1 Moving a divisor away from some points

Theorem 1 (see [Sha94, p. 158]). *For any divisor D on a smooth projective surface $S \subset \mathbb{P}^n$, and for any points $x_1, \dots, x_m \in S$, there exists a divisor D' such that $D' \sim D$ and $x_i \notin Supp(D')$ for $i = 1, \dots, m$.*

Proof. First we can choose a hyperplane $H \subset \mathbb{P}^n$ that does not contain any of the x_i: The space of hyperplanes in $\mathbb{P}^n$ is again a $\mathbb{P}^n$, and the condition "passing through a point" defines a $\mathbb{P}^{n-1} \subset \mathbb{P}^n$.
We can also assume that D is a hypersurface: the general case follows by moving each component of D.
We proceed by induction on m: we assume $x_1, \dots, x_{m-1} \notin D$, and $x_m \in D$. We want to move D away from x_m, without meeting again one of the other x_i.
Let $U = \mathbb{A}^n \cap S = S \setminus H$ be the complement of H, and let g be a local equation of D at x_m. We can write $g = f_1/f_2$ with $f_1, f_2 \in \mathbb{C}[U]$ and $f_2(x_m) \neq 0$, so replacing g by f_1 we can assume that g is regular on U.
Then if $(g) = (g)_0 - (g)_\infty$ is the principal divisor defined by g, we have $(g)_\infty \subset H$, and $(g)_0 = D + R$ where R is effective. Thus $D \sim D - (g) = (g)_\infty - R$.
We are done if we can show that the support of R does not contain any of the x_i. For $i = 1, \dots, m-1$ we choose $g_i \in \mathbb{C}[U]$ such that $g_i(x_i) \neq 0$, and $g_i \equiv 0$ on the algebraic set $D \cup \{x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_{m-1}\}$.
We replace g by $g + \sum a_i g_i^2$, with $a_i \neq -g(x_i)/g_i(x_i)^2$. With this condition we obtain that R does not contain any x_i, $i = 1, \dots, m-1$.
On the other hand $g + \sum a_i g_i^2$ is still a local equation of D: each g_i is a multiple of g in the local ring $\mathcal{O}_{x_m}$, hence we have $g_i = g u_i$ with u_i regular at x_m. Then

$$g + \sum a_i g_i^2 = g + \sum a_i g^2 u_i^2 = g(1 + \sum a_i g u_i^2)$$

and $1 + \sum a_i g u_i^2$, which is a local equation of R at x_m, is invertible in $\mathcal{O}_{x_m}$: Thus $x_m \notin R$. □

2 A principal divisor on a curve has degree 0

Theorem 2 (see [Sha94, p. 168] for an alternative proof). *Let $f \colon C \to \mathbb{P}^1$ be a morphism from a smooth projective curve to the projective line. Then the divisor $(f) = (f)_0 - (f)_\infty$ has degree 0; in other words f has as many zeros as poles, when counted with multiplicities.*

Proof. I give a simple proof over $\mathbb{C}$, using the residue theorem. The proof in [Sha94] works on any algebraically closed field, but involves some not so elementary commutative algebra...
First recall that if g is a meromorphic function in a neighborhood of $0 \in \mathbb{C}$, we can write $g(z) = z^n h(z)$ for some $n \in \mathbb{Z}$ and h holomorphic at 0 with $h(0) \neq 0$. So we have

$$\frac{g'(z)}{g(z)} = \frac{n}{z} + \frac{h'(z)}{h(z)}$$

and we remark that the residue of $\frac{g'}{g}$ at 0 is n.
Now consider C as a Riemann surface, and for each zero or pole p_i of f take B_i a small neighborhood of p_i diffeomorphic to a disk, with B_i all disjoint. We denote S the complement of the B_i in C, and $\omega = df/f$, which is an holomorphic 1-form on S. In particular $d\omega = 0$ on S.
On the one hand, by Stokes Theorem

$$\int_{\delta S} \omega = \int_S d\omega = 0.$$

On the other hand by the previous remark

$$\int_{\delta S} \omega = -\sum \int_{\delta_{B_i}} \omega = \deg(f),$$

hence the result. □

3 Local intersection

Definition 3.1. *If D_1, D_2 are two effective divisors on a smooth projective surface S, with no common component through a point $p \in S$, and if f_1, f_2 are the respective local equations at p, we say that the number*

$$(D_1 \cdot D_2)_p = \dim \mathcal{O}_p/(f_1, f_2)$$

*is the **local intersection number** of D_1 and D_2 at p.*

3.1 Transverse intersections

First we want to check that this definition correspond to the intuitive notion of intersection in the case of two transverse curves. We need the notion of local parameters.

Definition 3.2 (see [Sha94, p. 98])**.** *Let $p \in X$ a smooth point in X projective variety of dimension n. We say that $u_1, \cdots, u_n \in \mathfrak{m}_p$ are **local parameters** at p if one of the following equivalent conditions holds:*

1. *The u_i form a basis of $\mathfrak{m}_p/\mathfrak{m}_p^2$;*
2. *The varieties $\mathbf{V}(u_i)$ are smooth and transverse at p;*
3. *The differentials $d_p u_i$ are linearly independent.*

To show the equivalence between this definitions, consider the morphism $d_p \colon u \in \mathfrak{m}_p \to d_p u \in T_pX^*$. It is clear that $\mathfrak{m}_p^2 \subset \ker(d_p)$, in fact one can prove that there is equality and d_p induces an isomorphism $\mathfrak{m}_p/\mathfrak{m}_p^2 \cong T_pX^*$.
One can also prove (Nakayama Lemma) that any choice of local parameters generate the ideal $\mathfrak{m}_p$ (see [Sha94, p. 99]).

Now let D_1, D_2 be two irreducible curves on a smooth projective surface S, and let f_1, f_2 be their local equations at a point $p \in S$. Then $(D_1, D_2)_p = 1$ iff f_1, f_2 are local parameters at x. Indeed if f_1, f_2 are local parameters, then $\dim \mathcal{O}_p/(f_1, f_2) = \dim \mathcal{O}_p/\mathfrak{m}_p = 1$; and if $\bar{f} \in \mathfrak{m}_p/\mathfrak{m}_p^2$ is not a linear combination of $\bar{f}_1, \bar{f}_2$, then the functions $1, f$ are linearly independent in $\mathcal{O}_p/(f_1, f_2)$.

3.2 Order of a zero

Next we show that the notions of local intersection number and multiplicities of a zero are closely related.

Definition 3.3. *Let $p \subset C$ a smooth point on a projective curve, and u a local parameter at p. If $f \not\equiv 0$ is a rational function regular at p, there exists $k \in \mathbb{N}$ such that $f \in (u^k)$ but $f \notin (u^{k+1})$. In other words $k = \dim \mathcal{O}_{C,p}/(f)$. We call this integer k the* ***order of the zero*** *of f at p, and we denote it by $k = \nu_p(f)$. If f is an arbitrary rational function on C, we write $f = f_1/f_2$ with f_1, f_2 regular at p and we define $\nu_p(f) = \nu_p(f_1) - \nu_p(f_2)$.*

Proposition 3.4. *Let $p \in C \subset S$ a smooth point of a curve C in a smooth projective surface S. Let D be a divisor on S with local equation f at p. Then*

$$(C \cdot D)_p = \nu_p(f).$$

Proof. If g is a local equation of C at p,

$$(C \cdot D)_p = \dim \mathcal{O}_p/(g, f) = \dim \mathcal{O}_{C,p}/(f) = \nu_p(f).$$

□

3.3 Linearity

We can extend Definition 3.1 by linearity to the case of of non effective divisors (but still without common component). This makes sense because of the following property:

Proposition 3.5. *Let D_1, D_2, D_3 be effective divisors. Assume that D_1, D_2 and D_1, D_3 do not have a common component. Then $(D_1 \cdot (D_2 + D_3))_p = (D_1 \cdot D_2)_p + (D_1 \cdot D_3)_p$.*

Proof. Let f_2, f_3 be local equations of D_2 and D_3. We note $\mathcal{O}_{D_1,p} = \mathcal{O}_p/(f_1)$. We want to prove

$$\dim \mathcal{O}_{D_1,p}/(f_2.f_3) = \dim \mathcal{O}_{D_1,p}/(f_2) + \dim \mathcal{O}_{D_1,p}/(f_3).$$

The exact sequence

$$0 \to (f_3)/(f_2.f_3) \to \mathcal{O}_{D_1,p}/(f_2.f_3) \to \mathcal{O}_{D_1,p}/(f_3) \to 0$$

gives $\dim \mathcal{O}_{D_1,p}/(f_2.f_3) = \dim((f_3)/(f_2.f_3)) + \dim \mathcal{O}_{D_1,p}/(f_3)$, and multiplication by f_3 gives an isomorphism $\mathcal{O}_{D_1,p}/(f_2) \cong (f_3)/(f_2.f_3)$, hence the result.

□

4 Definition of the intersection number

Definition 4.1. *Let $C \subset S$ be a (possibly reducible, singular) curve in a smooth projective surface. We say that a divisor $D = \sum a_i D_i$ on S is in* ***good position*** *with respect to C if no component of C is one of the D_i, and the support of D does not contain any of the singular point of C.*

Lemma 4.2. *Let D_1, D_2 be two divisors on a smooth projective surface S. Then there exists a divisor D_3 linearly equivalent to D_2, such that D_3 is in good position with respect to the support of D_1.*

Proof. Choose a collection of point p_i which contains all singular points of the support of D_1, and such that any component of D_1 contains at least one of the p_i. Then by Theorem 1 one can find $D_3 \sim D_2$ such that the support of D_3 does not contain any of the p_i, hence the result. □

We are now in position to make the following definitions.

Definition 4.3.

1. *If C_1, C_2 are two curves such that C_2 is in good position with respect to C_1, then we defines*
$$C_1 \cdot C_2 = \sum_{p \in C_1 \cap C_2} (C_1 \cdot C_2)_p.$$
2. *If $D_1 = \sum a_{1,i} D_{1,i}$, $D_2 = \sum a_{2,i} D_{2,i}$ are divisors such that D_2 is in good position with respect to each $D_{1,i}$, then we extend the previous definition by linearity:*
$$D_1 \cdot D_2 = \sum_{p \in D_{1,i} \cap D_{2,j}} a_{1,i} a_{2,j} (D_{1,i} \cdot D_{2,j})_p.$$
3. *if D_1 and D_2 are arbitrary divisors, we choose $D_3 \sim D_2$ such that D_3 is in good position with respect to D_1, and we use the previous definition to define the* ***intersection number*** *of D_1 and D_2:*
$$D_1 \cdot D_2 = D_1 \cdot D_3.$$

The last definition is well defined by the following result.

Theorem 3. *Let D_1, D_2, D_3 be divisors on a smooth projective surface S. Assume that $D_2 \sim D_3$ and D_2 and D_3 are in good position with respect to D_1. Then*
$$D_1 \cdot D_2 = D_1 \cdot D_3.$$

Proof. By assumption $D_2 - D_3 = (f)$ is a principal divisor, and we want to show that $D_1 \cdot (f) = 0$. By linearity we can assume that D_1 is an irreducible curve C.
If $p \in C \cap (f)$, and f_α is a local equation of C at p, we have
$$(C.(f))_p = \dim \mathcal{O}_{S,p}/(f_\alpha, f) = \dim \mathcal{O}_{C,p}/(f).$$
Since by assumption C is smooth at each $p \in C \cap (f)$, we have $\dim \mathcal{O}_{C,p}/(f) = \nu_p(f)$, and so we want to prove that the divisor $(f) = \sum \nu_p(f) p$ on C has degree 0. This is precisely the content of Theorem 2 if C is smooth; if C is singular we apply the same argument to the desingularization of C. □

5 Pull-back, exceptional divisor

Proposition 5.1 (see [Sha94, p. 252])**.** *Let $f\colon X \to Y$ be a birational morphism between surfaces. Then:*

1. *If D_1, D_2 are divisors on Y, we have $f^*D_1 \cdot f^*D_2 = D_1 \cdot D_2$.*
2. *If D is a divisor on Y, and E is a divisor on X all components of which are contracted by f, then $E \cdot f^*D = 0$.*

Proof. Move divisors! □

Some numerology related to the blow-up map:

Proposition 5.2. *Let $\pi\colon S' \to S$ be the blow-up of a point p on a surface S. We denote by E the exceptional divisor. Let $C \subset S$ be a curve with multiplicity k at p. Then $\pi^*C = C' + k.E$.*

Proof. Let x, y be local parameters at p. The curve C admit locally an equation of the form $P(x, y)+$ monomials of degree $> k = 0$, with P homogeneous of degree k. In the blow-up chart $(x', y') = (x, y/x)$, the equation of π^*C is $0 = P(x', x'y') + \cdots = {x'}^k(P(1, y') + \cdots) = 0$. Since $x' = 0$ is a local equation of E, we have the result. □

The curve C' is the **strict transform** of C, we have $C' = \overline{\pi^{-1}(C \setminus \{x\})}$

Corollary 5.3 (voir [S, p. 253])**.** *Let $\pi\colon S' \to S$ be the blow-up of a point p on a surface S. We denote by E the exceptional divisor. Then:*

1. *$E^2 = E \cdot E = -1$;*
2. *If $C \subset S$ is a curve with multiplicity k at p, and if C' is the strict transform of C, then $C' \cdot E = k$;*
3. *If C_1, C_2 are curves with multiplicities k_1, k_2 at p, then $C'_1 \cdot C'_2 = C_1 \cdot C_2 - k_1k_2$;*
4. *In particular, if p is a smooth point of C, then ${C'}^2 = C^2 - 1$.*

Proof.

1. Consider C with equation $y = 0$, then in the chart (x', y') the curves C' and E have local equation $y' = 0$ and $x' = 0$ respectively. Hence we have $E \cdot C' = 1$. We have $0 = \pi^*C \cdot E = C'.E + E^2 = 1 + E^2$, hence $E^2 = -1$.
2. We have $0 = \pi^*C \cdot E = (C' + k.E) \cdot E = C' \cdot E - k$.
3. We have $C_1 \cdot C_2 = \pi^*C_1 \cdot \pi^*C_2 = (C'_1 + k_1E) \cdot (C'_2 + k_2E) = C'_1 \cdot C'_2 + k_1k_2 + k_1k_2 - k_1k_2$.

□

6 Birational maps between surfaces

The aim of this section is to prove that any birational map between smooth surfaces is a sequence of blow-ups and inverses of blow-ups.

The first result applies to any rational map between surfaces, and admit generalisations in higher dimensions, even if the proof is harder:

Theorem 4 (see [Sha94, p. 254]). *Let S be a smooth projective surface, and $f\colon S \dashrightarrow Y \subset \mathbb{P}^n$ be a rational map. Then there exists a sequence of blow-ups $\pi\colon V \to S$ that solves the base locus of f. In other words in the diagram*

$$\begin{tikzcd} & V \arrow[dl, "\pi"'] \arrow[dr, "\bar{f}"] & \\ S \arrow[rr, dashed, "f"'] & & Y \end{tikzcd}$$

$\bar{f}$ is a morphism.

Proof. Around a point $p \in S$ the map f has the form $q \in S \dashrightarrow [f_0(q) : \cdots : f_n(q)] \in Y \subset \mathbb{P}^n$, with $f_i \in \mathcal{O}_p$ without common factor.
The point p is a base point of f if $f_i(p) = 0$ for all i. Remark that the base locus of f does not contain a curve through p (since then the f_i would admit a local equation of this curve as a common factor): thus f is regular outside a finite number of points.
Consider now H the strict transform by f of a hyperplane section of Y. The curve H satisfies $H^2 \geq 0$, and if p is a base point of f then $p \in H$. After blowing-up the point p, the strict transform H' of H satisfies $H' \cdot H' < H \cdot H$. By induction on H^2, we conclude that after a finite number of such blow-ups, we get rid of all the base points. $\square$

The next result is specific to *birational* morphisms between *surfaces*:

Theorem 5 (see [Sha94, p. 256]). *Let S, S' be two smooth projective surfaces, and $f\colon S \to S'$ be a birational morphism. Then f is a finite sequence of blow-ups.*

For the proof we need the following lemma.

Lemma 6.1 (see [Sha94, p. 256 and 119]). *Let $f\colon X \dashrightarrow Y$ be a birational map between smooth surfaces. If $y \in Y$ is a base point of f^{-1}, then there exists a curve $C \subset X$ such that $f(C) = \{y\}$.*

Proof of the lemma. Assume first that f is a morphism (hence f is surjective). We can work on affine charts, and write locally around y

$$f^{-1}\colon z \in Y \dashrightarrow (t_1, \cdots, t_n) = (g_1(z), \cdots, g_n(z)) \in X \subset \mathbb{A}^n.$$

One of the g_i must be non regular at y: we write $g_i = u/v$ with $v(y) = 0$, and we can assume that u and v does not have a common factor. We have $t_i = g_i \circ f = \frac{u \circ f}{v \circ f}$. Consider

$C = \mathbf{V}(v \circ f)$. We have $y \in f(C) \subset \mathbf{V}(v)$, furthermore the identity $t_i.v \circ f = u \circ f$ implies $f(C) \subset \mathbf{V}(u)$. Thus $f(C) \subset \mathbf{V}(v) \cap \mathbf{V}(u)$, which is locally equal to y (if it was a curve this would contradict u, v without common factor).

For the general case we consider the closure $Z \subset X \times Y$ of the graph of the map f. Denote by $p\colon Z \to X$ and $q\colon Z \to Y$ the two projections. We have $f^{-1} = p \circ q^{-1}$, so y is a base point of q^{-1}. By the previous argument, there exists a curve $D \subset Z$ such that $q(D) = y$. Then $C = p(D)$ is the curve we were looking for. Indeed if $p(D) = x$ was a point, then we would have $D \subset (x, y) \in X \times Y$: contradiction. $\square$

Proof of the theorem. If $y \in S'$ is a base point of f^{-1}, consider $\pi\colon V \to S'$ the blow-up of y. We have to show that $h = \pi^{-1} \circ f$ is still a morphism. If not, there exists $x \in S$ a base point of h. Then on the one hand $f(x) = y$ and f is not locally invertible at x; on the other hand there exists a curve in V that is contracted on x by h^{-1}. This curve must be the exceptional divisor E associated with π. Take p and q two distinct points of E where h^{-1} is regular, and C, C' two germs of smooth curves transverse to E in p and q respectively. Then $\pi(C)$ and $\pi(C')$ are two transverse germs of smooth curves at y, images by f of two germs of curves at x. Thus the differential $D_x f$ has rank 2, which contradicts the fact that f is not locally invertible at x. $\square$

The two previous theorem together give the following result.

Theorem 6 (Zariski)**.** *Let S, S' be smooth projective surfaces, and $f\colon S \to S'$ be a birational map. Then there exists a third surface V and two sequences of blow-ups $\pi\colon V \to S$ and $\sigma\colon V \to S'$ such that the following diagram commutes:*

$$\begin{tikzcd} & V \arrow[dl, "\pi"'] \arrow[dr, "\sigma"] & \\ S \arrow[rr, dashed, "f"'] & & S' \end{tikzcd}$$

References

[Sha94] Igor R. Shafarevich. *Basic algebraic geometry. 1.* Springer-Verlag, Berlin, second edition, 1994. Varieties in projective space, Translated from the 1988 Russian edition and with notes by Miles Reid.