{bir}}$ (pretending it exists, compare (2.4)). Let $\bar{M}$ be the semi-normalization of the reduced scheme underlying $M_g^{\xi,\mathrm{bir}}$, $\bar{\mathcal{U}}_M := \mathcal{U}_M \times_{M_g^{\xi,\mathrm{bir}}} \bar{M}$, and $\bar{\Phi} : \bar{\mathcal{U}}_M \to S \times \bar{M}$ the induced morphism of $\bar{M}$-schemes. I claim that the scheme-theoretic image $\bar{\Phi}(\bar{\mathcal{U}}_M)$ is flat over $\bar{M}$.

Indeed, the morphism $\varpi := \mathrm{pr}_2 : \bar{\Phi}(\bar{\mathcal{U}}_M) \to \bar{M}$ is a well-defined family of codimension 1 algebraic cycles of S in the sense of [6, I.3.11]. Since $\bar{M}$ is normal, the claim follows from [6, I.3.23.2].

It follows that there is a morphism from $\bar{M}$ to the Hilbert scheme of curves on S. It obviously factorizes through $V_g^\xi(S)$, and actually through its normalization $\bar{V}_g^\xi$ by the universal property of the normalization. Since by definition the semi-normalization morphism $\bar{M} \to M_g^{\xi,\mathrm{bir}}$ is $1:1$, two points $[\phi : C \to S], [\phi' : C' \to S] \in \bar{M}$ are mapped to the same point $[\Gamma] \in V_g^\xi$ if and only if there exists an isomorphism $\iota : C \cong C'$ such that $\phi = \phi' \circ \iota$. To draw a precise conclusion about the morphism $\bar{M} \to V_g^\xi$ from this, it is necessary to specify precisely who is $M_g^{\xi,\mathrm{bir}}$; we will need and state less, see Prop. (2.7).

(2.6) From curves to maps. On the other hand, consider the universal family $\mathcal{U}_V \to V_g^\xi$ of curves gotten from the universal family over the Hilbert scheme of curves on S. Let $\bar{V}$ be the normalization of V_g^ξ, and $\bar{\mathcal{U}}_V$ the normalization of $\mathcal{U}_V \times_{V_g^\xi} \bar{V}$. Teissier's résolution simultanée theorem [8] asserts that $\bar{\mathcal{U}}_V \to \bar{V}$ is a family of smooth genus g curves ; it comes with a morphism of $\bar{V}$-schemes

$$\bar{\mathcal{U}}_V \to \mathcal{U}_V \times_{V_g^\xi} \bar{V} \subset S \times \bar{V}.$$

It follows that there is a morphism from $\bar{V}$ to the space $M_g^{\xi,\mathrm{bir}}$. It is generically injective, because the universal family of curves over V_g^ξ is nowhere isotrivial.

From the considerations in §(2.5) and (2.6), one deduces the following.

(2.7) Proposition. *Let $[\phi : C \to S] \in M_g^{\xi,bir}(S)$ be a general point (i.e., a general point of any irreducible component of $M_g^{\xi,bir}(S)$). Let $\Gamma := \phi(C)$, $\xi \in \mathrm{NS}(S)$ the homology class of Γ, and g its geometric genus. Then $[\Gamma]$ belongs to a unique irreducible component of V_g^ξ and*

$$\dim_{[\Gamma]} V_g^\xi = \dim R_\phi.$$

(Recall that R_{ϕ_0} is the complete local $\mathbf{C}$-algebra that prorepresents $\mathrm{Def}_{\phi/S}$).

(2.7.1) *Remark.* If we take $M_g^{\xi,\mathrm{bir}}$ as the moduli space of maps, then $\dim R_\phi = \dim_{[\phi]} M_g^{\xi,\mathrm{bir}}$, whereas if one defines it as in [5] one has $\dim R_\phi = \dim_{[\phi]} M_g^{\xi,\mathrm{bir}} - \dim(\mathrm{Aut}\, C)$.

The next result provides a sharper upper bound on the dimension of the Severi varieties than that given by the inequality (2.2.2) $\dim R_\phi \leqslant h^0(N_\phi)$.

Let $\phi : C \to S$ be a morphism from a smooth projective curve C, birational onto its image Γ. Let $\xi \in \mathrm{NS}(S)$ be the homology class of Γ, and g its geometric genus. We consider $\Phi : \mathcal{C} \to S \times B$ a deformation of ϕ over a pointed normal connected scheme $(B, 0)$. Then $\Phi(\mathcal{C}) \subset S \times B$ is a deformation of Γ over $(B, 0)$, see (2.5). There are thus two classifying morphisms κ and γ from $(B, 0)$ to $M_g^{\xi,\mathrm{bir}}(S)$ (or R_ϕ) and $\mathrm{Curve}(S)$ respectively, with differentials

$$d\kappa : T_{B,0} \to H^0(C, N_\phi) \quad \text{and} \quad d\gamma : T_{B,0} \to H^0(\Gamma, N_{\Gamma/X}).$$

(2.8) Lemma. *The inverse image by $d\kappa$ of the torsion $H^0(C, \mathcal{H}_\phi) \subset H^0(C, N_\phi)$ is contained in the kernel of $d\gamma$.*

Proof. Given a non-zero section $\sigma \in H^0(C, N_\phi)$, the first order deformation of ϕ defined by σ can be described in the following way : consider an affine open cover $\{U_i\}_{i\in I}$ of C, and for each $i \in I$ consider a lifting $\theta_i \in C(U_i, \phi^* T_X)$ of the restriction $\sigma_{|U_i}$. Each θ_i defines a morphism

$$\tilde{\phi}_i : U_i \times \mathrm{Spec}(\mathbf{C}[\varepsilon]) \to S$$

extending $\phi_{|U_i} : U_i \to X$. The morphisms $\tilde{\phi}_i$ are then made compatible after gluing the trivial deformations $U_i \times \mathrm{Spec}(\mathbf{C}[\varepsilon])$ into the first order deformation of C defined by the coboundary $\partial(\sigma) \in H^1(C, T_C)$ of the exact sequence (2.2.1). In case $\sigma \in \mathrm{H}^0(C, \mathcal{H}_\phi)$, everyone of the maps $\tilde{\phi}_i$ is the trivial deformation of $\sigma_{|U_i}$ over an open subset. This implies that the corresponding first order deformation of ϕ leaves the image fixed, hence the vanishing of $dq_0(\sigma)$. □

(2.9) Corollary. *Let $g \in \mathbf{Z}$, $\xi \in \mathrm{NS}(S)$. Let $[C]$ be a general point of V_g^ξ, and $\phi : \bar{C} \to C \subset S$ its normalization. Then*

$$\dim V_g^\xi \leqslant h^0(\bar{C}, \bar{N}_\phi).$$

Proof. By generality we may assume that $[C]$ is a smooth point of V_g^ξ. Then $\dim V_g^\xi = \dim T_{[C]} V_g^\xi$, and by (2.6) there is a map

$$d\kappa_{[\phi]} : T_{[C]} V_g^\xi \to H^0(\bar{C}, N_\phi).$$

It is injective because to every tangent vector $\theta \in T_{[C]}V$ corresponds a non-trivial deformation of C. On the other hand, it follows from Lemma (2.8) that $\operatorname{Im} d\kappa_{[\phi]} \subset H^0(\bar{C}, \bar{N}_\phi)$. $\square$

Lemma (2.8) is a crucial observation (and indeed, the cornerstone of the proof of Theorem (1.4)) that was made by Arbarello and Cornalba [1, p. 26], who deemed it a *fenomeno assai curioso*. They write : « nel caso in cui ϕ sia una birazionalità tra C e Γ, la presenza di "cuspidi" su Γ, comporta l'esistenza, dal punto di vista infinitesimo, di più di un modello liscio della curva Γ, se così ci possiamo esprimere. » [1]

Next, paraphrasing them, in order to use this phenomenon constructively they establish [1, Cor. 6.11] : in the above notation, if B is the complex unit disk and if the family of curves is *equisingular*, then $d\kappa(\partial/\partial t)$ belongs to $H^0(C, \mathcal{H}_\phi)$ if and only if it is zero. Later, Caporaso and Harris (together with J. de Jong, they write) state and prove [3, Lem. 2.3]. They add the remark that this is linked to the notion of equisingularity, even though they make absolutely no use of this, neither in the statement nor in its proof.

The treatment I give here is that of Sernesi and myself in [5]. Although essentially equivalent to that of [3], it slightly differs in its formulation. This formulation, I hope, sheds some light on what is actually going on and in particular displays that equisingularity has very little to do in the argument for Corollary (2.9).

2.3 – Tangency conditions with respect to a fixed curve

In general we consider R a fixed reduced curve on S. In this subsection we study deformations of curves on S satisfying some tangency conditions with R ; it follows from our Definition (1.3) that it suffices to treat the case when R is smooth.

(2.10) Let m be a non-negative integer. Let $\phi : C \to S$ be a non-constant morphism from a smooth projective curve C. A deformation of ϕ with fixed target *preserving a tangency of order m with R* over a pointed base $(B, 0)$ is a deformation $\Phi : \mathcal{C} \to S \times B$ of ϕ with fixed target as in (2.1), such that there exists a section Q of $\mathcal{C} \to B$ such that the pulled-back divisor $\Phi^* R$ contains Q with multiplicity m (i.e., $\Phi^* R - mQ \geqslant 0$).

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\Phi} & S \times B \\ {\scriptstyle Q} \uparrow \downarrow {\scriptstyle \pi} & \swarrow {\scriptstyle \mathrm{pr}_2} & \\ B & & \end{array}$$

The tangency is said to be respectively at a *fixed point* $p \in R$ if $\Phi(Q) = \{p\} \times B$, and at a *variable point* if $\mathrm{pr}_1 \circ \Phi(Q)$ is a curve.

We say that a family of maps $\Phi : \mathcal{C} \to S \times B$ preserves a tangency of order m with R if for all $b \in B$ it is locally around b a deformation of maps preserving a tangency of order m.

The following result displays the additional conditions the class of a deformation of maps has to meet for this deformation to preserve a tangency with R. It is [3, Lem. 2.6]. Let B be a reduced scheme and $\Phi : \mathcal{C} \to S \times B$ be a family of maps preserving a tangency with R of order exactly m. Let $b \in B$ be a general point, and $\phi : C \to S$ be the corresponding map. This comes with a classifying map κ, with differential $d\kappa : T_0 B \to H^0(C, N_\phi)$; we call $\bar{d}\kappa$ its composition with the projection $H^0(C, N_\phi) \to H^0(C, \bar{N}_\phi)$.

1. a very curious phenomenon ; in the case when ϕ is birational between C and Γ, the presence of "cusps" on Γ comports the existence, at the infinitesimal level, of more than one smooth model of the curve Γ, if we may say so.

Let $q := Q \cap C$, and $l-1$ be the order of vanishing of the differential $d\phi$ at q (i.e., l is the multiplicity of the point $p := \phi(q)$ in the local branch of $\phi(C)$ corresponding to q). Note that necessarily $l \leqslant m$.

(2.11) Lemma. *Let $\sigma \in \mathrm{Im}(\bar{d}\kappa)$ be a non-zero section, and denote by $v_q(\sigma)$ its order of vanishing at $q = Q \cap C$.*
(a) *One has $v_q(\sigma) \in \{m-l\} \cup [\![m, +\infty[\![$.*[2]
(b) *If the tangency is at a fixed point of R, then actually $v_q(\sigma) \in [\![m, +\infty[\![$.*

Proof. This is a local computation. Let (x, y) be (analytic) local coordinates on S at $p = \phi(q)$, such that R is defined by the equation $y = 0$. Then the vector fields $\partial/\partial x$ and $\partial/\partial y$ generate T_S near p, and their pull-backs generate $\phi^* T_S$ near q; by abuse of notation we shall denote them by $\partial/\partial x$ and $\partial/\partial y$ as well.

We may assume that B is a curve. Let ε be a local coordinate on B centered at b, and t be a local equation of the section Q near q. Thus (t, ε) are local coordinates on $\mathcal{C}$ at q. We may assume that t gives a local coordinate on C at q, in such a way that ϕ is given locally by

$$\phi(t) = \begin{cases} \left(t^l + a_{l+1}t^{l+1} + \cdots, t^m\right), & \text{if } l < m \\ \left(t^n + a_{n+1}t^{n+1} + \cdots, t^m\right) \text{ for some } n \geqslant m, & \text{if } l = m. \end{cases}$$

From now on we assume $l < m$ and leave the other, similar, case to the reader. Then the differential of ϕ at q is

$$\begin{aligned} \tfrac{\partial}{\partial t} \longmapsto & (lt^{l-1} + (l+1)a_{l+1}t^l + \cdots) \cdot \tfrac{\partial}{\partial x} + mt^{m-1} \cdot \tfrac{\partial}{\partial y} \\ & = t^{l-1}\big((l + (l+1)a_{l+1}t + \cdots) \cdot \tfrac{\partial}{\partial x} + mt^{m-l} \cdot \tfrac{\partial}{\partial y}\big). \end{aligned}$$

We see that around q on C, the torsion part $\mathcal{H}_\phi$ of N_ϕ is a skycraper sheaf of length $l-1$ concentrated at q, generated by the section

$$\tau : t \mapsto \big(l + (l+1)a_{l+1}t + \cdots\big) \cdot \tfrac{\partial}{\partial x} + mt^{m-l} \cdot \tfrac{\partial}{\partial y}.$$

The torsion-free quotient $\bar{N}_\phi$ is an invertible sheaf, generated by $\partial/\partial y$. Observe that modulo τ, $\partial/\partial x$ is $t^{m-l} \cdot \partial/\partial y$ times an invertible, hence the image of $\partial/\partial x$ in $\bar{N}_\phi$ vanishes to the order exactly $m - l$ at q.

In turn the family Φ is given locally by

$$\Phi(t, \varepsilon) = \big((t^l + a_{l+1}t^{l+1} + \cdots) + \varepsilon(u_0 + u_1 t + \cdots) + O(\varepsilon^2), t^m, \varepsilon\big),$$

as $\Phi^* R$ contains Q with multiplicity m. By definition, the corresponding section $\bar{d}\kappa(\partial/\partial\varepsilon)$ of $\bar{N}_\phi$ is

$$\bar{d}\kappa(\tfrac{\partial}{\partial \varepsilon}) = (u_0 + u_1 t + \cdots) \cdot \tfrac{\partial}{\partial x} \mod \tau.$$

Since $\partial/\partial x$ itself vanishes modulo τ to the order $m-l$ as we have seen, one has $v_q\big(\bar{d}\kappa(\partial/\partial\varepsilon)\big) \geqslant m - l$ in any event.

Moreover, by generality of $b \in B$ we may assume that all $\Phi(\varepsilon, \cdot)$ have their differential vanishing to the order l at $Q \cap \mathcal{C}_\varepsilon$, which translates into the fact that $u_1 = \cdots = u_{l-1} = 0$. Then $\bar{d}\kappa(\partial/\partial\varepsilon)$ vanishes either to the order $m - l$, if $u_0 \neq 0$, or to some order larger than m, if $u_0 = 0$. When the tangency is maintained at a fixed point we are necessarily in the latter case. The lemma is proved. □

2. I use the (probably french) notation $[\![a, b]\!] = [a, b] \cap \mathbf{Z}$ for $a, b \in \mathbf{R} \cup \{\pm\infty\}$.

Let $\mathcal{C} \subset S \times B$ be a family of a reduced curves over a reduced base B. It is said to preserve a tangency of order m with R if the corresponding family of normalization maps over the normalization $\bar{B}$ of B (see (2.6)) does.

(2.12) Corollary. *Let $V \subset \mathrm{Curve}(S)$ be a family of curves of genus g having a tangency of order m with the divisor R. Let $[C]$ be a general member of V, $\phi : \bar{C} \to C \subset S$ be the normalization of C, $q \in \bar{C}$ the tangency point, and $l-1$ the order of vanishing of the differential $d\phi$ at q. Then*

$$\dim V \leqslant h^0\big(\bar{C}, \bar{N}_\phi(-a \cdot q)\big),$$

where $a \geqslant m - l$. If the tangency is at a fixed point of R, then actually $a \geqslant m$.

Proof. As in the proof of Cor. (2.9), we have $\dim V = \dim T_{[C]}V$ by generality of $[C]$, and there is an injective map $T_{[C]}V \to H^0(\bar{C}, \bar{N}_\phi)$. By Lem. (2.11) the image of this map is contained in $H^0\big(\bar{C}, \bar{N}_\phi(-a \cdot q)\big)$ with a as in the statement. This ends the proof. □

3 – Proof of the Main Theorem

In this section we prove Theorem (1.4). The proof itself is in Subsection 3.2, after we give some lemmas in Subsection 3.1.

3.1 – Applications of the Riemann–Roch formula

Lemma (3.1) below is standard, but we will also use the more clever Lemma (3.2).

(3.1) Lemma. *Let X be a smooth (possibly disconnected) projective curve, and L a line bundle on X. Let $k \in \mathbf{N}^*$. If $\deg\big(L \otimes \omega_X^{-1}\big|_{X_i}\big) > k$ for all irreducible component X_i of X, then the linear system $|L|$ separates any k points on X.*

Proof. Let Z be a subscheme of X of length k, and let Z' be another subscheme of X such that $Z' \subsetneq Z$. The assumption on the degree of L ensures that both $L(-Z)$ and $L(-Z')$ are non-special, hence by the Riemann–Roch formula $h^0\big(L(-Z)\big) < h^0\big(L(-Z')\big)$. □

(3.2) Lemma. *Let X be a smooth (possibly disconnected) projective curve of genus $g = 1 - \chi(\mathcal{O}_X)$, and L, M two line bundles on X such that*

$$\forall\, X_i \text{ component of } X : \quad \deg(M|_{X_i}) \leqslant \deg(L|_{X_i}).$$

(a) *If* $\deg L \otimes \omega_X^{-1}\big|_{X_i} > 0$ *for every component X_i of X, then*

$$h^0(X, M) \leqslant h^0(X, L) = \deg(L) - g + 1. \tag{3.2.1}$$

(b) *If* $\deg L \otimes \omega_X^{-1}\big|_{X_i} > 1$ *for every component X_i of X, then actually equality holds in* (3.2.1) *if and only if* $\deg M = \deg L$.

Proof. Assumption (a) ensures that L is non-special, hence the right-hand-side equality in (3.2.1) by the Riemann–Roch formula. If M is non-special as well, then $h^0(L) - h^0(M) = \deg(L) - \deg(M)$ again by the Riemann–Roch formula, which gives the result.

Let us assume from now on that M is special. Then,

$$h^0(X, M) \leqslant h^0(X, \omega_X) = g + n - 1.$$

Under Assumption (a),

$$\deg(L) = \sum_{i=1}^{n} \deg(L|_{X_i}) \geqslant \sum_{i=1}^{n} (2g_i - 1) = 2g - 2 + n,$$

where n is the number of components of X and g_i is the genus of X_i for $i = 1, \ldots, n$; recall that $g = \sum g_i - n + 1$. In this case, one has

$$h^0(X, L) = \deg(L) - g + 1 \geqslant g - 1 + n$$

hence $h^0(X, L) \geqslant h^0(X, M)$.

Under Assumption (b) one has in the same fashion $\deg(L) \geqslant 2g - 2 + 2n$, hence

$$h^0(X, L) \geqslant g + 2n - 1 > h^0(X, M).$$

This ends the proof as the specialty of M implies $\deg(M) < \deg(L)$ under the general assumption of the Lemma. □

3.2 – Proof of Theorem (1.4)

(3.3) *We start by proving that V has expected dimension*

(3.3.1) $$\operatorname{expdim} V_g^{\xi}(\alpha, \beta)(\Omega) = -(K_S + R) \cdot \xi + g - 1 + |\beta|.$$

By (2.2.2) the expected dimension of V_g^{ξ} is $\chi(N_\phi)$, which by the Riemann–Roch formula and the exact sequence (2.2.1) equals

(3.3.2) $$\operatorname{expdim} V_g^{\xi} = \deg \omega_{\bar{C}} - \deg \phi^* \omega_S + 1 - g = -K_S \cdot C + g - 1.$$

Now requiring that a curve C have tangency of order m with R at a specified point p is m linear conditions on the coefficients of the equation of C, and if we let the point p vary along R the expected codimension of the corresponding locus of curves C is one less, i.e. $m - 1$. We thus end up with

$$\begin{aligned} \operatorname{expdim}\big(V_g^{\xi}(\alpha, \beta)(\Omega)\big) &= \operatorname{expdim}\big(V_g^{\xi}\big) - \sum_i \sum_{1 \leqslant j \leqslant \alpha_i} i - \sum_i \sum_{1 \leqslant j \leqslant \beta_i} (i - 1) \\ &= \operatorname{expdim}\big(V_g^{\xi}\big) - I\alpha - (I\beta - |\beta|), \end{aligned}$$

which together with (3.3.2) gives the required equality (3.3.1) after one remarks that $I\alpha + I\beta = R \cdot \xi$.

Note that this proves that in any event

(3.3.3) $$\dim(V) \geqslant -(K_S + R) \cdot \xi + g - 1 + |\beta|.$$

□

(3.4) *We now turn to the proof that the dimension of V equals its expected dimension under assumption (1.4.0).*

Note that the points $q_{i,j}$ are necessarily pairwise distinct because they have distinct images $p_{i,j} \in R$. Let us first assume in addition that the points $q_{i,j}$ and $r_{i,j}$ are all together pairwise distinct; the case when this does not hold will be dealt with in (3.5).

We set

$$D := \sum_{1\leqslant j\leqslant \alpha_i} i\, q_{i,j} + \sum_{1\leqslant j\leqslant \beta_i} (i-1)\, r_{i,j} \tag{3.4.1}$$

the divisor on $\bar{C}$ of "infinitesimal tangency conditions with R" (compare (3.3)), and

$$D_0 := \sum_{1\leqslant j\leqslant \beta_i} (l_{i,j}-1)\, r_{i,j}, \quad \text{where} \quad l_{i,j} := v_{r_{i,j}}(d\phi) \tag{3.4.2}$$

is the order of vanishing of the differential $d\phi$ at the point $r_{i,j}$, i.e. D_0 is the ramification divisor of ϕ "in the points $r_{i,j}$". We then decompose the difference of these two divisors as

$$D - D_0 = D_1 - D_1' \tag{3.4.3}$$

where D_1 and D_1' are non-negative divisors on $\bar{C}$ with disjoint supports; note that D_1' may be nonzero only at the points $r_{i,j}$, and that it is so if and only if $l_{i,j} > i$.

It follows from Cor. (2.12) that

$$\dim(V) \leqslant h^0\big(\bar{C}, \bar{N}_\phi(-D_1)\big). \tag{3.4.4}$$

Let Z_0 be the non-negative divisor on $\bar{C}$ such that the ramification divisor of ϕ is $D_0 + Z_0$. Then by (2.3.3) we have $\bar{N}_\phi \cong \omega_{\bar{C}} \otimes \phi^*\omega_S^{-1}(-D_0 - Z_0)$, and therefore (3.4.4) above reads

$$\begin{aligned}
\dim(V) &\leqslant h^0\big(\bar{C}, \omega_{\bar{C}} \otimes \phi^*\omega_S^{-1}(-D_0 - Z_0 - D_1)\big) && (3.4.5)\\
&= h^0\big(\bar{C}, \omega_{\bar{C}} \otimes \phi^*\omega_S^{-1}(-D - D_1' - Z_0)\big) && (3.4.6)\\
&\leqslant h^0\big(\bar{C}, \omega_{\bar{C}} \otimes \phi^*\omega_S^{-1}(-D)\big). && (3.4.7)
\end{aligned}$$

Now by assumption (1.4.0) the line bundle $\omega_{\bar{C}} \otimes \phi^*\omega_S^{-1}(-D)$ is non-special, hence

$$\begin{aligned}
h^0\big(\bar{C}, \omega_{\bar{C}} \otimes \phi^*\omega_S^{-1}(-D)\big) &= h^0\big(\bar{C}, \omega_{\bar{C}} \otimes \phi^*\omega_S(R)^{-1}(\phi^*R - D)\big) && (3.4.8)\\
&= 2g - 2 - (K_S + R)\cdot\xi + |\beta| + 1 - g && (3.4.9)\\
&= \operatorname{expdim} V_g^\xi(\alpha,\beta)(\Omega). && (3.4.10)
\end{aligned}$$

We thus have $\dim V \leqslant \operatorname{expdim} V_g^\xi(\alpha,\beta)(\Omega)$ which, together with (3.3.3) implies that V has the expected dimension if indeed the points $q_{i,j}$ and $r_{i,j}$ are all together pairwise distinct.

(3.5) Now if it is not true that the points $q_{i,j}$ and $r_{i,j}$ are all together pairwise distinct, then V is actually a component of some Severi variety $V_g^\xi(\alpha',\beta')(\Omega')$ with $|\beta'| < |\beta|$ for which the corresponding points $q'_{i,j}$ and $r'_{i,j}$ are indeed pairwise disjoint (as sets $\Omega = \Omega'$, i.e., $\bigcup_i \Omega_i = \bigcup_i \Omega'_i$, and $\Omega_i \subseteq \bigcup_{k\geqslant i} \Omega'_k$).

Then, setting correspondingly D' as in (3.4.1), one gets $\dim V \leqslant h^0(\bar{C}, \omega_{\bar{C}} \otimes \phi^*\omega_S^{-1}(-D'))$ exactly as in (3.4). Now $\deg D' > \deg D$ because $|\beta'| < |\beta|$, and it therefore follows from Lemma (3.2), part (a) that

$$h^0\big(\bar{C}, \omega_{\bar{C}} \otimes \phi^*\omega_S^{-1}(-D')\big) \leqslant h^0\big(\bar{C}, \omega_{\bar{C}} \otimes \phi^*\omega_S^{-1}(-D)\big), \tag{3.5.1}$$

so that it still holds that $\dim V \leqslant \operatorname{expdim} V_g^\xi(\alpha,\beta)(\Omega)$, hence V has the expected dimension. □

Note that we have proved the additional fact that the tangent space at $[C]$ of V identifies with

$$H^0\big(\bar{C}, \omega_{\bar{C}} \otimes \phi^*\omega_S^{-1}(-D)\big) \cong H^0\big(\bar{C}, \bar{N}_\phi(-D_1)\big) \cong H^0\big(\bar{C}, \bar{N}_\phi(D_0 - D)\big) \subset H^0\big(\bar{C}, \bar{N}_\phi\big).$$

(3.6) *We now prove that under Assumption (1.4.1) the assertions* ($a^{\flat\flat}$)*,* (b)*,* ($c^\flat$)*, and* (d) *hold.*

Suppose by contradiction that (b) doesn't hold. Then we argue as in (3.5). In this case, part (b) of Lemma (3.2) applies thanks to Assumption (1.4.1), and we get that the inequality (3.5.1) is strict, which is in contradiction with (3.3.3).

The same argument shows that none of the points $\phi(r_{i,j})$ can be fixed on R. This implies in particular that ($c^\flat$) holds.

The proof of (d) is similar : if C were tangent to G, then it would belong to an irreducible component W of some Severi variety of the pair $(S, R+G)$. Assumption (1.4.1) implies that W is liable for part (1.4.0) of Theorem (1.4), hence $\dim(W) < \dim(V)$, in contradiction with the fact that $[C]$ is a general member of V. The same argument shows that C avoids Γ (pick some random curve on S containing Γ).

Eventually, we note that equality holds in (3.4.7) if and only if $D_1' = Z_0 = 0$ since the line bundle $\omega_{\bar{C}} \otimes \phi^*\omega_S^{-1}(-D)$ is globally generated by assumption (1.4.1). Now it is indeed the case that equality holds in (3.4.7) since we have proved that $\dim(V) = \operatorname{expdim}\big(V_g^\xi(\alpha,\beta)\big)$. We conclude that $D_1' = Z_0 = 0$, which means that (i) ϕ is an immersion outside of the points $r_{i,j}$ (this is assertion ($a^{\flat\flat}$)) and (ii) $l_{i,j} \leqslant i$ for $1 \leqslant j \leqslant \beta_i$. □

Remark. It is not enough to assume that $\omega_{\bar{C}} \otimes \phi^*\omega_S^{-1}(-D)$ is non-special and globally generated because of the argument we made to assume that the points $q_{i,j}$ and $r_{i,j}$ are pairwise distinct. We actually need to know something about every possible $\omega_{\bar{C}} \otimes \phi^*\omega_S^{-1}(-D')$, where $D' = \sum iq'_{i,j} + \sum(i-1)r'_{i,j}$ in the notation used for this argument.

(3.7) *We now prove that, under the assumption (1.4.2), ϕ is an immersion also at the points $r_{i,j}$, i.e. that $l_{i,j} = 1$ for $1 \leqslant j \leqslant \beta_i$, thus completing the proof of assertion* ($a^\flat$).

Let $i \geqslant 1$ and $1 \leqslant j \leqslant \beta_i$. It follows from the assumption that the linear series $\big|\omega_{\bar{C}} \otimes \phi^*\omega_S^{-1}(-D)\big|$ separates any two points, so there exists a section $\sigma \in H^0\big(\bar{C}, \omega_{\bar{C}} \otimes \phi^*\omega_S^{-1}(-D)\big)$ with vanishing order $v_{r_{i,j}}(\sigma) = 1$ at the point $r_{i,j}$. Seen as a section $\tilde{\sigma} \in H^0(\bar{C}, \bar{N}_\phi)$, it vanishes at $r_{i,j}$ with order $v_{r_{i,j}}(\tilde{\sigma}) = 1 + (i - l_{i,j})$ (see (2.3.3)). By Lemma (2.11) this implies

$$1 + i - l_{i,j} \in \{i - l_{i,j}\} \cup [\![i, +\infty [\![$$

and therefore $l_{i,j} = 1$ as required. □

(3.8) *Let us prove that Assumption (1.4.2) implies Assertion* (c)*, i.e., the points $p_{i,j}$ and $s_{i,j} = \phi(r_{i,j})$ are pairwise distinct.*

By (3.6), we already know that ($c^\flat$) holds, i.e., none of the $s_{i,j} = \phi(r_{i,j})$ belongs to $\Omega = \{p_{i,j}\}$. We thus only need to show that no two of the points $s_{i,j}$ coincide.

Suppose there exist (i,j) and (i',j') distinct such that $\phi(r_{i,j}) = \phi(r_{i',j'})$. Assumption (1.4.2) implies that the series $|\omega_{\bar{C}} \otimes \phi^*\omega_S^{-1}(-D)| = |T_{[C]}V|$ separates any two points. Therefore there exists a section $\sigma \in T_{[C]}V \cong H^0(\bar{N}_\phi(-D))$ such that

$$v_{r_{i,j}}(\sigma) = 1 \quad \text{and} \quad v_{r_{i',j'}}(\sigma) = 0.$$

This implies the existence of a deformation of C in which the points $\phi(r_{i,j})$ and $\phi(r_{i',j'})$ no longer coincide, a contradiction to the generality of $[C]$ in V. □

(3.9) *Let us prove that Assumption (1.4.2) implies Assertion* (e)*, i.e., C is smooth at its intersection points with R.*

At this point we know that ($a^\flat$) and (c) under Assumption (1.4.2), i.e., the curve C is immersed and the points $p_{i,j}$ and $s_{i,j}$ are pairwise distinct. Because the intersection $C \cap R$ is set-theoretically the union of all the points $p_{i,j}$ and $s_{i,j}$, this implies that C is smooth at its intersection points with R. □

(3.10) *We eventually prove that under the Assumption (1.4.3) the curve C is nodal, which is Assertion* (a) of Thm. (1.4).

Since we already know that the curve C is immersed, it is enough to show that for all point $p \in C$, C has neither 3 or more local branches, nor 2 or more tangent local branches.

To exclude the former possibility, suppose by contradiction that there exist $a, b, c \in \bar{C}$ pairwise distinct such that $\phi(a) = \phi(b) = \phi(c)$. The assumption (1.4.3) implies that the linear series $\big|\omega_{\bar{C}} \otimes \phi^*\omega_S^{-1}(-D)\big|$ separates any three points, so there exists a section $\sigma \in H^0\big(\bar{C}, \omega_{\bar{C}} \otimes \phi^*\omega_S^{-1}(-D)\big)$ such that

$$\sigma(a) = \sigma(b) = 0 \quad \text{and} \quad \sigma(c) \neq 0;$$

it corresponds to a first-order deformation of ϕ leaving both $\phi(a)$ and $\phi(b)$ fixed while moving $\phi(c)$. By generality of $[C]$, there is correspondingly an actual deformation of the curve C for which the 3 local branches under consideration are no longer intersecting in a common point, a contradiction to the generality of $[C]$ in V (see, e.g., [5, Prop. 1.4]).

We exclude the second possibility in a similar fashion. Suppose by contradiction that there exist $a, b \in \bar{C}$ distinct such that $\phi(a) = \phi(b)$ and $\operatorname{Im} d\phi_a = \operatorname{Im} d\phi_b$. Again since $\big|\omega_{\bar{C}} \otimes \phi^*\omega_S^{-1}(-D)\big|$ separates any three points, there exists $\sigma \in H^0\big(\bar{C}, \omega_{\bar{C}} \otimes \phi^*\omega_S^{-1}(-D)\big)$ such that $\sigma(a) = 0$ and $\sigma(b) \notin \operatorname{Im} d\phi_a$. It corresponds to a first-order deformation of ϕ leaving $\phi(a)$ fixed while moving $\phi(b)$ in a direction transverse to the common tangent to the 2 local branches of C under consideration. This contradicts the generality of $[C]$ as before. □

4 – A characterization of logarithmic Severi varieties

We consider as before a pair (S, R) consisting of a smooth surface S and a smooth divisor R. In this Section we give an upper bound for the dimension of families of (not necessarily reduced) curves in S with prescribed homology class and genus. To be moved to intro, probably...

This is an easy corollary, but the proof is quite tricky.

Let V be an irreducible locally closed subset of the Hilbert scheme of curves on S with homology class $\xi \in \mathrm{NS}(S)$. Assume that

$$-(K_S + R) \cdot \xi \geqslant 1$$

so that everything is OK (? ?).

We suppose moreover that V parametrizes genus g curves in the following sense : let $[X]$ be a general member of V ; there exists a smooth curve C and a morphism $\phi : C \to X$, not constant on any component of C and such that the push-forward in the sense of cycles $\phi_*[C]$ equals the fundamental cycle of X.

(4.1) Proposition. *Let $[X]$ be a general member of V, and $\phi : C \to X$ a morphism as above. For every finite subset $\Omega \subset R$, one has*

(4.1.a) $$\dim V \leqslant -(K_S + R) \cdot \xi + g - 1 + \operatorname{Card}\big((X \cap R) \setminus \Omega\big)$$

(note that the last number is defined purely set-theoretically).

If equality holds, then V is a dense subset of a component of a log-Severi variety $V_g^\xi(\alpha, \beta)(\Omega)$ if and only if

(4.1.b) $$\operatorname{Card}\big(\phi^{-1}(R)\big) = \operatorname{Card}(X \cap R).$$

(4.1.1) *Remark.* (a) This is really a result about families of embedded curves in S, not families of maps. Indeed, if X is not reduced then ϕ involves multiple covers, and there is in general a positive dimensional family of maps giving the same X.

In the equality case, the map ϕ is necessarily a birational isomorphism on each component of C.

(b) A straightforward consequence of Proposition (4.1) which may be useful for the applications is that the inequality (4.1.a) still holds if we only assume C to be reduced and replace g by the *arithmetic* genus of C ; this alternative inequality is always strict when C is not smooth.

(4.1.2) *Remark.* (a) Assumption (4.1.b) ensures that the normalization of X is unibranch over the points in $X \cap R$.

Example. Set $(S, R) = (\mathbf{P}^2, L)$ whith L a line, and $\xi = 3[H]$ with $[H]$ the hyperplane class. The family of V of plane cubics with one node on L and otherwise smooth has dimension 7, which equals

$$\underbrace{-(K_S + R) \cdot \xi}_{=2[H]\cdot 3[H]=6} + \underbrace{g}_{=0} -1 + \underbrace{\mathrm{Card}\big((X \cap R) \setminus \Omega\big)}_{=2} \quad \text{with } \Omega = \emptyset.$$

It is a divisor in the log-Severi variety $V_0^{3[H]}(0,3)(\emptyset)$, but it is not a component of the family $V_0^{3[H]}(0,[1,1])(\emptyset)$ of rational plane cubics with one variable tangency along L, which has dimension 7 as well.

(b) In the equality case, if one replaces (4.1.b) with the weaker condition that

$$\mathrm{Card}\big(\phi^{-1}(R \setminus \Omega)\big) = \mathrm{Card}\big((X \cap L) \setminus \Omega\big),$$

the families that we get that are not log-Severi varieties may be considered as "log-Severi varieties with Ω containing multiple points" (these are not log-Severi varieties according to Definition (1.3)).

Example. Set $(S, R) = (\mathbf{P}^2, L)$ as above, and $\xi = 4[H]$, and fix $p \in L$. The family of plane quartics with one triple point at p has dimension $\binom{6}{2} - 1 - 6 = 8$, which equals

$$\underbrace{-(K_S + R) \cdot \xi}_{=2[H]\cdot 4[H]=8} + \underbrace{g}_{=0} -1 + \underbrace{\mathrm{Card}\big((X \cap L) \setminus \Omega\big)}_{=1} \quad \text{with } \Omega = \{p\}.$$

One may wish to consider it as $V_0^{3[H]}(3,1)(p,p,p)$.

Proof of Proposition (4.1). We divide it into several steps which correspond to paragraphs (4.2)–(4.4). We first treat the case when X is irreducible ; the general case is taken care of by induction in (4.4).

(4.2) We first prove the Proposition under the assumption that X is reduced and irreducible ; in this case ϕ is the normalization of X and $C \cong \bar{X}$.

If $e := \mathrm{Card}\big((X \cap R) \setminus \Omega\big) = 0$, then the statement is a slight variant of part (1.4.0) of the main Theorem (1.4) : we get the required inequality (4.1.a) exactly as in paragraph (3.4), with $D = \phi^* R$ and $D_0 = 0$ in (3.4.1) and (3.4.2) respectively. The only difference with the setting of (3.4) is that here two distinct points of the support of $\phi^* R \subset \bar{X}$ may have the same image by ϕ in X ; this makes absolutely no difference in the argument.

Now if (4.1.b) holds, then $\bar{X}$ is unibranch over the points of $X \cap R$. This implies that V is contained in a certain log-Severi variety $V_g^{\xi}(\alpha, 0)(\Omega)$. If moreover equality holds in (4.1.a), then V is dense in an irreducible component of this same log-Severi variety by (1.4.0).

In the case when $e > 0$, we consider the map

$$\rho : V \to \mathrm{Sym}^e R$$

sending a curve to its reduced intersection scheme with $R \setminus \Omega$; this may not be well-defined everywhere since e may drop along certain closed subschemes of V, but it is in a neighbourhood of $[X]$.

Then we can apply the $e = 0$ case of the Proposition to the fibres of ρ; for a general $\Sigma \in \mathrm{Sym}^e R$, setting $\Omega' = \Omega \cup \mathrm{Supp}\, \Sigma$ one gets that the fibre $\rho^{-1}(\Sigma)$ has dimension at most $-(K_S + R)\cdot\xi + g - 1$. Inequality (4.1.a) follows, and the equality case of the Proposition as well, again applying the $e = 0$ case to the fibres of ρ.

Remark. Note that as a byproduct of the above reasoning, one gets that when V is dense in a suitable irreducible component of a log-Severi variety, the map $\rho : V \to \mathrm{Sym}^e R$ is dominant.

(4.3) Let us now consider the case when X is non-reduced, but still irreducible; we shall show that inequality (4.1.a) holds, and that it is always strict.

We have to consider $\phi : C \to X$ where X is non-reduced but irreducible, and C may be reducible. We let m be the degree of ϕ, i.e. the sum of the degrees of the various $\phi_i : C_i \to X$. The key is to write "Riemann–Hurwitz" correctly : we have

$$2g - 2 = \deg \omega_C \geqslant m \deg \omega_{X_{\mathrm{red}}} = m(2h - 2),$$

which gives $g \geqslant mh - m + 1$.

(4.4) It remains to consider the case when X is reducible. Proceeding by induction on the number of irreducible components, we may assume that $X = X_1 \cup X_2$ where X_1 and X_2 move in two families V_1 and V_2 such that $V \subseteq V_1 \times V_2$ and the Proposition holds for V_1 and V_2. Adding the two corresponding inequalities readily gives

$$\begin{aligned} \dim V &\leqslant \dim V_1 + \dim V_2 \\ &= -(K_S + R)\cdot\xi + g - 1 + \mathrm{Card}\big((X_1 \cap R) \setminus \Omega\big) + \mathrm{Card}\big((X_2 \cap R) \setminus \Omega\big). \end{aligned} \tag{4.4.1}$$

If $(X_1 \cap X_2 \cap R) \setminus \Omega$ is empty then this is the required inequality (and the equality case follows). If not, let us assume for simplicity that it consists of only one point p (the general case is strictly similar).

If p is a fixed point of either one of the two families V_1 or V_2, then it is a fixed point of the two of them by the generality of X. Applying the Proposition to V_1 and V_2 with $\Omega' := \Omega \cup \{p\}$ one thus gets

$$\dim V \leqslant -(K_S + R)\cdot\xi + g - 1 + \underbrace{\mathrm{Card}\big((X_1 \cap R) \setminus \Omega'\big) + \mathrm{Card}\big((X_2 \cap R) \setminus \Omega'\big)}_{=\mathrm{Card}\left((X \cap R)\setminus\Omega\right) - 1}$$

and the result follows.

Otherwise p is variable for both V_1 and V_2; in this case V necessarily has codimension at least 1 in $V_1 \times V_2$ (this may be proved for instance as in (4.2) by applying the Proposition to the fibres of the projection $V \to V_1$), and therefore (4.4.1) gives the required inequality. Equality may hold but in any event condition (4.1.b) will not be fulfilled (see Remark (4.1.2)).

□

5 – Examples

5.1 – Logarithmic $K3$ surfaces

(5.1) Let us first consider the case of "absolute" $K3$ surfaces : let S be a $K3$ surface, and $R = 0$. In this case Theorem (1.4) is not quite accurate, a prominent problem being that the expected dimension given in (1.4.0) is not the actual dimension.

Suppose S is equipped with a polarisation L of genus p (i.e., $L^2 = 2p - 2$). The expected dimension of V_g^L is $g - 1$ whereas its actual dimension is g (if $0 \leqslant g \leqslant p$ and S is very general, say). Technically, the deformations of $[C] \in V_g^L$ are governed by the invertible sheaf $L|_C \cong \omega_C$, hence the obstruction space is $H^1(C, \omega_C)$ which is 1-dimensional, although the equigeneric deformations of C are in fact unobstructed. In some sense, the reason behind this is that there exist non-algebraic $K3$ surfaces. I refer to [5, §4.2] for a detailed account.

(5.2) In this subsection we shall describe some analogous phenomena for $K3$ pairs, by looking at a typical example. From now on we let $S = \mathbf{P}^2$, and R be a smooth cubic ; note that in this case one has $K_S + R = 0$. Let C be a smooth curve of degree $d \geqslant 4$ on S, and set $Z = C \cap R$ (for simplicity we shall assume that C and R intersect transversely). Then the blow-up S' of $\mathbf{P}^2$ at Z is a smooth surface having a unique anticanonical divisor, namely the proper transform of R. The linear system $|C'|$ of the proper transform C' of C gives a birational model of S' in $\mathbf{P}^g$, whose hyperplane sections are the canonical models of the degree d plane curves passing through Z (we let $g = \mathrm{p}(d)$ be the genus of smooth plane d-ic). In many aspects the surface S' behaves like a $K3$ surface.

(5.3) We may view the linear system $|C'|$ as the Severi variety $V_g^d(3d, 0)(Z)$ of $(\mathbf{P}^2, R)$. Then its expected dimension given in (1.4.0) is $g - 1$, whereas its actual dimension is g. In this case the discrepancy is readily explained : the points in Z are not general points on R (and therefore, according to our Definition (1.3) $V_g^d(3d, 0)(Z)$ is not a Severi variety).

To realize $|C'|$ as a genuine Severi variety, we may replace $V_g^d(3d, 0)(Z)$ by $V_g^d(3d-1, 1)(Z-p)$ for an arbitrary point $p \in Z$. The latter has both expected and actual dimension equal to g, and all its members automatically pass through p as well.

Let us illustrate this in the concrete case $d = 4$ (then, $g = 3$). The linear system of plane quartics has dimension 14. If we take a set Ω of 12 general points on R, then the Severi variety $V_3^4(12, 0)(\Omega)$ is empty : Ω imposes 12 independent conditions on quartics, the linear system of quartics through Ω is 2-dimensional, and all its members are made of R plus a line. On the other hand if we take Z the complete intersection of R with a smooth quartic C, then one sees using the restriction exact sequence

$$0 \to \mathcal{O}_{\mathbf{P}^2}(1) \to \mathcal{O}_{\mathbf{P}^2}(4) \to \mathcal{O}_R(4) \to 0$$

that the linear system of quartics through Ω is 3-dimensional, generated by C and the net of reducible quartics containing R. If we take a set Ω of 11 general points on R, it imposes 11 independent conditions on quartics, and the linear system of quartics through Ω is 3-dimensional with a 12-th base point on R.

(5.4) We may consider the linear system $|2C'|$ in a similar fashion, although there is no completely convincing way of realizing it as a genuine Severi variety. Seen on $\mathbf{P}^2$, it is the system of plane $(2d)$-ics with a node at each of the $3d$ points of $Z = C \cap R$.

Again we shall work this out in the case $d = 4$. One has $(2C')^2 = 4 \cdot (C')^2 = 16 = 2 \cdot 9 - 2$, so the adjunction formula on S', which is essentially the same as on a $K3$ surface, tells us that curves in $|2C'|$ have genus 9. Moreover the Riemann–Roch formula, which works as on a $K3$ as well, tells us that $|2C'|$ has dimension 9.

On the other hand plane octics have arithmetic genus 21, so an octic with 12 nodes has geometric genus 9, confirming the above computations carried out on S'. The linear system of plane octics has dimension 44. Since a node at a prescribed point is 3 conditions, the expected dimension of a linear system of octics with 12 nodes at prescribed points is $44 - 36 = 8$. We thus find once again that there is 1 extra dimension when the 12 nodes are at the complete intersection $C \cap R$.

This may be verified directly using a resolution of the ideal $\mathcal{I}_Z^2$, where $\mathcal{I}_Z$ is the ideal of $Z \subset \mathbf{P}^2$. Let r and f be homogeneous equations of the curves R and C respectively. While for $\mathcal{I}_Z$ there is the Koszul resolution, for its square we have the exact sequence

$$0 \to \mathcal{O}(-10) \oplus \mathcal{O}(-11) \xrightarrow{\begin{pmatrix} -f & 0 \\ r & -f \\ 0 & r \end{pmatrix}} \mathcal{O}(-6) \oplus \mathcal{O}(-7) \oplus \mathcal{O}(-8) \xrightarrow{(r^2, rf, f^2)} \mathcal{I}_Z^2 \to 0,$$

which gives $h^0(\mathcal{O}_{\mathbf{P}^2}(8) \otimes \mathcal{I}_Z^2) = 10$ as required.

(5.5) This carries over to all systems $|kC'|$, to the effect that the condition of having a k-uple point at all $3d$ points of $Z = C \cap R$ imposes one less condition on plane d-ics than if the $3d$ points of Z were in general position. We leave this to the reader.

5.2 – Superabundant log Severi varieties coming from double covers

In this subsection we observe that linear systems on double covers of the projective plane provide examples of superabundant logarithmic Severi varieties. This is taken from [4].

(5.6) We shall use the following elementary facts about double covers. Let d be a positive integer, and B be a degree $2d$ curve in $\mathbf{P}^2$. We consider $\pi : S \to \mathbf{P}^2$ the double cover branched over B. Let H be the line class on $\mathbf{P}^2$, and L be its pull-back to S. For all $k \in \mathbf{N}$ we have

$$H^0(S, kL) = \pi^* H^0(\mathbf{P}^2, kH) \oplus \pi^* H^0(\mathbf{P}^2, kH - \tfrac{1}{2}B),$$

which is the isotypic decomposition of $H^0(S, kL)$ as a representation of $\mathbf{Z}/2$. The first summand corresponds to divisors that are double covers of degree k curves in $\mathbf{P}^2$, and the second to divisors that decompose as B (seen as the ramification divisor in S) plus the double cover of a degree $k - d$ curve in $\mathbf{P}^2$.

(5.7) Proposition. *For $k \geqslant d$, the general member C of $|kL|$ is not a double cover of some hypersurface in $\mathbf{P}^2$, the restriction $\pi|_C$ is birational on its image, a degree $2k$ hypersurface $C^\flat$ in $\mathbf{P}^2$ everywhere tangent to B, with a node at every point of a complete intersection Z of type $(k, k-d)$.*

Proof. The divisor C belongs to a unique pencil $\langle A', B + D' \rangle$, with A' and D' the double covers of curves A and D in $\mathbf{P}^2$ of respective degrees k and $k - d$. Thus $C^\flat := \pi(C)$ belongs to the pencil $\langle 2A, B + 2D \rangle$, from which it follows that $C^\flat$ is double along $Z := A \cap D$, and touches B doubly along $A \cap B$, which accounts for the whole intersection scheme of $C^\flat$ and B. The base locus of this pencil is the scheme defined by the ideal $\mathcal{I}_Z^2(\mathcal{I}_A^2 + \mathcal{I}_B)$.

The pull-back $\pi^* C^\flat \in |2kL|$ splits as $C + i(C)$, with i the involution on V associated to π; it has a double singularity along $Z' := \pi^{-1}(Z)$ and $\pi^{-1}(B \cap A)$, with at each point one local sheet belonging to C and another to $i(C)$. The union $Z' \cup \pi^{-1}(B \cap A)$ is the base locus of the pencil $\langle A', B + D' \rangle$. □

(5.8) We consider the image $V_{B,k}$ in $|2kH|$ of the linear system $|kL|$ on S. It has dimension

$$h^0(S, kL) - 1 = h^0(\mathbf{P}^2, kH) + h^0(\mathbf{P}^2, (k-d)H) - 1,$$

and parametrizes curves of geometric genus

$$g_{k,d} = \tfrac{1}{2}(2k-1)(2k-2) - k(k-d)$$

everywhere tangent to B, the number of contact points is thus $2kd$.

The family of curves $V_{B,k}$ is therefore contained in the log-Severi variety $V^{2k}_{g_{k,d}}(0, [0, 2kd])$ of the pair $(\mathbf{P}^2, B)$, which has the expected dimension

$$-(K_{\mathbf{P}^2} + B) \cdot 2kH + g_{k,d} - 1 + 2kd = k(k+3-d).$$

By (1.4.0) a component of the Severi variety has the expected dimension if it has an irreducible member and

$$-K_{\mathbf{P}^2} \cdot 2kH - 2kd \geqslant 1 \quad \Longleftrightarrow \quad 2k(3-d) \geqslant 1,$$

the latter inequality holding if and only if $d \leqslant 2$.

It turns out that the dimension of our family $V_{B,k}$ exceeds the expected dimension of the log-Severi variety. Indeed a direct computation shows that

$$\begin{aligned} \dim(V_{B,k}) - \operatorname{expdim}\bigl(V^{2kH}_{g_{k,d}}(0, [0, 2kd])\bigr) &= \tfrac{(d-1)(d-2)}{2} \\ &= p_g(S) \end{aligned}$$

(cf. [2, V.22 p.237] for the last equality).

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