

III.4.2) The lower bound

We prove here the following stronger result

Proposition LB: On the survival event,  $\mathcal{T}_t \geq \lambda_c t - \frac{3}{2\lambda_c} \log t + O_p(1)$ , which means

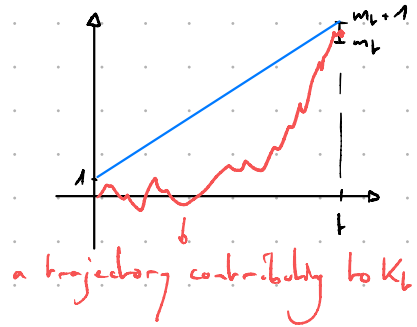
that, for any  $\varepsilon > 0$ , there exists  $\eta > 0$  such that for  $t$  large enough

$$P(\mathcal{T}_t \geq \lambda_c t - \frac{3}{2\lambda_c} \log t - \eta \mid \text{survival}) \geq 1 - \varepsilon.$$

We define  $m_t := \lambda_c t - \frac{3}{2\lambda_c} \log t$ ,

$\lambda_t := \frac{m_t}{t}$  the slope of our barrier,

$$K_t := \sum_{u \in \mathcal{W}_t} \mathbb{1}_{X_u(t) \geq m_t \text{ and } \max_{s \in [0,t]} X_u(s) - \lambda_t s \leq 1}.$$



We apply the 1<sup>st</sup> and 2<sup>nd</sup> moment argument to  $K_t$ .

We first check in the two following lemmas that  $E[K_t]$  and  $E[K_t^2]$  are of order 1 as  $t \rightarrow \infty$ .

Lemma 1:  $\frac{e^{-\lambda_c}}{\sqrt{2\pi}} \leq \liminf_{t \rightarrow \infty} E[K_t] \leq \limsup_{t \rightarrow \infty} E[K_t] \leq \frac{1}{\sqrt{2\pi}}$

Proof: By the many-to-one lemma,

$$E[K_t] = e^{m_t} P(\underline{B}_t \geq m_t, \max_{s \in [0,t]} B_s - \lambda_t s \leq 1)$$

Girsanov  $\downarrow$   
 $\Leftrightarrow B_t - \lambda_t t \geq 0$

$$= e^{m_t} E \left[ e^{-\lambda_t B_t - \frac{\lambda_t^2}{2} t} \mathbb{1}_{B_t \geq 0, \max_{s \in [0,t]} B_s \leq 1} \right] \rightarrow B_t \in [0,1]$$

Note that  $\frac{\lambda_t^2}{2} t = \frac{t}{2} \left( \frac{\lambda_c^2}{2} - 3 \frac{\log t}{t} + \left( \frac{3}{2\lambda_c} \frac{\log t}{t} \right)^2 \right) = m_t - \frac{3}{2} \log t + o(1)$   
 $= 2m$

Moreover, on  $\{B_t \in [0,1]\}$ , we have  $e^{-\lambda_t B_t} \in [e^{-\lambda_t}, 1] \subset [e^{-\lambda_c}, 1]$ , so we get

$$e^{-\lambda_c} t^{3/2} P(B_t \geq 0, \max_{s \in [0,t]} B_s \leq 1) \leq E[K_t] \leq t^{3/2} P(B_t \geq 0, \max_{s \in [0,t]} B_s \leq 1)$$

But  $P(B_t \geq 0, \max_{s \in [0,t]} B_s \leq 1) \sim \frac{1}{\sqrt{2\pi} t^{3/2}}$  by Corollary 2 of Lecture 8 (with  $a=g=1$ )

so we get the result. ▣

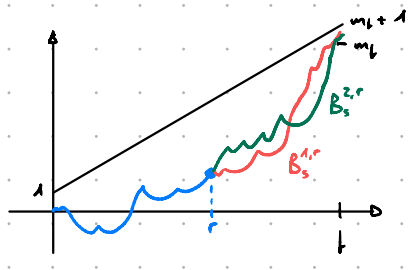
Lemma 2: There exists  $C > 0$  such that for  $t$  large enough  $E[K_t^2] \leq C$ .

Proof: We decompose  $E[K_t^2] = E[K_t] + E[K_t(K_t - 1)]$  and we already know that  $E[K_t]$  is bounded for  $t$  large. Then, we bound

$$E[K_t(K_t - 1)] = E\left[\sum_{\substack{u, r \in \mathcal{D}_t \\ u \neq r}} \mathbb{1}_{X_u(t) \geq m_t, \max_{s \in [0, t]} X_u(s) - \lambda_t s \leq 1} \mathbb{1}_{X_r(t) \geq m_t, \max_{s \in [0, t]} X_r(s) - \lambda_t s \leq 1}\right]$$

many-to-two

$$= E[L(L-1)] \int_0^t e^{2mb - mr} \mathbb{P}\left(\left\{ \begin{array}{l} B_{t-r}^{1,r} \geq m_t, \max_{s \in [0, t]} B_s^{1,r} - \lambda_t s \leq 1 \\ B_{t-r}^{2,r} \geq m_t, \max_{s \in [0, t]} B_s^{2,r} - \lambda_t s \leq 1 \end{array} \right\}\right) dr$$



see more details at the end of this note  $\ominus E\left[\mathbb{1}_{\max_{s \in [0, r]} B_s - \lambda_t s \leq 1} \varphi(B_r)^2\right]$

with  $\varphi(x) := \mathbb{P}\left(\max_{s \in [0, t-r]} B_s + x - \lambda_t(s+r) \leq 1, B_{t-r} + x \geq m_t\right)$

We first bound  $\varphi(x)$  for  $x \leq \lambda_t r + 1$  (note that  $\varphi(x) = 0$  for  $x > \lambda_t r + 1$ )

$$\varphi(x) = \mathbb{P}\left(\max_{s \in [0, t-r]} B_s - \lambda_t s \leq 1 + \lambda_t r - x, B_{t-r} - \lambda_t(t-r) \geq \underbrace{m_t - x - \lambda_t(t-r)}_{= \lambda_t t - x - \lambda_t t + \lambda_t r = \lambda_t r - x}\right)$$

Girsanov  $\downarrow$

$$= E\left[e^{-\lambda_t B_{t-r} - \frac{\lambda_t^2}{2}(t-r)} \mathbb{1}_{\max_{s \in [0, t-r]} B_s \leq 1 + \lambda_t r - x, B_{t-r} \geq \lambda_t r - x}\right]$$

$$\leq e^{\lambda_t(x - \lambda_t r) - \frac{\lambda_t^2}{2}(t-r)} \mathbb{P}\left(\max_{s \in [0, t-r]} B_s \leq \underbrace{1 + \lambda_t r - x}_a, B_{t-r} \geq \underbrace{\lambda_t r - x}_b \Rightarrow y=1\right)$$

Corollary 2 in lecture 8

$$\leq e^{\lambda_t(x - \lambda_t r) - \frac{\lambda_t^2}{2}(t-r)} \left(\frac{1 + \lambda_t r - x}{(t-r)^{3/2}} \wedge 1\right)$$

we can add this because  $\mathbb{P}(\dots) \leq 1$  (useful to make the function integrable at  $r=t$ )

$$E[K_t(K_t - 1)] \leq C \int_0^t e^{2mb - mr} E\left[\mathbb{1}_{\max_{s \in [0, r]} B_s - \lambda_t s \leq 1} \left(e^{-\lambda_t(B_r - \lambda_t r) - \frac{\lambda_t^2}{2}(t-r)} \left(\frac{1 + \lambda_t r - B_r}{(t-r)^{3/2}} \wedge 1\right)\right)^2\right] dr$$

$$= C \int_0^t e^{2mb - mr - \lambda_t^2(t-r)} E\left[\mathbb{1}_{\max_{s \in [0, r]} B_s - \lambda_t s \leq 1} e^{2\lambda_t(B_r - \lambda_t r)} \left(\frac{(1 + \lambda_t r - B_r)^2}{(t-r)^3} \wedge 1\right)\right] dr$$

$$E[\dots] = E\left[e^{-\lambda_t B_r - \frac{\lambda_t^2}{2}r} \mathbb{1}_{\max_{s \in [0, r]} B_s \leq 1} e^{2\lambda_t B_r} \left(\frac{(1 - B_r)^2}{(t-r)^3} \wedge 1\right)\right] \quad (\text{by Girsanov})$$

$$= e^{-\frac{\lambda_t^2}{2}r} E\left[e^{-\lambda_t B_r} \mathbb{1}_{\max_{s \in [0, r]} B_s \leq 1} \left(\frac{(1 - B_r)^2}{(t-r)^3} \wedge 1\right)\right]$$

we distinguish according to values of  $B_r$  (note that  $B_r \leq 1$  on the considered event)

$$\begin{aligned}
&= e^{-\frac{\lambda_k^2}{2}} \sum_{k \geq 0} \mathbb{E} \left[ \mathbb{1}_{B_r \in [-k, -k+1]} e^{\lambda_k B_r} \mathbb{1}_{\max_{s \in [0, r]} B_s \leq 1} \left( \frac{(1 - B_r)^2 \wedge 1}{(1-r)^3} \right) \right] \\
&\leq \left( \frac{1}{(1-r)^3} \wedge 1 \right) e^{-\frac{\lambda_k^2}{2}} e^{\lambda_k} \sum_{k \geq 0} (k+1)^2 e^{-\lambda_k k} \mathbb{P} \left( \max_{s \in [0, r]} B_s \leq 1, B_r \geq -k \right) \\
&\leq \left( \frac{1}{(1-r)^3} \wedge 1 \right) \left( \frac{1}{r^{3/2}} \wedge 1 \right) e^{-\frac{\lambda_k^2}{2}} \times e^{\lambda_k} \sum_{k \geq 0} (k+1)^4 e^{-\lambda_k k} \\
&\leq C' \left( \frac{1}{(1-r)^3} \wedge 1 \right) \left( \frac{1}{r^{3/2}} \wedge 1 \right) e^{-\frac{\lambda_k^2}{2}} \leq C' \quad \text{recall } \lambda_k = \lambda_c - \frac{3}{2} \frac{\log k}{k} \in \left[ \frac{\lambda_c}{2}, \lambda_c \right] \text{ for } k \text{ large enough}
\end{aligned}$$

Corollary 2 of Lecture 8  $\rightarrow \leq \frac{(k+1)^2}{r^{3/2}} \wedge 1 \leq \left( \frac{1}{r^{3/2}} \wedge 1 \right) (k+1)^2$   
 $\Rightarrow y = k+1$

$$\mathbb{E}[K_b(K_b - 1)] \leq C'' \int_0^b e^{2mb - mr - \lambda_k^2 b + \frac{\lambda_k^2}{2} r} \left( \frac{1}{(1-r)^3} \wedge 1 \right) \left( \frac{1}{r^{3/2}} \wedge 1 \right) dr$$

But  $\lambda_k^2 = \lambda_c^2 - 3 \frac{\log k}{k} + \left( \frac{3}{2} \frac{\log k}{k} \right)^2 \left. \begin{aligned} &\geq 2m - 3 \frac{\log k}{k} \\ &\leq 2m - 3 \frac{\log k}{k} + \frac{1}{k} \end{aligned} \right\} \text{ for } k \text{ large enough}$

so  $2mb - \lambda_k^2 b \leq 3 \log k$  and  $\frac{\lambda_k^2}{2} r - mr \leq -\frac{3}{2} \frac{r}{k} \log k + \frac{1}{2}$  which yields:

$$\begin{aligned}
\mathbb{E}[K_b(K_b - 1)] &\leq C''' \int_0^b t^{3 - \frac{3r}{2t}} \left( \frac{1}{(1-r)^3} \wedge 1 \right) \left( \frac{1}{r^{3/2}} \wedge 1 \right) dr \\
&\leq C''' \left( \int_0^{b/2} t^3 \frac{1}{(b/2)^3} \left( \frac{1}{r^{3/2}} \wedge 1 \right) dr + \int_{b/2}^b t^{3 - \frac{3r}{2t}} \left( \frac{1}{(1-r)^3} \wedge 1 \right) \frac{1}{(b/2)^{3/2}} dr \right) \\
&\leq C'''' \left( \int_0^{b/2} \left( \frac{1}{r^{3/2}} \wedge 1 \right) dr + \int_0^{b/2} t^{\frac{3s}{2t}} \left( \frac{1}{s^3} \wedge 1 \right) ds \right) \\
&\leq C'''' \leq cste \quad \leq \int_0^{\sqrt{t}} t^{\frac{3}{2\sqrt{t}}} \left( \frac{1}{s^3} \wedge 1 \right) ds + \int_{\sqrt{t}}^{b/2} t^{\frac{3}{4}} \frac{1}{s^3} ds \leq cste \\
&= t^{3/4} \left[ \frac{-1}{2s^2} \right]_{\sqrt{t}}^{b/2} \leq \frac{t^{3/4}}{2t}
\end{aligned}$$

Proof of Proposition LB:

Step 1: Lower bounding the probability to have a large maximum.

We use the following version of Paley-Zygmund inequality for integer-valued r.v.:

$$\mathbb{P}(K_b \geq 1) \geq \frac{\mathbb{E}[K_b]^2}{\mathbb{E}[K_b^2]}$$

(Proof:  $\mathbb{E}[K_b]^2 = \mathbb{E}[K_b \mathbb{1}_{K_b \geq 1}]^2 \leq \mathbb{E}[K_b^2] \mathbb{E}[\mathbb{1}_{K_b \geq 1}^2] = \mathbb{E}[K_b^2] \mathbb{P}(K_b \geq 1)$ )

Cauchy-Schwarz

Together with Lemma 1 and 2, we get: there exists  $c > 0$  and  $t_0 \geq 0$  such that for any  $t \geq t_0$ ,  $P(K_t \geq 1) \geq c$ .

In particular,  $\forall t \geq t_0$ ,  $P(\Gamma_t \geq m_t) \geq c$ .

Step 2: Letting branch at the beginning to improve the lower bound.

Let  $\varepsilon > 0$ . We want to prove that there exists  $\eta > 0$  such that for  $t$  large enough  $P(\Gamma_t < m_t - \eta | S) \leq \varepsilon$ , where  $S$  is the survival event.

Let  $n \geq 1$  to be chosen in terms of  $\varepsilon$  later.

On  $S$ , we have  $N_t \xrightarrow[t \rightarrow \infty]{a.s.} \infty$ , (more precisely,  $e^{-mt} N_t = W_t^\circ \xrightarrow[t \rightarrow \infty]{a.s.} W_\infty^\circ > 0$  on  $S$ )

so  $P(N_t \geq n | S) \xrightarrow[t \rightarrow \infty]{} 1$ .

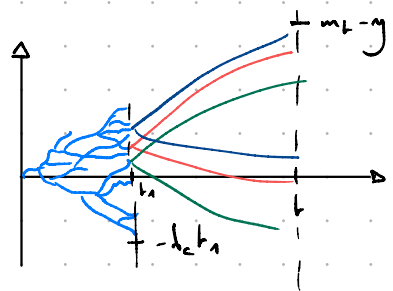
On the other hand,  $P(\min_{u \in \mathcal{D}_t} X_u(t) \geq -\lambda_c t) \xrightarrow[t \rightarrow \infty]{} 1$ , so there exists  $t_1 \geq 0$

such that  $P(A_{t_1} | S) \geq 1 - \frac{\varepsilon}{2}$  where  $A_{t_1} = \{N_{t_1} \geq n, \min_{u \in \mathcal{D}_{t_1}} X_u(t_1) \geq -\lambda_c t_1\}$

Now, let  $t \geq t_0 + t_1$ , we have

$$P(\Gamma_t < m_t - \eta | S) \leq \frac{\varepsilon}{2} + P(A_{t_1} \cap \{\Gamma_t < m_t - \eta\} | S)$$

$$\leq \frac{\varepsilon}{2} + \frac{1}{P(S)} P(\underbrace{A_{t_1}}_{\mathcal{F}_{t_1}\text{-measurable}} \cap \{\Gamma_t < m_t - \eta\})$$



Note that  $\{\Gamma_t < m_t - \eta\} = \{ \forall u \in \mathcal{D}_t, \max_{\substack{v \in \mathcal{D}_t \\ v \geq u}} X_v(t) < m_t - \eta \}$

$$= \{ \forall u \in \mathcal{D}_{t_1}, \max_{\substack{v \in \mathcal{D}_t \\ v \geq u}} X_v(t) - X_u(t_1) < \underbrace{m_t - \eta - X_u(t_1)}_{\leq m_t - \eta + \lambda_c t_1} \}$$

by the branching property, independent of  $\mathcal{F}_{t_1}$  and of each other, and  $\stackrel{(d)}{=} \Gamma_{t-t_1}$  on the event  $A_{t_1}$

So on  $A_{t_1}$ ,  $P(\Gamma_t \leq m_t - \eta | \mathcal{F}_{t_1}) \leq \prod_{u \in \mathcal{D}_{t_1}} P(\Gamma_{t-t_1} < m_t - \eta + \lambda_c t_1)$

Choosing  $\eta \geq 2\lambda_c t_1$ , we have  $m_t - \eta + \lambda_c t_1 \leq \lambda_c t - \frac{3}{2} \log t - \lambda_c t_1 \leq m_{t-t_1}$ ,

and so  $P(\Gamma_{t-t_1} < m_t - \eta + \lambda_c t_1) \leq P(\Gamma_{t-t_1} < m_{t-t_1}) \stackrel{\text{by Step 1}}{\leq} 1 - c$  ( $t-t_1 > t_0$ )

Moreover, on  $A_{k_1}$ , we have  $N_{k_1} \geq n$ , so  $P(\Pi_t \leq m_t - \gamma | \mathcal{F}_{k_1}) \leq (1-c)^n$ .

It follows that  $P(\Pi_t < m_t - \gamma | S) \leq \frac{\Sigma}{Z} + \frac{(1-c)^n}{P(S)} \leq \Sigma$  if  $n$  is chosen large enough.  $\blacksquare$

Additional details in the proof of Lemma 2 for the equality

$$P\left(\left\{\begin{array}{l} B_t^{1,r} \geq m_t, \max_{s \in [0,t]} B_s^{1,r} - \lambda_t s \leq 1 \\ B_t^{2,r} \geq m_t, \max_{s \in [0,t]} B_s^{2,r} - \lambda_t s \leq 1 \end{array}\right\}\right) = E\left[\mathbb{1}_{\max_{s \in [0,r]} B_s - \lambda_t s \leq 1} \Psi(B_r)^2\right]$$

$$\text{where } \Psi(z) := P\left(\max_{s \in [0,t-r]} B_s + z - \lambda_t(s+r) \leq 1, B_{t-r} + z \geq m_t\right)$$

First recall that  $B^{1,r}$  and  $B^{2,r}$  can be defined using 3 independent Brownian motions  $B^0, B^1, B^2$  by setting, for  $i \in \{1,2\}$ ,  $B_s^{i,r} = \begin{cases} B_s^i & \text{if } s \leq r, \\ B_{s-r}^i + B_r^0 & \text{if } s > r. \end{cases}$

Then we can rewrite

$$P\left(\left\{\begin{array}{l} B_t^{1,r} \geq m_t, \max_{s \in [0,t]} B_s^{1,r} - \lambda_t s \leq 1 \\ B_t^{2,r} \geq m_t, \max_{s \in [0,t]} B_s^{2,r} - \lambda_t s \leq 1 \end{array}\right\}\right) = \max_{p \in [0,t-r]} B_p^1 + B_r^0 - \lambda_t(p+r) \quad \text{by setting } p = s-r$$

$$= P\left(\max_{s \in [0,r]} B_s^0 - \lambda_t s \leq 1, \left\{\begin{array}{l} B_{t-r}^1 + B_r^0 \geq m_t, \max_{s \in [r,t]} B_{s-r}^1 + B_r^0 - \lambda_t s \leq 1 \\ B_{t-r}^2 + B_r^0 \geq m_t, \max_{s \in [r,t]} B_{s-r}^2 + B_r^0 - \lambda_t s \leq 1 \end{array}\right\}\right)$$

$\downarrow$  write  $P(\dots) = E[P(\dots | B^0)]$

$$= E\left[\mathbb{1}_{\max_{s \in [0,r]} B_s^0 - \lambda_t s \leq 1} P\left(\begin{array}{l} B_{t-r}^1 + B_r^0 \geq m_t, \max_{p \in [0,t-r]} B_p^1 + B_r^0 - \lambda_t(p+r) \leq 1 \\ B_{t-r}^2 + B_r^0 \geq m_t, \max_{p \in [0,t-r]} B_p^2 + B_r^0 - \lambda_t(p+r) \leq 1 \end{array} \middle| B^0\right)\right]$$

$\downarrow$   $B^1$  and  $B^2$  are independent and have the same law

$$= E\left[\mathbb{1}_{\max_{s \in [0,r]} B_s^0 - \lambda_t s \leq 1} \underbrace{P\left(B_{t-r}^1 + B_r^0 \geq m_t, \max_{p \in [0,t-r]} B_p^1 + B_r^0 - \lambda_t(p+r) \leq 1 \middle| B^0\right)^2}_{= \Psi(B_r^0)}\right]$$

We use here that: if  $X$  and  $Y$  are independent r.v. then for any  $f$  measurable positive,

$$E[f(X,Y) | X] = g(X) \text{ where } g(x) := E[f(x,Y)].$$

Proof:  $g(X)$  is  $\sigma(X)$ -measurable and, for any  $h$  measurable positive

$$E[g(X)h(X)] = \int g(x)h(x) dP_X(x) \stackrel{\text{law of } X}{=} \int \left(\int f(x,y) dP_Y(y)\right) h(x) dP_X(x) \stackrel{\text{Fubini}}{=} \iint f(x,y)h(x) d(P_X \otimes P_Y)(x,y)$$

$$\text{independence} \stackrel{\text{def}}{=} \iint f(x,y)h(x) dP_{(X,Y)}(x,y) = E[f(X,Y)] \rightarrow \text{this characterizes } E[f(X,Y) | X].$$