Lecture 8 : The Logarithmic correction

14/01/2026

III.3) Preliminaries on Brownian motion

For the proof of the second order of $M_t = \max_{u \in \mathcal{N}_t} X_u(t)$, we need estimates for Brownian motion staying below a barrier that we prove in this section.

We first need to state the strong Markov property for Brownian motion.

Theorem (strong Markov property): Let T be a stopping time for the canonical filtration of a Brownian motion $(B_t)_{t\geq 0}$. Define, for $t \geq 0$,

$$B_t^{(T)} = \begin{cases} B_{T+t} - B_T & \text{if } T < \infty \\ 0 & \text{otherwise.} \end{cases}$$

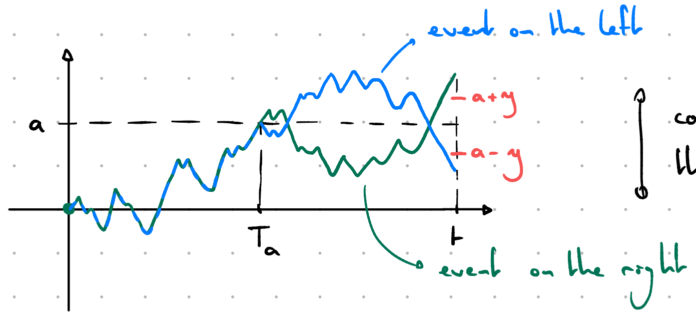
Assume $\mathbb{P}(T<\infty) > 0$. Then, under $\mathbb{P}(\cdot \mid T < \infty)$, the process $(B_t^{(T)})_{t\geq 0}$ is a Brownian motion started at 0 and independent of $\mathcal{F}_T$.

Proof: See Theorem 14.15 in Le Gall, "Measure Theory, Probability, and Stochastic Processes". It relies on a discretization of time by distinguishing according to the events $\{\frac{k-1}{2^n} < T < \frac{k}{2^n}\}$ for $k \geq 1$, in order to then apply the ordinary Markov property. ■

Proposition (reflection principle): For $t \geq 0$, $a > 0$ and $y \geq 0$, we have

$$\mathbb{P}\left(\max_{s\leq t} B_s \geq a,\ B_t \leq a-y\right) = \mathbb{P}(B_t \geq a+y).$$

A picture of the proof:



correspondence by reflecting the trajectory after T_a with respect to level a.

Proof: Let $T_a = \inf\{t \geq 0 : B_t = a\}$. Note that $\{\max_{s\leq t} B_s \geq a\} = \{T_a \leq t\}$.

So $$\mathbb{P}\left(\max_{s\leq t} B_s \geq a,\ B_t \leq \underbrace{a}_{= B_{T_a}} - y\right) = \mathbb{P}(T_a \leq t,\ B_t - B_{T_a} \leq -y)$$

$$= \mathbb{P}\left(T_a \leq t,\ B_{t-T_a}^{(T_a)} \leq -y\right).$$

↳ defined as in the theorem above

$$= \mathbb{P}\left((T_a, B^{(T_a)}) \in H\right)$$

where $H = \{(s,w) \in (\mathbb{R}_+ \cup \{\infty\}) \times \underbrace{\mathcal{C}(\mathbb{R}_+)}_{\substack{\text{continuous functions} \\ \text{from } \mathbb{R}_+ \to \mathbb{R}}} : s \leq t,\ w(t-s) \leq b-a\}$ is measurable (exercise!)

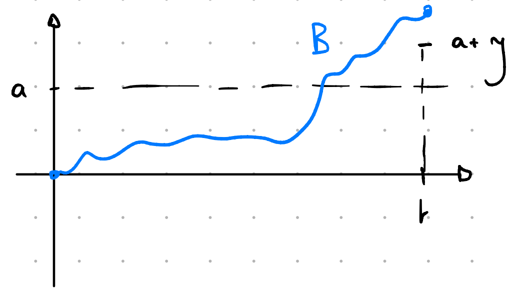
(hint: prove it is closed when $\mathcal{C}(\mathbb{R}_+)$ is equipped with the topology of uniform convergence on compact sets)

But, by the strong Markov property, we have, given $T_a < \infty$,

- $B^{(T_a)}$ is independent of $\mathcal{F}_{T_a}$ so it is independent of T_a.
- $B^{(T_a)}$ has the same law as $-B^{(T_a)}$ (both the Brownian motion has a law invariant by a change of sign).

So $(T_a, B^{(T_a)}) \overset{(d)}{=} (T_a, -B^{(T_a)})$ given $T_a < \infty$. But when $T_a = \infty$, we have $B^{(T_a)} = 0 = -B^{(T_a)}$. So we have $(T_a, B^{(T_a)}) \overset{(d)}{=} (T_a, -B^{(T_a)})$. (Remark: actually $T_a < \infty$ a.s. but we do not need it here)

Therefore
$$\begin{aligned}
\mathbb{P}\left((T_a, B^{(T_a)}) \in H\right) &= \mathbb{P}\left((T_a, -B^{(T_a)}) \in H\right) \\
&= \mathbb{P}\left(T_a \leq t,\ -(B_t - \underbrace{B_{T_a}}_{=a}) \leq -y\right) \\
&= \mathbb{P}\left(T_a \leq t,\ B_t \geq a+y\right) \\
&= \mathbb{P}(B_t \geq a+y)
\end{aligned}$$
because $\{T_a \leq t\} \subset \{B_t \geq a+y\}$ by the intermediate value theorem ($B_0 = 0 \leq a$ and $B_t \geq a+y \geq a$ + continuity of B) ∎



Corollary 1: For $t \geq 0$, $\max_{s \leq t} B_s \overset{(d)}{=} |B_t|$.

In particular, for $a > 0$, $\mathbb{P}\left(\max_{s \leq t} B_s < a\right) = \int_{-a}^{a} \frac{e^{-x^2/2t}}{\sqrt{2\pi t}}\, dx$

and, as $t \to \infty$, with a possibly depending on t such that $a = o(\sqrt{t})$,
$$\mathbb{P}\left(\max_{s \leq t} B_s < a\right) \underset{t \to \infty}{\sim} \sqrt{\frac{2}{\pi}} \frac{a}{\sqrt{t}}.$$

Proof: Let $a > 0$
$$\mathbb{P}\left(\sup_{s \leq t} B_s \geq a\right) = \mathbb{P}\left(\underbrace{\sup_{s \leq t} B_s \geq a}_{\subset \{B_t > a\}},\ B_t > a\right) + \underbrace{\mathbb{P}\left(\sup_{s \leq t} B_s \geq a,\ B_t \leq a\right)}_{\text{Proposition above}}$$
$$= \mathbb{P}(B_t > a) + \mathbb{P}(B_t \geq a) = \mathbb{P}(B_t \leq -a) + \mathbb{P}(B_t \geq a) = \mathbb{P}(|B_t| \geq a)$$

This characterizes the distribution (because the r.v. are positive).

Then $P\left(\max_{s\le t} B_s < a\right) = P(|B_t|<a) = \int_{-a}^{a} \frac{e^{-x^2/2t}}{\sqrt{2\pi t}}\,dx$

Note that $1-\frac{x^2}{2t} \le e^{-x^2/2t} \le 1$ which gives $\int_{-a}^{a} e^{-x^2/2t}\,dx = 2a + O\left(\frac{a^3}{t}\right) \sim 2a$ as $t\longrightarrow\infty$ because $a=o(\sqrt{t})$, and so $\int_{-a}^{a} \frac{e^{-x^2/2t}}{\sqrt{2\pi t}}\,dx \underset{t\to\infty}{\sim} \sqrt{\frac{2}{\pi}}\,\frac{a}{\sqrt{t}}$. ∎

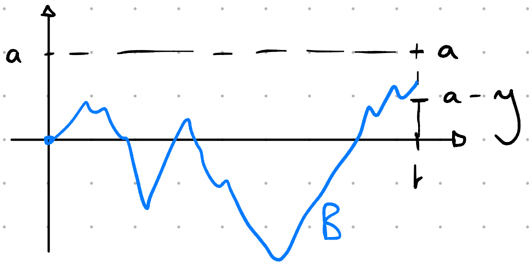
Corollary 2: Let $a>0$, $y\geq 0$ and $t\geq 0$.

$$P\left(\max_{s\le t} B_s < a,\ B_t \geq a-y\right) = \int_{-a}^{a} \frac{e^{-x^2/2t} - e^{-(x-y)^2/2t}}{\sqrt{2\pi t}}\,dx \le \frac{a y^2}{\sqrt{2\pi}\, t^{3/2}}.$$

Moreover, as $t\longrightarrow\infty$, with a,y possibly dependent on t with $y=o(\sqrt{t})$ and $a=o(\sqrt{t})$,

$$P\left(\max_{s< t} B_s < a,\ B_t \geq a-y\right) \underset{t\to\infty}{\sim} \frac{a y^2}{\sqrt{2\pi}\, t^{3/2}}.$$

A picture of the event:



Exercise 1: Proof of the corollary

1.| Exact formulas

1.a| Prove that $P\left(\max_{s\le t} B_s < a,\ B_t \le a-y\right) = P(B_t \le a-y) - P(B_t \geq a+y)$

1.b.| Deduce that $P\left(\max_{s\le t} B_s < a,\ B_t \le a-y\right) = P\left(B_t \in [y-a, y+a]\right)$

1.c.| Using Corollary 1, prove the equality in the statement.

1.d.| Prove that $P\left(\max_{s\le t} B_s < a,\ B_t \geq a-y\right) = \int_{a}^{a+y} \frac{e^{-(x-y)^2/2t} - e^{-x^2/2t}}{\sqrt{2\pi t}}\,dx$.

2.| Upper bound

2.a| Prove that for any $x,y\in\mathbb{R}$, $\left|e^{-(x-y)^2/2t} - e^{-x^2/2t}\right| \le \frac{y}{2t}|2x-y|$.

2.b| Prove the inequality in the statement.

Hint: Use formula in 1.d if $y\le 2a$ and formula in 1.c. if $y>2a$.

3.| Asymptotic equivalent

3.a.| Prove that $e^{-(x-y)^2/2t} - e^{-x^2/2t} = \frac{y(2x-y)}{2t} + O\left(\frac{(x+y)x^2y}{t^2}\right)$ uniformly in $x\in[-a-y, a+y]$.

3.b.| Prove the asymptotic equivalent.

Hint: same as in 2.b.

We conclude this section by recalling Girsanov's theorem for Brownian motion and by providing a proof in the following exercise.
The proof presented below is based on an elementary approach, whereas the general Girsanov's theorem uses martingale theory (for the general version see Section 5.5 of Le Gall, Brownian motion, Martingales, and Stochastic Calculus).

Exercise 2: Girsanov's theorem

Let $t \geq 0$ and $\mathcal{C}([0,t])$ denote the set of continuous functions from $[0,t] \longrightarrow \mathbb{R}$.
We equip $\mathcal{C}([0,t])$ with the norm $\|-\|_\infty$ and the associated Borelian σ-field.

1. We prove here the following simple version (the most used in this class):
For any $\lambda \in \mathbb{R}$ and $F : \mathcal{C}([0,t]) \longrightarrow \mathbb{R}_+$ measurable, we have

$$\mathbb{E}\left[e^{\lambda B_t - \frac{\lambda^2}{2}t} F\left((B_s)_{s\in[0,t]}\right)\right] = \mathbb{E}\left[F\left((B_s + \lambda s)_{s\in[0,t]}\right)\right]$$

1.a. Prove it for F of the form $F\left((B_s)_{s\in[0,t]}\right) = f(B_t)$ with $f : \mathbb{R} \to \mathbb{R}_+$ measurable.

Hint: Use the density of B_t to write the expectations as integrals.

1.b. Conclude.

Hint: The Borelian σ-field on $\mathcal{C}([0,t])$ is generated by cylinders so it is enough to consider F of the form $F\left((B_s)_{s\in[0,t]}\right) = \prod_{k=1}^{n} f_k\left(B_{t_k} - t_{k-1}\right)$ where $0 = t_0 < \dots < t_n = t$ and $f_1 \longrightarrow f_n : \mathbb{R} \longrightarrow \mathbb{R}_+$ are measurable.

2. Use the previous question to prove the more general form: for $h : [0,t] \to \mathbb{R}$ continuous and piecewise $\mathcal{C}^1$ and F as before,

$$\mathbb{E}\left[\exp\left(\int_0^t h'(s)\, dB_s - \int_0^t \frac{h'(s)^2}{2} ds\right) F\left((B_s)_{s\in[0,t]}\right)\right] = \mathbb{E}\left[F\left((B_s + h(s))_{s\in[0,t]}\right)\right]$$

Hint: Use that h can be approximated by piecewise affine functions, together with the hint of question 1.b.

III.4) The logarithmic correction

Our goal in this section is to prove the following theorem:

Theorem (Bramson 1978) On the survival event $\dfrac{M_t - \lambda_c t}{\log t} \xrightarrow[t\to\infty]{\mathbb{P}} -\dfrac{3}{2\lambda_c}$.

Remark: This has to be compared with the similar result in the iid case:

$$\frac{\tilde{M}_t - \lambda_c t}{\log t} \xrightarrow[t\to\infty]{\mathbb{P}} -\frac{1}{2\lambda_c}.$$

III.4.1) The upper bound

We need to show that, for any $\varepsilon > 0$, $\mathbb{P}\left(M_t \geq \lambda_c t - \left(\frac{3}{2\lambda_c} - \varepsilon\right)\log t\right) \xrightarrow[t\to\infty]{} 0$.

We prove the following stronger result.

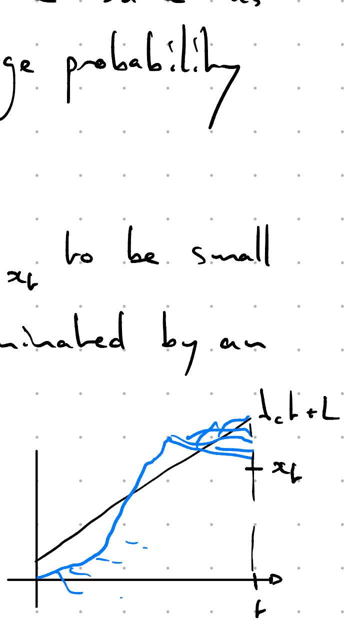
Proposition UB: $\mathbb{P}\left(M_t \geq \lambda_c t - \frac{3}{2\lambda_c}\log t + \frac{3}{\lambda_c}\log\log t\right) \xrightarrow[t\to\infty]{} 0$.

Idea of the proof: Let $x_t = \lambda_c t - \frac{3}{2\lambda_c}\log t + \frac{3}{\lambda_c}\log\log t$.

Then $\mathbb{P}(M_t \geq x_t) = \mathbb{P}(\exists u \in \mathcal{N}_t : X_u(t) \geq x_t) \leq \mathbb{E}\left[\sum_{u\in\mathcal{N}_t} \mathbb{1}_{X_u(t)\geq x_t}\right]$,

but this cannot work: by the many-to-one this expectation is the same as in the iid case where there are particles above x_t with large probability (because $\tilde{M}_t = \lambda_c t - \frac{1}{2\lambda_c}\log t + O(1)$).

So $\mathbb{E}\left[\sum_{u\in\mathcal{N}_t} \mathbb{1}_{X_u(t)\geq x_t}\right]$ is large even if we expect $\sum_{u\in\mathcal{N}_t} \mathbb{1}_{X_u(t)\geq x_t}$ to be small with high probability: this means that this expectation is dominated by an unlikely event on which $\sum_{u\in\mathcal{N}_t} \mathbb{1}_{X_u(t)\geq x_t}$ is very large.



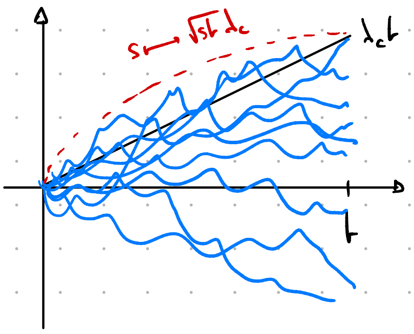
The issue is this type of event: one particle goes very high and then, by branching, drops many particles above x_t.

But this event is unlikely because M_s should be below $\lambda_c s + L$ for all s if L is large enough (already seen last time, see the lemma below).

So we first need to add the knowledge that particles stay below the line $s \mapsto \lambda_c s + L$ (we remove the bad event) before computing the first moment.

Such an argument could not be done in the iid case: at a time $s = bt$, $b \in (0,1)$, the maximum of the $\lfloor e^{mt} \rfloor$ particles is far above $\lambda_c s$:

$$\max_{1 \le i \le \lfloor e^{mt} \rfloor} B_s^i \overset{(d)}{=} \sqrt{\frac{s}{t}} \max_{1 \le i \le \lfloor e^{mt} \rfloor} B_t^i \underset{t\to\infty}{\sim} \sqrt{\frac{s}{t}} \times \lambda_c t = \frac{1}{\sqrt{b}} \lambda_c s \gg \lambda_c s \ !$$



This is because there are $\lfloor e^{mt} \rfloor$ particles from the beginning whereas in the BBM population grows progressively!

Lemma: For any $\varepsilon > 0$, there exists $L > 0$ such that $\mathbb{P}(E_L) \ge 1 - \varepsilon$ where

$$E_L = \{ \forall s \ge 0,\ M_s < \lambda_c s + L \}.$$

Proof: We have seen in the last lecture that $\limsup_{s\to\infty} M_s - \lambda_c s < \infty$ a.s.

But $s \mapsto M_s - \lambda_c s$ is continuous a.s. so $\sup_{s \ge 0} M_s - \lambda_c s < \infty$ a.s.,

so there exists $L > 0$ such that $\mathbb{P}\Big(\underbrace{\sup_{s\ge 0} M_s - \lambda_c s \ge L}_{\supset E_L^c} \Big) \le \varepsilon$. ∎

Proof of Proposition UB: Let $\varepsilon > 0$ and $L > 0$ such that $\mathbb{P}(E_L) \ge 1 - \varepsilon$.

Then $\mathbb{P}(M_t \ge x_t) \le \underbrace{\mathbb{P}(E_L^c)}_{\le \varepsilon} + \mathbb{P}(\{M_t \ge x_t\} \cap E_L)$

By inclusion of events,

$$\mathbb{P}(\{M_t \ge x_t\} \cap E_L) \le \mathbb{P}\big(\exists u \in \mathcal{N}_t : X_u(t) \ge x_t \text{ and } \forall s \in [0,t],\ X_u(s) \le \lambda_c s + L \big)$$

$$\le \mathbb{E}\Big[\sum_{u \in \mathcal{N}_t} \mathbb{1}_{X_u(t) \ge x_t \text{ and } \max_{s\in[0,t]} (X_u(s) - \lambda_c s) \le L} \Big]$$

many-to-one

$$= e^{mt}\, \mathbb{P}\Big(B_t \ge x_t,\ \max_{s\in[0,t]} (B_s - \lambda_c s) \le L \Big)$$

$$= e^{mt}\, \mathbb{P}\Big(B_t - \lambda_c t \ge x_t - \lambda_c t,\ \max_{s\in[0,t]} (B_s - \lambda_c s) \le L \Big)$$

Girsanov

$$= e^{mt}\, \mathbb{E}\Big[\underbrace{e^{-\lambda_c B_t}}_{\le e^{-\lambda_c (x_t - \lambda_c t)} = \frac{t^{3/2}}{(\log t)^3}} \cdot \underbrace{e^{\frac{\lambda_c^2}{2} t}}_{= mt} \mathbb{1}_{B_t \ge \underbrace{x_t - \lambda_c t}_{= -\frac{3}{2\lambda_c}\log t + \frac{3}{\lambda_c}\log\log t},\ \max_{s\in[0,t]} B_s \le L} \Big]$$

$$\le \frac{t^{3/2}}{(\log t)^3} \, \mathbb{P}\left(\max_{s\in[0,t]} B_s \le L \, , \, B_t \ge L - \left(\frac{3}{2\lambda_c} \log t - \frac{3}{\lambda_c} \log\log t + L \right) \right)$$

Corollary 2 in Section III.3

$$\sim \frac{t^{3/2}}{(\log t)^3} \, \frac{L \times \left(\frac{3}{2\lambda_c} \log t \right)^2}{\sqrt{2\pi}\, t^{3/2}} = O\left(\frac{1}{\log t} \right) \xrightarrow[t\to\infty]{} 0 .$$

III.4.2) The lower bound

We prove here the following stronger result

Proposition LB: On the survival event, $M_t \ge \lambda_c t - \frac{3}{2\lambda_c} \log t + O_P(1)$, which means that, for any $\varepsilon > 0$, there exists $\eta > 0$ such that for t large enough

$$\mathbb{P}\left(M_t \ge \lambda_c t - \frac{3}{2\lambda_c} \log t - \eta \,\middle|\, \text{survival} \right) \ge 1 - \varepsilon .$$

Remark: $O_P(1)$ is a notation for a random term depending on t which is tight for large t when seen as a process in t.

Idea of the proof:

Let K_t be the number of particles above $m_t = \lambda_c t - \frac{3}{2\lambda_c} \log t$ at time t and satisfying some trajectory condition to be chosen.

Then, use many-to-one and many-to-two to compute $\mathbb{E}[K_t]$ and $\mathbb{E}[K_t^2]$ and show that $\mathbb{E}[K_t^2] \le C\,\mathbb{E}[K_t]^2$ (this requires to choose the condition in the definition of K_t such that these moments are not dominated by unlikely events.

By Paley-Zygmund / Cauchy-Schwarz inequality, we deduce $\mathbb{P}(Z_t \ge 1) \ge \frac{\mathbb{E}[Z_t]^2}{\mathbb{E}[Z_t^2]} \ge \frac{1}{C}$.

To conclude, let particles branch at the beginning until there is a large number of particles: each of them has a probability at least $\frac{1}{C}$ to have a very high descendant at a time t in the future, so it is very likely that at least one of them does.