Exam - duration 3 hours

Handwritten notes and lecture notes are allowed. No electronic devices. Your answers may be in English or in French.

Exercise 1: Let $(G_n)_{n\geq 1}$ be a sequence of i.i.d. random variables with $G_1 \sim \mathcal{N}(0,1)$.

1) Show that for all $n \geq 1$ and all real number $x \geq 2$, we have $\mathbb{P}(|G_n| \geq x) \leq e^{-x^2/2}$.
2) Show that almost surely, for n large enough, $\frac{1}{n}|G_n| < 2$. *Hint: use Borel Cantelli lemma.*
3) Deduce that for all $\alpha > 0$, almost surely, $\sum_{n\geq 1} e^{-n\alpha}|G_n| < \infty$.
4) For all $\alpha > 0$ and all $t \in [0,1]$, we define
$$X_t^{(\alpha)} = \sum_{n\geq 1} e^{-\alpha n} G_n \cos(2\pi n t).$$
Show that almost surely, $(\alpha, t) \mapsto X_t^{(\alpha)}$ is continuous on $(0,+\infty)\times[0,1]$.
5) Let $\alpha > 0$ and denote by μ_α the distribution of $(X_t^{(\alpha)})_{t\in[0,1]}$. Let $(\alpha_n)_n$ be a sequence of positive numbers such that $\alpha_n \to \alpha$ as $n \to \infty$. Prove that $\mu_{\alpha_n} \overset{\text{weakly}}{\longrightarrow} \mu_\alpha$, as $n \to \infty$.

Solution 1.

1) Let $x \geq 2$. We have
$$\begin{aligned}\mathbb{P}(|G_n| \geq x) &= 2\,\mathbb{P}(G_n \geq x) = 2\int_x^\infty e^{-y^2/2}\frac{dy}{\sqrt{2\pi}} \\ &\leq 2\int_x^\infty \frac{y}{x} e^{-y^2/2}\frac{dy}{\sqrt{2\pi}} = \frac{2}{x}e^{-x^2/2}\frac{1}{\sqrt{2\pi}} \\ &\leq e^{-x^2/2},\end{aligned}$$
since $x \geq 2$ and $\sqrt{2\pi} \geq 1$.
2) By the previous question,
$$\sum_n \mathbb{P}(|G_n| \geq 2n) \leq \sum_n e^{-2n^2} < \infty.$$
By Borel Cantelli, we get
$$\mathbb{P}(\limsup_n \{|G_n| \geq 2n\}) = 0,$$
i.e. $\mathbb{P}(\liminf_n \{|G_n| < 2n\} = 1$. This means that a.s. there exists n_0 such that for all $n \geq n_0$, $\frac{1}{n}|G_n| < 2$.
3) A.s., for n large enough, $e^{-n\alpha}|G_n| \leq e^{-n\alpha}2n$, which is summable in n, so
$$\sum_n e^{-n\alpha}|G_n| < \infty \quad \text{a.s.}$$
4) Let $\beta > 0$ and let $\alpha > \beta$. Then,
$$e^{-n\alpha}|G_n\cos(2\pi n t)| \leq e^{-n\beta}|G_n|,$$
which is integrable with respect to the counting measure by the previous question. Hence, by dominated convergence theorem (more precisely by continuity under the integral sign), one obtains that $(\alpha,t) \mapsto X_t^\alpha$ is continuous on $(\beta,\infty)\times[0,1]$. This holds for all $\beta > 0$, hence $(\alpha,t) \mapsto X_t^\alpha$ is continuous on $(0,\infty)\times[0,1]$.
Alternative proof: Let $\beta > 0$ and let $\alpha > \beta$. Then,
$$\sum_n e^{-n\alpha}|G_n\cos(2\pi n t)| \leq C\sum_n e^{-n\beta} n < \infty$$

Since $\sum_n e^{-n\alpha}|G_n \cos(2\pi nt)|$ is a normaly convergent series of continuous functions, it converges uniformly to a continuous function. Hence, $(\alpha, t) \mapsto X_t^\alpha$ is continuous on $(\beta, \infty)\times[0,1]$. This holds for all $\beta > 0$, hence $(\alpha, t) \mapsto X_t^\alpha$ is continuous on $(0, \infty)\times[0,1]$.

5) It is enough to prove the convergence of the finite dimensional distributions and tightness.

By the continuity of $(\alpha, t) \mapsto X_t^{(\alpha)}$, we have for all $t_1 < \cdots t_k$,

$$\left(X_{t_1}^{(\alpha_n)}, \ldots, X_{t_k}^{(\alpha_n)}\right) \to \left(X_{t_1}^{(\alpha)}, \ldots, X_{t_k}^{(\alpha)}\right)$$

as $n \to \infty$, almost surely. Hence, in distribution!

For the tightness, we apply Kolmogorov tightness criterion. The first condition is that $(X_0^{(\alpha_n)})_{n\geq 1}$ is tight, but this follows directly from the fact that this sequence converges in distribution by the convergence of finite dimensional distributions. Let η a lower bound of the $(\alpha_n)_n$.

$$\begin{aligned}
\mathbb{E}\left[|X_t^{(\alpha_n)} - X_s^{(\alpha_n)}|^2\right] &= \mathbb{E}\left[\left(\sum_k e^{-k\alpha_n} G_k(\cos(2\pi kt) - \cos(2\pi ks)\right)^2\right] \\
&\leq 4\pi^2 (t-s)^2\, \mathbb{E}\left[\left(\sum_k k e^{-k\eta} G_k\right)^2\right] \\
&\leq 4\pi^2 \sum_k k^2 e^{-k\eta}(t-s)^2 \\
&= C(t-s)^2.
\end{aligned}$$

By Kolmogorov tightness criterion, with $\beta = 1$ and $p = 1$, we get tightness of $(X^{(\alpha_n)})_{n\geq 1}$.

Exercise 2: Let $\mathcal{P}_2(\mathbb{R}^d)$ be the set of probability measures on $\mathbb{R}^d$ with finite second moment:

$$\mathcal{P}_2(\mathbb{R}^d) = \left\{\mu \in \mathcal{M}_1(\mathbb{R}^d) \mid \int_{\mathbb{R}^d} |x|^2 \mu(dx) < \infty\right\}.$$

Denote by W_2 the 2-Wasserstein distance on $\mathcal{P}_2(\mathbb{R}^d)$, i.e. for $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$,

$$W_2(\mu,\nu) = \left(\inf_{\pi\in\Pi(\mu,\nu)} \int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^2 d\pi(x,y)\right)^{1/2}.$$

The aim of this exercise is to prove the triangle inequality for W_2.

1) Let $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$. Show that there exists $\pi \in \Pi(\mu,\nu)$ and a pair of functions $(\alpha, \beta) \in L^1(\mu) \times L^1(\nu)$ with

$$\alpha(x) = \inf_{y\in\mathbb{R}^d}\left(|x-y|^2 - \beta(y)\right), \quad \beta(y) = \inf_{x\in\mathbb{R}^d}\left(|x-y|^2 - \alpha(x)\right)$$

for all $x, y \in \mathbb{R}^d$, such that

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^2 d\pi(x,y) = \int_{\mathbb{R}^d} \alpha d\mu + \int_{\mathbb{R}^d} \beta d\nu.$$

2) Define, for all $\eta > 0$, and all $y \in \mathbb{R}^d$,

$$\alpha_\eta(y) = \inf_{z\in\mathbb{R}^d}\left(\left(1+\frac{1}{\eta}\right)|y-z|^2 - \beta(z)\right).$$

Let γ be another probability measure in $\mathcal{P}_2(\mathbb{R}^d)$. Show that $\alpha_\eta \in L^1(\gamma)$.

3) Show that for all $x, y \in \mathbb{R}^d$,

$$\alpha(x) - \alpha_\eta(y) \leq (1+\eta)|x-y|^2.$$

Hint: show that for all $a, b \geq 0$ and $\eta > 0$, $(a+b)^2 \leq (1+\eta)a^2 + (1+\frac{1}{\eta})b^2$.

4) Prove that

$$\int \alpha(x) d\mu(x) - \int \alpha_\eta(y)\gamma(dy) \leq (1+\eta) W_2(\mu,\gamma)^2.$$

5) Prove that
$$\int \alpha_\eta(y)\gamma(dy) + \int \beta(z)\nu(dz) \leq (1+\frac{1}{\eta})W_2(\gamma,\nu)^2.$$
6) Deduce that
$$W_2(\mu,\nu)^2 \leq (1+\eta)W_2(\mu,\gamma)^2 + (1+\frac{1}{\eta})W_2(\gamma,\nu)^2.$$
7) By choosing η appropriately, deduce the triangle inequality for W_2.

Solution 2.

1) By Kantorovich duality, there exists π and (φ,ψ) conjugate to each others such that
$$\int \langle x,y\rangle d\pi(x,y) = \int \varphi d\mu + \int \psi d\nu.$$
Using $|x-y|^2 = |x|^2 + |y|^2 - 2\langle x,y\rangle$, we get
$$\begin{aligned}\int |x-y|^2 d\pi &= \int |x|^2 d\mu + \int |y|^2 d\nu - 2\int |x-y|^2 d\pi \\ &= \int |x|^2 d\mu + \int |y|^2 d\nu - 2\left(\int \varphi d\mu + \int \psi d\nu\right) \\ &= \int \left(|x|^2 - 2\varphi(x)\right) d\mu(x) + \int \left(|y|^2 - 2\psi(y)\right) d\nu(y).\end{aligned}$$
Define
$$\alpha(x) = |x|^2 - 2\varphi(x) \quad \text{and} \quad \beta(y) = |y|^2 - 2\psi(y).$$
Then, using that φ is the Legendre transform of ψ,
$$\begin{aligned}\alpha(x) &= |x|^2 - 2\sup_y \left(\langle x,y\rangle - \psi(y)\right) \\ &= \inf_y \left(|x|^2 - 2\langle x,y\rangle + 2\psi(y)\right) \\ &= \inf_y \left(|x-y|^2 - (|y|^2 - 2\psi(y))\right) \\ &= \inf_y \left(|x-y|^2 - \beta(y)\right)\end{aligned}$$
The same holds for β. Note that $(\alpha,\beta) \in L^1(\mu)\times L^1(\nu)$ since $(\varphi,\psi) \in L^1(\mu)\times L^1(\nu)$ and μ and ν have finite second moment.

2) The strategy consists in bounding α_η from above and below by a quadratic function, which is in $L^1(\gamma)$ because γ has a finite second moment.

Upper bound. Since $\beta \in L^1(\nu)$, there exists y_0 such that $|\beta(z_0)| < \infty$. Then, for any $x \in \mathbb{R}$,
$$\alpha_\eta(x) \leq (1+\frac{1}{\eta})|x-y_0|^2 - \beta(y_0).$$

Lower bound. Since $\alpha \in L^1(\nu)$, there exists x_0 such that $|\alpha(x_0)| < \infty$. Then, for any $y \in \mathbb{R}$,
$$\beta(y) \leq |y-x_0|^2 - \alpha(x_0).$$
Therefore, for any $x \in \mathbb{R}$,
$$\alpha_\eta(x) = \inf_y \left((1+\frac{1}{\eta})|x-y|^2 - \beta(y)\right) \geq \inf_y \left((1+\frac{1}{\eta})|x-y|^2 - |y-x_0|^2\right) + \alpha(x_0).$$
The infimum above can be computed by studying the function f defined by
$$f(y) = (1+\frac{1}{\eta})|x-y|^2 - |y-x_0|^2.$$

One has $\nabla f(y) = (1+\frac{1}{\eta})2(y-x) - 2(y-x_0)$ and therefore f has only one critical point, which is at $y = (1+\eta)x - \eta x_0$, and since f tends to $+\infty$ at infinity, f reaches its global minimum at this point. We deduce that $\inf f = -(1+\eta)|x-x_0|^2$, which gives

$$\alpha_\eta(x) \geq -(1+\eta)|x-x_0|^2 + \alpha(x_0),$$

proving the lower bound. Alternatively, one can prove $\inf f \geq -(1+\eta)|x-x_0|^2$ by using the hint of the next question:

$$|y-x_0|^2 \leq (|x-x_0| + |x-y|)^2 \leq (1+\eta)|x-x_0|^2 + (1+\frac{1}{\eta})|x-y|^2.$$

3) We first prove the hint: using $2xy \leq x^2 + y^2$,

$$(x+y)^2 = x^2 + y^2 + 2xy = x^2 + y^2 + 2 \cdot \sqrt{\eta}x \cdot y/\sqrt{\eta} \leq x^2 + y^2 + \eta x^2 + y^2/\eta.$$

Now we prove the main statement in the question, i.e. that for all $x, y \in \mathbb{R}^d$,

$$\alpha(x) \leq \alpha_\eta(y) + (1+\eta)|x-y|^2.$$

By definition of α and α_η it is enough to prove that for any $z \in \mathbb{R}^d$,

$$|x-z|^2 - \beta(z) \leq \Big(1+\frac{1}{\eta}\Big)|y-z|^2 - \beta(z) + (1+\eta)|x-y|^2,$$

because then taking the infimum on z yields the result. But this last inequality follows from the hint and the triangle inequality:

$$|x-z|^2 \leq (|x-y| + |y-z|)^2 \leq (1+\eta)|x-y|^2 + (1+\frac{1}{\eta})|y-z|^2,$$

4) Consider an optimal transport plan $\pi' \in \Pi(\mu,\gamma)$ between μ and γ. Then, integrating the previous inequality with respect to π' gives

$$\int (\alpha(x) - \alpha_\eta(y))d\pi \leq (1+\eta)\int |x-y|^2 d\pi.$$

By optimality of π', we have $\int |x-y|^2 d\pi = W_2(\mu,\gamma)^2$. On the other hand, using that $\alpha \in L^1(\mu)$, $\alpha_\eta \in L^1(\gamma)$ and $\pi' \in \Pi(\mu,\gamma)$, we have

$$\int (\alpha(x) - \alpha_\eta(y))d\pi = \int \alpha(x)d\mu(x) - \int \alpha_\eta(y)\gamma(dy).$$

This gives the desired inequality.

5) Same idea. Since

$$\alpha_\eta(y) + \beta(z) \leq (1+\frac{1}{\eta})|y-z|^2, \quad \forall y, z$$

integrating with respect to an optimal transport plan $\pi'' \in \Pi(\mu,\gamma)$ gives the result.

6) By choice of α and β, we have

$$W_2(\mu,\nu)^2 = \int \alpha d\mu + \int \beta d\nu = \int \alpha d\mu - \int \alpha_\eta d\gamma + \int \alpha d\gamma + \int \beta d\nu$$

and using the two previous inequality gives that:

$$W_2(\mu,\nu)^2 \leq (1+\eta)W_2(\mu,\gamma)^2 + (1+\frac{1}{\eta})W_2(\gamma,\nu)^2.$$

7) Assume $\mu \neq \gamma$ (if not the inequality is trivial). Then $W_2(\mu,\gamma) > 0$. Then, choosing

$$\eta = \frac{W_2(\gamma,\nu)}{W_2(\mu,\gamma)}$$

gives the result.

Exercise 3: Let (X, d_X) and (Y, d_Y) be two Polish spaces. Let $(c_k)_k$ be a sequence of continuous and bounded costs on $X \times Y$ that converges uniformly to some cost c. Let $(\mu_k)_{k\geq 1}$ be a sequence in $\mathcal{M}_1(X)$ converging weakly to μ and $(\nu_k)_{k\geq 1}$ be a sequence in $\mathcal{M}_1(Y)$ converging weakly to ν.

1) Let $(\pi_k)_{k\geq 1}$ be a sequence of transport plans such that $\pi_k \in \Pi(\mu_k, \nu_k)$ for all $k \geq 1$. Show that $(\pi_k)_{k\geq 1}$ is relatively compact.
2) Show that any accumulation point of $(\pi_k)_{k\geq 1}$ is a transport plan between μ and ν.
3) a) Let γ be a probability measure on a separable metric space E. Let $U \subset E$ be an open set such that $\gamma(U) > 0$. Show that $U \cap \operatorname{supp}\gamma \neq \varnothing$.
   b) Suppose that for all $k \geq 1$, π_k is optimal for the cost c_k. We assume that the sequence $(\pi_k)_{k\geq 1}$ converges weakly to some π. Let $(x_i, y_i)_{i=1,\dots,N}$ be a sequence of points in $\operatorname{supp}\pi$. Prove that for all $n \geq 1$, there exists k_n, with $k_n \underset{n\to\infty}{\longrightarrow} \infty$ and $(x_{n,i}, y_{n,i})_{i=1,\dots,N}$ in $\operatorname{supp}\pi_{k_n}$ such that
$$d_X(x_i, x_{n,i}) + d_Y(y_i, y_{n,i}) < \frac{1}{n}.$$
4) Deduce that $\operatorname{supp}\pi$ is c-cyclically monotone.

Solution 3.

1) Since (μ_n) and $(\nu_n)_n$ converge, they are tight sequences: there exists K_X compact, K_Y compact such that for all k
$$\mu_k(K_X^c) \leq \varepsilon/2, \quad \nu_k(K_Y^c) \leq \varepsilon/2.$$
Remark: one uses here that X and Y are Polish to use that weak convergence implies tightness!
Define $K = K_X \times K_Y$ which is compact. Then,
$$\pi_k(K^c) = \pi_k(K_X^c \times Y \cup X \times K_Y^c) \leq \mu_k(K_X^c) + \nu_k(K_Y^c) \leq \varepsilon.$$
Hence, the sequence $(\pi_k)_k$ is tight, hence relatively compact.
2) Up to some extraction, suppose that π_k converges to some π. But continuity of the canonical projections, and the continuous mapping theorem, we get
$$(p_X)_\# \pi_k \xrightarrow[k\to\infty]{\text{w}} (p_X)_\# \pi, \quad (p_Y)_\# \pi_k \xrightarrow[k\to\infty]{\text{w}} (p_Y)_\# \pi.$$
But $\mu_k = (p_X)_\# \pi_k$ so we get $\mu = (p_X)_\# \pi$. Same for ν. Hence $\pi \in \Pi(\mu, \nu)$.
3) a) We use the fact that $\mu((\operatorname{supp}\mu)^c) = 0$ It follows that any Borel set $A \subset (\operatorname{supp}\mu)^c$ satisfies $\mu(A) = 0$, so U is not included in $(\operatorname{supp}\mu)^c$, which means $U \cap \operatorname{supp}\mu \neq \varnothing$.
   b) Consider the open ball
$$U_{n,i} = B((x_i, y_i), 1/n) = \{(x, y) \mid d(x, x_i) + d(y, y_i) < \frac{1}{n}\}.$$
Since $(x_i, y_i) \in \operatorname{supp}\pi$, by definition of the support, $\pi(U_{n,i}) > 0$. By the Portmanteau theorem and the fact that π_k converges weakly to π,
$$0 < \pi(U_{n,i}) \leq \liminf_k \pi_k(U_{n,i})$$
for all n and all i. Hence, for all n and all i, there exists $K_{n,i}$ such that for all $k \geq K_{n,i}$,
$$\pi_k(U_{n,i}) > 0.$$
Now define $K_n = \max_{i=1,\dots,N} K_{n,i}$ and $k_n = \max\{K_n, n\}$. Thus we have, for all $i = 1, \dots, N$, and all n
$$\pi_{k_n}(U_{n,i}) > 0.$$
In particular, $U_{n,i} \cap \operatorname{supp}\pi_{k_n} \neq \varnothing$. Hence choose $(x_{n,i}, y_{n,i}) \in U_{n,i} \cap \operatorname{supp}\pi_{k_n}$, which gives the result.
4) Since $c_k \to c$ uniformly, and since the support of π_k is c_k-cyclically monotone by optimality, one has
$$0 \geq \sum_{i=1}^N c_{k_n}(x_{n,i+1}, y_{n,i}) - c_{k_n}(x_{n,i}, y_{n,i}) \underset{n\to\infty}{\longrightarrow} \sum_{i=1}^N c(x_{i+1}, y_i) - c(x_i, y_i),$$

which proves that $\operatorname{supp}\pi$ is c-cyclically monotone (recall that if $f_n \to f$ uniformly and $x_n \to x$, then $f_n(x_n) \to f(x)$, which is not true if the convergence is not uniform (take $f_n(x) = x^n$ for instance)).

Exercise 4: Let $v\colon \mathbb{R} \to \mathbb{R}$ be a bounded Borel function, such that the set D_v of its discontinuities is at most countable. We denote by λ the Lebesgue measure on $\mathbb{R}$ and by $\mathbb{W}$ the Wiener measure on $\mathscr{C} = C([0,1])$, that is $\mathbb{W}$ is the distribution of a standard Brownian motion $B = (B_t)_{t\in[0,1]}$.

1) a) Show that
$$\mathbb{E}\left(\int_0^1 \mathbb{1}_{\{B_t\in D_v\}}dt\right) = 0,$$
and deduce that there exists $N \subset \mathscr{C}$ with $\mathbb{W}(N) = 0$ such that for all $\omega \notin N$, $\lambda(\{t \in [0,1] \mid \omega(t) \in D_v\}) = 0$.
   b) Deduce that if $\omega \notin N$, then $\omega(t) \notin D_v$ for λ-almost all $t \in [0,1]$.
   c) Show that the functional Ψ defined on $\mathscr{C}$ by
$$\Psi(\omega) = \int_0^1 v(\omega(t))dt, \quad \omega \in \mathscr{C},$$
is continuous $\mathbb{W}$-almost everywhere.

2) Let $(S_n)_{n\geq 0}$ be a simple random walk, i.e. $S_0 = 0$ and for $n \geq 1$, $S_n = \xi_1 + \cdots + \xi_n$, where $(\xi_k)_{k\geq 1}$ is a sequence of i.i.d. random variables with $\xi_1 \sim \frac{1}{2}\delta_{+1} + \frac{1}{2}\delta_{-1}$. Let $(X_t^{(n)})_{t\in[0,1]}$ be the rescaled interpolated random walk.
   a) Prove that
$$\frac{1}{n}\#\{k \leq n \mid S_k = 0\}$$
converges in probability to 0 as $n \to \infty$. *We recall Stirling formula:* $n! \sim n^n e^{-n}\sqrt{2\pi n}$.
   b) Deduce that
$$\left|\int_0^1 \mathbb{1}_{\{X_t^{(n)}>0\}}dt - \frac{1}{n}\#\{k \leq n \mid S_k > 0\}\right|$$
converges in probability to 0 as $n \to \infty$.
   c) Show that $\frac{1}{n}\#\{k \leq n \mid S_k > 0\}$ converges in distribution to $\int_0^1 \mathbb{1}_{\{B_t>0\}}dt$.

Solution 4.

1) a) By Fubini theorem
$$\mathbb{E}\int_0^1 \mathbb{1}_{B_t\in D_v}dt = \int_0^1 \mathbb{P}(B_t \in D_v)dt = 0$$
since B_t has a continuous distribution and D_v is countable. Hence, $\mathbb{W}$-almost surely, $\lambda(t \mid B_t \in D_v) = 0$, that is there exists N with $\mathbb{W}(N) = 0$ such that for all $\omega \notin N$, $\lambda(t \mid \omega(t) \in D_v) = 0$.
   b) Hence, for all $\omega \notin N$, $\omega(t) \notin D_v$ for λ-almost all t.
   c) Let $\omega_n \to \omega$. Then, for all t, $\omega_n(t) \to \omega(t)$. Since v is continuous on D_v^c, for all $\omega \notin N$, we have
$$v(\omega_n(t)) \to v(\omega(t)),$$
for λ-almost all t. Since v is bounded, by dominated convergence theorem, we get that except on N, $\Psi(\omega_n) \to \Psi(\omega)$.

2) a) Let $L_n = \#\{k \leq n \mid S_k = 0\}$. It is enough to prove that $L_n/n \to 0$ in L^1. For this, we compute By Markov inequality,
$$\mathbb{E}\left[\frac{L_n}{n}\right] = \frac{1}{n}\sum_{k=0}^{n}\mathbb{P}(S_k = 0) = \frac{1}{n}\sum_{k=0}^{n}\binom{2k}{k}\frac{1}{2^{2k}}.$$

And by Stirling formula:

$$\binom{2k}{k} \sim \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{k}}.$$

By a simple integral/series comparaison, we have

$$\sum_{k=0}^{n} \frac{1}{\sqrt{k}} \leq \int_0^n \frac{1}{\sqrt{x}} dx = 2\sqrt{n}.$$

Thus we get that

$$\mathbb{E}\left[\frac{L_n}{n}\right] = O\left(\frac{1}{\sqrt{n}}\right),$$

and thus converges to 0.

b) Write

$$\int_0^1 \mathbb{1}_{X_t^{(n)}>0} dt = \sum_{k=0}^{n-1} \int_{k/n}^{(k+1)/n} \mathbb{1}_{X_t^{(n)}>0} dt.$$

For $t \in (\frac{k}{n}, \frac{k+1}{n})$, we have

$$\sqrt{n} X_t^{(n)} = S_k + (t-k)\xi_{k+1}.$$

Note that then $X_t^{(n)} > 0$ can only occur if $S_k > 0$ or $S_k = 0$ and $S_{k+1} > 0$ (i.e. $\xi_{k+1} = 1$). Thus,

$$\int_0^1 \mathbb{1}_{X_t^{(n)}>0} dt = \sum_{k=0}^{n-1} \int_{k/n}^{(k+1)/n} \mathbb{1}_{S_k>0} dt + \sum_{k=0}^{n-1} \int_{k/n}^{(k+1)/n} \mathbb{1}_{S_k=0,\xi_{k+1}=1} dt$$

so we get

$$\int_0^1 \mathbb{1}_{X_t^{(n)}>0} dt - \frac{1}{n} \sum_{k=0}^{n-1} \mathbb{1}_{S_k>0}$$

is bounded by $\frac{1}{n}\sum_{k=0}^{n-1} \mathbb{1}_{S_k=0}$ which converges to 0 in probability by the previous question.

c) By Donsker theorem and the extension of the continuous mapping theorem to a.e. continuous functions (see Question 2 of Exercise 24 in the notes), and Question 1.c here,

$$\int_0^1 \mathbb{1}_{X_t^{(n)}>0} dt$$

converges in distribution to $\int_0^1 \mathbb{1}_{B_t>0} dt$. Then recall that, by Slutsky theorem, if $X_n \to X$ in distribution and $Y_n \to 0$ in probability, then $X_n + Y_n \to X$ in distribution. Thus we conclude using the previous question.