

Lecture 7: Starting the study of extremal particles

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III/Extremal particles of the BBM

In this chapter, we study the asymptotic behavior of $\Gamma_t := \max_{u \in \mathcal{N}(t)} X_u(t)$.

We always assume that $\underline{E}[L^2] < \infty$.

III.1) First order

We have seen that $\frac{\Gamma_t}{t} \xrightarrow{P} \lambda_c = \sqrt{2m}$ (when $P(L=0) = 0$, i.e. no extinction).

We prove here the following stronger result:

Theorem: $\frac{\Gamma_t}{t} \xrightarrow[t \rightarrow \infty]{a.s.} \lambda_c = \sqrt{2m}$ on the survival event.

Recall the definition of the additive martingale: $W_t^\lambda := \sum_{u \in \mathcal{N}(t)} e^{\lambda X_u(t) - (\frac{\lambda^2}{2} + m)t}$.

Proof:

① Upper bound: Keeping only the highest particle, we get $W_t^\lambda \geq e^{\lambda \Gamma_t - (\frac{\lambda^2}{2} + m)t}$.

With $\lambda = \lambda_c = \sqrt{2m}$, it yields $W_t^{\lambda_c} \geq \exp(\sqrt{2m}(\Gamma_t - \sqrt{2m}t))$.

But $W_t^{\lambda_c} \rightarrow W_\infty^{\lambda_c} < \infty$ a.s. so $\limsup_{t \rightarrow \infty} \Gamma_t - \sqrt{2m}t < \infty$ a.s.

This proves $\limsup_{t \rightarrow \infty} \frac{\Gamma_t}{t} \leq \sqrt{2m}$ a.s.

(Remark: Using that $W_\infty^{\lambda_c} = 0$ a.s. we can even deduce $\lim_{t \rightarrow \infty} \Gamma_t - \sqrt{2m}t = -\infty$ a.s.)

② Lower bound: Consider $\lambda \in (0, \lambda_c)$ and $\varepsilon > 0$ such that $\lambda + \varepsilon < \lambda_c$. Then

$$W_t^{\lambda + \varepsilon} = \sum_{u \in \mathcal{N}(t)} e^{(\lambda + \varepsilon)X_u(t) - (\frac{(\lambda + \varepsilon)^2}{2} + m)t} \leq e^{\varepsilon \Gamma_t} e^{\frac{(\lambda^2 - (\lambda + \varepsilon)^2)}{2}t} W_t^\lambda = e^{\varepsilon(\Gamma_t - \lambda t)} e^{-\frac{\varepsilon^2}{2}t} W_t^\lambda$$

On the survival event, $W_t^{\lambda + \varepsilon} \rightarrow W_\infty^{\lambda + \varepsilon} > 0$ a.s. and $e^{-\frac{\varepsilon^2}{2}t} W_t^\lambda \rightarrow 0$ a.s.

so $e^{\varepsilon(\Gamma_t - \lambda t)} \rightarrow \infty$ a.s., which implies $\liminf_{t \rightarrow \infty} \frac{\Gamma_t}{t} \geq \lambda$ a.s.

This concludes the proof by letting $\lambda \rightarrow \lambda_c = \sqrt{2m}$.

(Remark: A key idea behind this proof is that if $\Gamma_t \leq \lambda t$ then $W_t^{\lambda + \varepsilon}$ should be small.

Indeed W_t^λ is mainly supported by particles with a position $\lambda t + O(\sqrt{t})$: note that

$$\begin{aligned} \mathbb{E} \left[\sum_{u \in \mathcal{N}(t)} e^{\lambda X_u(t) - (m + \frac{\lambda^2}{2})t} \mathbb{1}_{X_u(t) \notin [\lambda t - K\sqrt{t}, \lambda t + K\sqrt{t}]} \right] &= \mathbb{E} \left[e^{\lambda B_1 - \frac{\lambda^2}{2}t} \mathbb{1}_{B_1 \notin [\lambda t - K\sqrt{t}, \lambda t + K\sqrt{t}]} \right] \text{ by the many-to-one} \\ &= P(B_1 \notin [-K\sqrt{t}, K\sqrt{t}]) \text{ by Girsanov (see next lecture} \\ &= P(B_1 \in [-K, K]) \xrightarrow{K \rightarrow \infty} 0 \text{ for a proof!)} \end{aligned}$$

Exercise 1: A proof without additive martingales

Recall that $N_t(at) := \sum_{u \in \mathcal{N}_t} \mathbb{1}_{X_u(t) \geq at}$ and $E[N_t(at)] \underset{t \rightarrow \infty}{\sim} \frac{e^{-at - \frac{a^2}{2}t}}{a\sqrt{2\pi t}}$ for any $a > 0$.

1. Upper bound along a subsequence

1.a. Let $a > \sqrt{2m}$. Prove that $N_k(ak) \xrightarrow[k \rightarrow \infty]{} 0$ a.s., where $k \rightarrow \infty$ along integers.

1.b. Deduce that $\limsup_{k \rightarrow \infty} \frac{\Gamma_k}{k} \leq \sqrt{2m}$ a.s., where $k \rightarrow \infty$ along integers.

2. Lower bound along a subsequence (harder!)

2.a. Let $a \in (0, \sqrt{2m})$. Prove that there exist $s > 0$ such that $E[N_s(as)] > 1$.

2.b. Let $\tilde{\mathcal{N}}_0 = \mathcal{N}_0$ and by induction $\tilde{\mathcal{N}}_{(k+1)s} = \left\{ u \in \mathcal{N}_{(k+1)s} : X_u((k+1)s) - X_u(ks) \geq as \text{ and } \exists v \in \tilde{\mathcal{N}}_{ks}, u \geq v \right\}$.

Let $p := P(\forall k \in \mathbb{N}, \tilde{\mathcal{N}}_{ks} \neq \emptyset)$. Prove that $p > 0$.

(Hint: Note that $(\#\tilde{\mathcal{N}}_{ks})_{k \geq 0}$ is a Galton-Watson process)

2.c. Prove that $P(\forall k \geq 0, \Gamma_{ks} \geq aks) \geq p$.

2.d. Deduce that, for any $\varepsilon > 0$, there exists k_0 and $C > 0$ such that

$$P(\forall k \geq k_0, \Gamma_{ks} \geq aks - C \mid \text{survival}) \geq 1 - \varepsilon.$$

Hint: Use that, on the survival event, $N_t \xrightarrow[t \rightarrow \infty]{\text{a.s.}} \infty$ to choose k_0 such that there are many particles alive at time k_0 with high probability conditioned on survival.

Apply 2.c. to sub-BBTs starting at time k_0 .

2.e. Conclude that $\liminf_{k \rightarrow \infty} \frac{\Gamma_{ks}}{ks} \geq a$ a.s. on the survival event.

3. Filling the gaps

3.a. Let $s, \varepsilon > 0$. Show that $E\left[\#\{u \in \mathcal{N}_t : \exists r \in [t-s, t], |X_u(t) - X_u(r)| > \varepsilon t\}\right] = O\left(e^{-\frac{m\varepsilon^2}{2s}}\right)$.

(Hint: You can use that $\sup_{r \in [0, s]} B_r \stackrel{(d)}{=} -\inf_{r \in [0, s]} B_r \stackrel{(d)}{=} |B_s|$)

3.b. Deduce that a.s., for k large enough, $\forall u \in \mathcal{N}_{ks}, \forall r \in [(k-1)s, ks], |X_u(ks) - X_u(r)| \leq \varepsilon ks$.

3.c. Conclude.

III.2) A reference model: the i.i.d. case

To see the influence of the tree structure on the maximal position at time t , we compare it to the case we would consider $\lfloor e^{nt} \rfloor$ particles with independent Brownian trajectories of length t .

Let $(B_t^i)_{t \geq 0}$ for $i \geq 1$ be iid. Brownian motions.

We compare $(X_u(t), u \in \mathcal{NP}(t))$ with $(B_t^i, i \in \{1, \dots, \lfloor e^{nt} \rfloor\})$.

Remark: We have the same result for the many-to-one (up to the integer part):

$$\mathbb{E} \left[\sum_{i=1}^{\lfloor e^{nt} \rfloor} F((B_s^i)_{s \in [0,t]}) \right] = \lfloor e^{nt} \rfloor \mathbb{E} [F((B_s)_{s \in [0,t]})].$$

Correlations can only be seen at the level of many-to-two: here we have

$$\mathbb{E} \left[\sum_{\substack{i,j=1 \\ i \neq j}}^{\lfloor e^{nt} \rfloor} F((B_s^i)_{s \in [0,t]}, (B_s^j)_{s \in [0,t]}) \right] = \lfloor e^{nt} \rfloor (\lfloor e^{nt} \rfloor - 1) \mathbb{E} [F(\underbrace{(B_s^1)_{s \in [0,t]}, (B_s^2)_{s \in [0,t]}}_{\text{independent}})]$$

which is different from the BBT case

Our goal is to compare the maximal position in the BBT and the i.i.d. cases.

$$\text{Let } \tilde{\pi}_t = \max_{1 \leq i \leq \lfloor e^{nt} \rfloor} B_t^i.$$

Exercise 2: Prove that $\frac{\tilde{\pi}_t}{t} \xrightarrow{\text{a.s.}} \lambda_c = \sqrt{2m}$. Hint: Prove it for t integer first. Then fill the gaps as in Ex 1.

The first order is the same! We prove here the more precise expansion:

Theorem: $\tilde{\pi}_t - \lambda_c t + \frac{1}{2\lambda_c} \log t \xrightarrow[t \rightarrow \infty]{(d)} \text{Gumbel} \left(-\frac{1}{\lambda_c} \log(\lambda_c \sqrt{2\pi}), \frac{1}{\lambda_c} \right)$,

where Gumbel (c, b) for $c \in \mathbb{R}$ and $b > 0$ has cumulative distribution function $x \in \mathbb{R} \mapsto \exp(-e^{-(x-c)/b})$.

Remark: When looking at $\max(X_1, \dots, X_n)$ as $n \rightarrow \infty$ where $(X_i)_{i \geq 1}$ are iid random variables, there are three possible families of limiting distributions (after proper recentring): Gumbel, Fréchet and Weibull distributions.

Remark: Note that the tail of $G \sim \text{Gumbel}(c, b)$ is asymmetric: we have

$$P(G \geq x) \sim e^{-(x-c)/b} \quad \text{and} \quad P(G \leq -x) \sim e^{-e^{(x+c)/b}} \quad \text{as } x \rightarrow \infty$$

so the left tail is much thinner (double exponential) than the right tail (exponential). This is not surprising for the limit of a maximum: having a larger maximum only requires one r.v. to be large whereas having a smaller one requires all r.v. to be small enough.

Proof: We prove convergence of the cumulative distribution function.

Fix $y \in \mathbb{R}$. Write $x_t = \lambda_c t - \frac{1}{2\lambda_c} \log t + y$.

$$P(\hat{\Gamma}_t - \lambda_c t + \frac{1}{2\lambda_c} \log t \leq y) = P(\hat{\Gamma}_t \leq x_t) = P(B_t \leq x_t)^{\lfloor L e^{-\lambda_c t} \rfloor} = (1 - P(B_t > x_t))^{\lfloor L e^{-\lambda_c t} \rfloor}$$

$$\text{Then } P(B_t > x_t) = P(B_1 > \frac{x_t}{\sqrt{t}}) \sim \frac{\sqrt{t}}{\sqrt{2\pi} x_t} e^{-x_t^2/2t} \text{ as } t \rightarrow \infty \text{ (because } \frac{x_t}{\sqrt{t}} \rightarrow \infty)$$

$$\sim \frac{1}{\sqrt{2\pi t} \lambda_c} \exp\left(-\frac{1}{2t} \left(\lambda_c^2 t^2 + \frac{1}{4\lambda_c^2} (\log t)^2 + y^2 - t \log t + 2\lambda_c t y - \frac{1}{\lambda_c} \log t\right)\right)$$

$$\sim \frac{1}{\sqrt{2\pi t} \lambda_c} \exp\left(-\lambda_c t + \frac{1}{2} \log t - \lambda_c y\right)$$

$$\sim \frac{e^{-\lambda_c y}}{\lambda_c \sqrt{2\pi}} e^{-\lambda_c t}$$

$$P(\hat{\Gamma}_t - \lambda_c t + \frac{1}{2\lambda_c} \log t \leq y) = \exp\left(\underbrace{\lfloor L e^{-\lambda_c t} \rfloor}_{\sim e^{-\lambda_c t}} \times \underbrace{\log(1 - P(B_t > x_t))}_{\sim -P(B_t > x_t)}\right) \xrightarrow{t \rightarrow \infty} \exp\left(-\frac{e^{-\lambda_c y}}{\lambda_c \sqrt{2\pi}}\right). \quad \blacksquare$$

Exercise 3: Recall N_t is the number of particles in the BBSI at time t and assume the Brownian motions B^i for $i \geq 1$ are independent of $(N_t)_{t \geq 0}$.

Assume $P(L=0) = 0$ so that there is a.s. survival.

Recall $e^{-\lambda_c t} N_t \xrightarrow{\text{a.s.}} W_\infty \in (0, \infty)$. Consider now $\hat{\Gamma}_t := \max_{1 \leq i \leq N_t} B_t^i$.

Prove that $\hat{\Gamma}_t - \lambda_c t + \frac{1}{2\lambda_c} \log t \xrightarrow[t \rightarrow \infty]{(d)} \frac{1}{\lambda_c} \log W_\infty + G$,

where G is Gumbel $(-\frac{1}{\lambda_c} \log(\lambda_c \sqrt{2\pi}), \frac{1}{\lambda_c})$ distributed and independent of W_∞ .

Remark: Another reference model would be the case of fully correlated particles: at time t , it consists of $\lfloor e^{nt} \rfloor$ particles following the same Brownian motion of length t . In that case the maximum is simply the position of one Brownian motion, which is of order \sqrt{t} (much smaller than $t_c t$).

In the next lectures, we will get the 2nd order term in the expansion of Γ_t for the BBT. Spoiler: logarithmic correction with a different constant.

Note that we have the following general result telling us that more correlations implies a smaller maximum (if the variances are the same!)

Slepian's lemma: let $n \geq 1$. let (X_1, \dots, X_n) and (Y_1, \dots, Y_n) be centered Gaussian vectors. If $\forall i, \mathbb{E}[X_i^2] = \mathbb{E}[Y_i^2]$ and $\forall i, j, \mathbb{E}[X_i X_j] \leq \mathbb{E}[Y_i Y_j]$, then $\max(Y_1, \dots, Y_n)$ is stochastically dominated by $\max(X_1, \dots, X_n)$.

(We say that Z is stochastically dominated by Z' if $\forall z \in \mathbb{R}, P(Z \leq z) \geq P(Z' \leq z)$.)

Proof: We first prove that if $f \in C^2(\mathbb{R}^n)$ satisfies $\forall i \neq j, \frac{\partial^2 f}{\partial x_i \partial x_j} \geq 0$ and has bounded second derivatives, then $\mathbb{E}[f(X)] \leq \mathbb{E}[f(Y)]$ where $X = (X_1, \dots, X_n)$ and $Y = (Y_1, \dots, Y_n)$.

For this let $Z(t) := \sqrt{t} X + \sqrt{1-t} Y$ where we assume w.l.o.g. that X and Y are independent. Then $Z(0) = Y$ and $Z(1) = X$ so it is enough to prove that $\frac{d}{dt} \mathbb{E}[f(Z(t))] \leq 0$. (This technique is called Gaussian interpolation)

$$\begin{aligned} \frac{d}{dt} \mathbb{E}[f(Z(t))] &= \mathbb{E} \left[\frac{d}{dt} (f(\sqrt{t} X + \sqrt{1-t} Y)) \right] \\ &= \sum_{i=1}^n \mathbb{E} \left[\left(\frac{X_i}{2\sqrt{t}} - \frac{Y_i}{2\sqrt{1-t}} \right) \frac{\partial f}{\partial x_i} (Z(t)) \right] \\ &= \sum_{i=1}^n \left(\frac{1}{2\sqrt{t}} \mathbb{E} \left[X_i \frac{\partial f}{\partial x_i} (Z(t)) \right] - \frac{1}{2\sqrt{1-t}} \mathbb{E} \left[Y_i \frac{\partial f}{\partial x_i} (Z(t)) \right] \right) \\ &= \sum_{j=1}^n \mathbb{E}[X_i X_j] \mathbb{E} \left[\sqrt{t} \frac{\partial^2 f}{\partial x_j \partial x_i} (Z(t)) \right] = \sum_{j=1}^n \mathbb{E}[Y_i Y_j] \mathbb{E} \left[\sqrt{1-t} \frac{\partial^2 f}{\partial x_j \partial x_i} (Z(t)) \right] \\ &\quad \text{by Gaussian integration by part (see below) conditionally on } Y. \end{aligned}$$

$$= \frac{1}{2} \sum_{i,j=1}^n (\underbrace{E[X_i X_j]}_{\leq 0 \text{ if } i \neq j, = 0 \text{ if } i=j}) E \left[\underbrace{\frac{\partial^2 f}{\partial x_j \partial x_i}}_{\geq 0 \text{ if } i \neq j} (Z(i)) \right]$$

$$\leq 0$$

This proves the first claim.

Now to prove the lemma, consider $x \in \mathbb{R}$. We aim at showing

$$P(\max(X_1, \dots, X_n) \leq x) \leq P(\max(Y_1, \dots, Y_n) \leq x)$$

$$\Leftrightarrow E \left[\prod_{i=1}^n \mathbb{1}_{(-\infty, x]}(X_i) \right] \leq E \left[\prod_{i=1}^n \mathbb{1}_{(-\infty, x]}(Y_i) \right]. \quad (*)$$

Let $h_k: \mathbb{R} \rightarrow [0, 1]$ be \mathcal{C}^2 non-increasing functions such that $h_k \xrightarrow{k \rightarrow \infty} \mathbb{1}_{(-\infty, x]}$.

Then $f_k: x \in \mathbb{R}^n \mapsto \prod_{i=1}^n h_k(x_i)$ satisfies the assumptions of the claim

so $E[f_k(X)] \leq E[f_k(Y)]$, which gives (*) by letting $k \rightarrow \infty$. \square

Lemma (Gaussian integration by part): Let $n \geq 1$ and $g \in \mathcal{C}^1(\mathbb{R}^n)$ such that

∇g is bounded. Let $X = (X_1, \dots, X_n)$ be a centered Gaussian vector.

Then, for any $i \in \{1, \dots, n\}$, $E[X_i g(X)] = \sum_{j=1}^n E[X_i X_j] E \left[\frac{\partial g}{\partial x_j}(X) \right]$.

Exercise 4 (Proof)

1.] Prove the result for $n=1$ using the usual integration by parts.

2.] Fix $i \in \{1, \dots, n\}$. For any $j \in \{1, \dots, n\}$, let $Z_j = X_j - \frac{E[X_i X_j]}{E[X_i^2]} X_i$.

Prove that $Z = (Z_1, \dots, Z_n)$ is independent of X_i .

3.] Conclude.

Remark: To use Stepan's lemma to compare two models, we need exactly the same number of variables, so it cannot be used directly to say

$\Pi_t^{(st)} \leq \tilde{\Pi}_t$, where " \leq " means stochastically dominated, but we can

get $\Pi_t^{(st)} \leq \hat{\Pi}_t$ by applying Stepan's lemma conditionally on the tree (recall $\hat{\Pi}_t$ is defined in exercise 3).