

Exam - duration 3 hours

Handwritten notes and lecture notes are allowed. No electronic devices. Your answers may be in English or in French.

Exercise 1: Let $(G_n)_{n \geq 1}$ be a sequence of i.i.d. random variables with $G_1 \sim \mathcal{N}(0, 1)$.

- 1) Show that for all $n \geq 1$ and all real number $x \geq 2$, we have $\mathbb{P}(|G_n| \geq x) \leq e^{-x^2/2}$.
- 2) Show that almost surely, for n large enough, $\frac{1}{n}|G_n| < 2$. *Hint: use Borel Cantelli lemma.*
- 3) Deduce that for all $\alpha > 0$, almost surely, $\sum_{n \geq 1} e^{-n\alpha}|G_n| < \infty$.
- 4) For all $\alpha > 0$ and all $t \in [0, 1]$, we define

$$X_t^{(\alpha)} = \sum_{n \geq 1} e^{-n\alpha} G_n \cos(2\pi n t).$$

Show that almost surely, $(\alpha, t) \mapsto X_t^{(\alpha)}$ is continuous on $(0, +\infty) \times [0, 1]$.

- 5) Let $\alpha > 0$ and denote by μ_α the distribution of $(X_t^{(\alpha)})_{t \in [0, 1]}$. Let $(\alpha_n)_n$ be a sequence of positive numbers such that $\alpha_n \rightarrow \alpha$ as $n \rightarrow \infty$. Prove that $\mu_{\alpha_n} \xrightarrow{\text{weakly}} \mu_\alpha$, as $n \rightarrow \infty$.

Exercise 2: Let $\mathcal{P}_2(\mathbb{R}^d)$ be the set of probability measures on $\mathbb{R}^d$ with finite second moment:

$$\mathcal{P}_2(\mathbb{R}^d) = \left\{ \mu \in \mathcal{M}_1(\mathbb{R}^d) \mid \int_{\mathbb{R}^d} |x|^2 \mu(dx) < \infty \right\}.$$

Denote by W_2 the 2-Wasserstein distance on $\mathcal{P}_2(\mathbb{R}^d)$, i.e. for $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$,

$$W_2(\mu, \nu) = \left(\inf_{\pi \in \Pi(\mu, \nu)} \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^2 d\pi(x, y) \right)^{1/2}.$$

The aim of this exercise is to prove the triangle inequality for W_2 .

- 1) Let $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$. Show that there exists $\pi \in \Pi(\mu, \nu)$ and a pair of functions $(\alpha, \beta) \in L^1(\mu) \times L^1(\nu)$ with

$$\alpha(x) = \inf_{y \in \mathbb{R}^d} (|x - y|^2 - \beta(y)), \quad \beta(y) = \inf_{x \in \mathbb{R}^d} (|x - y|^2 - \alpha(x))$$

for all $x, y \in \mathbb{R}^d$, such that

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^2 d\pi(x, y) = \int_{\mathbb{R}^d} \alpha d\mu + \int_{\mathbb{R}^d} \beta d\nu.$$

- 2) Define, for all $\eta > 0$, and all $y \in \mathbb{R}^d$,

$$\alpha_\eta(y) = \inf_{z \in \mathbb{R}^d} \left(\left(1 + \frac{1}{\eta}\right) |y - z|^2 - \beta(z) \right).$$

Let γ be another probability measure in $\mathcal{P}_2(\mathbb{R}^d)$. Show that $\alpha_\eta \in L^1(\gamma)$.

- 3) Show that for all $x, y \in \mathbb{R}^d$,

$$\alpha(x) - \alpha_\eta(y) \leq (1 + \eta) |x - y|^2.$$

Hint: show that for all $a, b \geq 0$ and $\eta > 0$, $(a + b)^2 \leq (1 + \eta)a^2 + (1 + \frac{1}{\eta})b^2$.

- 4) Prove that

$$\int \alpha(x) d\mu(x) - \int \alpha_\eta(y) \gamma(dy) \leq (1 + \eta) W_2(\mu, \gamma)^2.$$

- 5) Prove that

$$\int \alpha_\eta(y) \gamma(dy) + \int \beta(z) \nu(dz) \leq (1 + \frac{1}{\eta}) W_2(\gamma, \nu)^2.$$

6) Deduce that

$$W_2(\mu, \nu)^2 \leq (1 + \eta)W_2(\mu, \gamma)^2 + (1 + \frac{1}{\eta})W_2(\gamma, \nu)^2.$$

7) By choosing η appropriately, deduce the triangle inequality for W_2 .

Exercise 3: Let (X, d_X) and (Y, d_Y) be two Polish spaces. Let $(c_k)_k$ be a sequence of continuous and bounded costs on $X \times Y$ that converges uniformly to some cost c . Let $(\mu_k)_{k \geq 1}$ be a sequence in $\mathcal{M}_1(X)$ converging weakly to μ and $(\nu_k)_{k \geq 1}$ be a sequence in $\mathcal{M}_1(Y)$ converging weakly to ν .

- 1) Let $(\pi_k)_{k \geq 1}$ be a sequence of transport plans such that $\pi_k \in \Pi(\mu_k, \nu_k)$ for all $k \geq 1$. Show that $(\pi_k)_{k \geq 1}$ is relatively compact.
- 2) Show that any accumulation point of $(\pi_k)_{k \geq 1}$ is a transport plan between μ and ν .
- 3) a) Let γ be a probability measure on a separable metric space E . Let $U \subset E$ be an open set such that $\gamma(U) > 0$. Show that $U \cap \text{supp } \gamma \neq \emptyset$.
b) Suppose that for all $k \geq 1$, π_k is optimal for the cost c_k . We assume that the sequence $(\pi_k)_{k \geq 1}$ converges weakly to some π . Let $(x_i, y_i)_{i=1, \dots, N}$ be a sequence of points in $\text{supp } \pi$. Prove that for all $n \geq 1$, there exists k_n , with $k_n \xrightarrow{n \rightarrow \infty} \infty$ and $(x_{n,i}, y_{n,i})_{i=1, \dots, N}$ in $\text{supp } \pi_{k_n}$ such that

$$d_X(x_i, x_{n,i}) + d_Y(y_i, y_{n,i}) < \frac{1}{n}.$$

- 4) Deduce that $\text{supp } \pi$ is c -cyclically monotone.

Exercise 4: Let $v: \mathbb{R} \rightarrow \mathbb{R}$ be a bounded Borel function, such that the set D_v of its discontinuities is at most countable. We denote by λ the Lebesgue measure on $\mathbb{R}$ and by $\mathbb{W}$ the Wiener measure on $\mathcal{C} = C([0, 1])$, that is $\mathbb{W}$ is the distribution of a standard Brownian motion $B = (B_t)_{t \in [0, 1]}$.

- 1) a) Show that

$$\mathbb{E} \left(\int_0^1 \mathbb{1}_{\{B_t \in D_v\}} dt \right) = 0,$$

and deduce that there exists $N \subset \mathcal{C}$ with $\mathbb{W}(N) = 0$ such that for all $\omega \notin N$, $\lambda(\{t \in [0, 1] \mid \omega(t) \in D_v\}) = 0$.

- b) Deduce that if $\omega \notin N$, then $\omega(t) \notin D_v$ for λ -almost all $t \in [0, 1]$.
- c) Show that the functional Ψ defined on $\mathcal{C}$ by

$$\Psi(\omega) = \int_0^1 v(\omega(t)) dt, \quad \omega \in \mathcal{C},$$

is continuous $\mathbb{W}$ -almost everywhere.

- 2) Let $(S_n)_{n \geq 0}$ be a simple random walk, i.e. $S_0 = 0$ and for $n \geq 1$, $S_n = \xi_1 + \dots + \xi_n$, where $(\xi_k)_{k \geq 1}$ is a sequence of i.i.d. random variables with $\xi_1 \sim \frac{1}{2}\delta_{+1} + \frac{1}{2}\delta_{-1}$. Let $(X_t^{(n)})_{t \in [0, 1]}$ be the rescaled interpolated random walk.

- a) Prove that

$$\frac{1}{n} \# \{k \leq n \mid S_k = 0\}$$

converges in probability to 0 as $n \rightarrow \infty$. We recall Stirling formula: $n! \sim n^n e^{-n} \sqrt{2\pi n}$.

- b) Deduce that

$$\left| \int_0^1 \mathbb{1}_{\{X_t^{(n)} > 0\}} dt - \frac{1}{n} \# \{k \leq n \mid S_k > 0\} \right|$$

converges in probability to 0 as $n \rightarrow \infty$.

- c) Show that $\frac{1}{n} \# \{k \leq n \mid S_k > 0\}$ converges in distribution to $\int_0^1 \mathbb{1}_{\{B_t > 0\}} dt$.