Lecture 11: Traveling waves, back to the max and derivative martingale

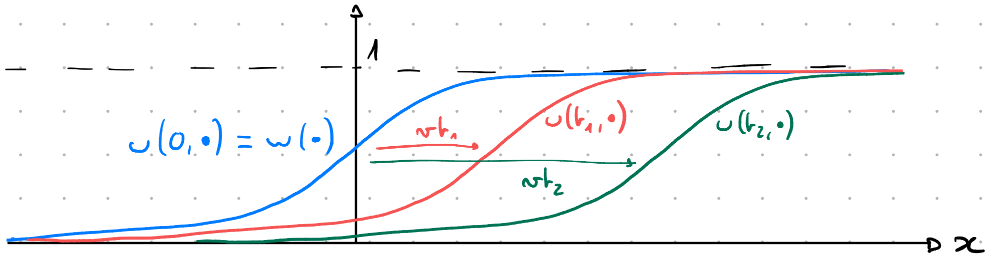
28/01/2026

III.5.2) Traveling wave solutions

A traveling wave solution to the F-KPP equation is a solution of the form $u(t,x) = w(x - vt)$ for some $\mathcal{C}^2$ function $w : \mathbb{R} \to [0,1]$ and a speed $v \in \mathbb{R}$.

Note that $\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + f(u) - u$ if and only if $\frac{1}{2}w'' + v w' + f(w) - w = 0$.

At any time, such a solution has the same shape w as a function of x but this shape is "traveling" at speed v.



Kolmogorov, Petrovsky and Piskounov (1937) proved that there exists a non-constant traveling wave solution at speed $v \geq 0$ if and only if $v \geq \lambda_c$. Moreover, for a given $v \geq \lambda_c$, the traveling wave solution is unique up to translation by a constant.

Here we prove existence for $v > \lambda_c$ using the BBM (idea from McKean 1975).

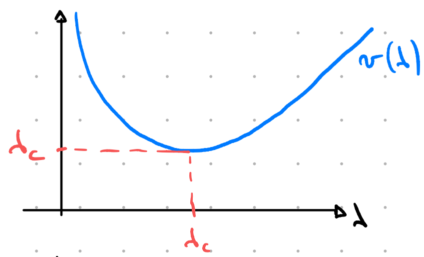
Theorem: Let $\lambda \in (0, \lambda_c)$, $v(\lambda) = \frac{\lambda}{2} + \frac{m}{\lambda}$ and, for $x \in \mathbb{R}$, $w(x) = \mathbb{E}\left[\exp(-e^{-\lambda x} W_\infty^\lambda)\right]$.

Assume $\mathbb{E}[L^2] < \infty$.

Then $u(t,x) = w(x - v(\lambda)t)$ is solution of $\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + f(u) - u$.

This covers all speeds $v > \lambda_c = \sqrt{2m}$:

$$v(\lambda_c) = \frac{\sqrt{2m}}{2} + \frac{m}{\sqrt{2m}} = \sqrt{2m} = \lambda_c$$



Note that w is non-constant because $W_\infty^\lambda > 0$ a.s. on survival for $\lambda < \lambda_c$.

More precisely, w is increasing with $w(x) \xrightarrow[x \to \infty]{} 1$ and $w(x) \xrightarrow[x \to -\infty]{} \mathbb{P}(\text{extinction})$.

For $\lambda \geq \lambda_c$, the proof below works but $w(x) = 1$ because $W_\infty^\lambda = 0$ a.s.

Proof: It is enough to prove that, for any $t \geq 0$ and $x \in \mathbb{R}$,

$w(x - v(\lambda)t) = \mathbb{E}\left[\prod_{v \in \mathcal{N}(t)} w(X_v(t) + x)\right]$ because the right-hand side is a solution of the PDE by the first theorem of this lecture.

This equality follows from the following decomposition for W_∞^λ: for $t, s \geq 0$,

$$W_{t+s}^\lambda = \sum_{v \in \mathcal{N}(t)} \sum_{\substack{u \in \mathcal{N}(t+s) \\ \text{s.t. } u \geq v}} e^{\lambda X_u(t+s) - (\frac{\lambda^2}{2} + m)(t+s)}$$

$$= \sum_{v \in \mathcal{N}(t)} e^{\lambda X_v(t) - (\frac{\lambda^2}{2}+m)t} \underbrace{\sum_{\substack{u \in \mathcal{N}(t+s) \\ \text{s.t. } u \geq v}} e^{\lambda (X_u(t+s) - X_v(t)) - (\frac{\lambda^2}{2}+m)s}}$$

by the branching property, conditionally on $\mathcal{F}_t$, these are independent random variables with the same law as W_s^λ

Taking $s \to \infty$, we get

$$W_\infty^\lambda = \sum_{v \in \mathcal{N}(t)} e^{\lambda X_v(t) - (\frac{\lambda^2}{2}+m)t} W_\infty^\lambda(v,t) \quad \text{where } W_\infty^\lambda(v,t) = \lim_{s \to \infty} W_s^\lambda(v,t) \text{ (a.s. limit).}$$

We can now prove the desired equality:

$$w(x - v(\lambda)t) = \mathbb{E}\left[\exp\left(-e^{-\lambda x + \lambda v(\lambda)t} W_\infty^\lambda\right)\right]$$

decomposition above and $\lambda v(\lambda) = \frac{\lambda^2}{2} + m$

$$= \mathbb{E}\left[\prod_{v \in \mathcal{N}(t)} \exp\left(-e^{-\lambda(x - X_v(t))} W_\infty^\lambda(v,t)\right)\right]$$

$$= \mathbb{E}\left[\mathbb{E}\left[\prod_{v \in \mathcal{N}(t)} \exp\left(-e^{-\lambda(x - X_v(t))} \underline{W_\infty^\lambda(v,t)}\right) \middle| \mathcal{F}_t\right]\right]$$

$\mathcal{F}_t$-measurable

conditionally on $\mathcal{F}_t$, these are independent r.v. with the same law as W_∞^λ

$$= \mathbb{E}\left[\prod_{v \in \mathcal{N}(t)} w(x - X_v(t))\right]$$

$$= \mathbb{E}\left[\prod_{v \in \mathcal{N}(t)} w(x + X_v(t))\right]$$

by symmetry of the BBM

∎

For more general initial conditions $u(0,\cdot)$, the solution also looks like a front moving at some speed v (which depends on the initial condition) and the shape of the front now varies with time, but converges to the shape of the traveling wave with speed v. We only see one example of this fact:

Exercice 1: Let $\lambda \in (0, \lambda_c)$ and let u be the solution of $\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + f(u) - u$ with initial condition $u(0,x) = \exp(-e^{-\lambda x})$ given by the first theorem.

(Rk: such a solution is unique but we haven't proved it).

Let $v(\lambda) = \frac{\lambda}{2} + \frac{m}{\lambda}$ and, for $x \in \mathbb{R}$, $w(x) = \mathbb{E}[\exp(-e^{-\lambda x} W_\infty^\lambda)]$.

Prove that, for any $x \in \mathbb{R}$, $u(t, x - v(\lambda)t) \xrightarrow[t\to\infty]{} w(x)$.

III.6) Bramson's results on the maximum

Recall we proved $\frac{M_t - \lambda_c t}{\log t} \xrightarrow[t\to\infty]{\mathbb{P}} -\frac{3}{2\lambda_c}$ stated as a theorem by Bramson in 1978.

Bramson's result was actually stronger: (recall $m_t = \lambda_c t - \frac{3}{2\lambda_c}\log t$)

Theorem (Bramson '78): Assume $\mathbb{E}[L^2] < \infty$.

Then, $M_t = m_t + O_P(1)$ on the survival event S, i.e.

$$\forall \varepsilon > 0, \exists y > 0, \exists t_0 \geq 0, \forall t \geq t_0, \mathbb{P}(M_t - m_t \in [-y, y] \mid S) \geq 1 - \varepsilon.$$

We proved this precision for the lower bound but not for the upper bound and it would require a bit more work to prove $M_t \leq m_t + O_P(1)$.

Bramson's proof in 1978 was probabilistic, based on many-to-one and many-to-two lemmas with barriers.

He then refined his result in 1983, but using this time the link with the F-KPP equation. He studies the asymptotic behavior of $u(t,x)$ for various initial conditions, but in the case $u(0,x) = \mathbb{1}_{x \geq 0}$, his result reads:

Theorem (Bramson 1983): Assume $\mathbb{E}[L^2] < \infty$ and $\mathbb{P}(L = 0) = 0$.

Then, for any $x \in \mathbb{R}$, $\mathbb{P}(M_t - m_t \leq x) \xrightarrow[t\to\infty]{} w(x)$, where $w : \mathbb{R} \to [0,1]$ is the unique solution, up to translation, of

$$\frac{1}{2} w'' + \lambda_c w' + f(w) - w = 0 \text{ with } w(-\infty) = 0 \text{ and } w(+\infty) = 1.$$

Moreover, there is $C_w > 0$ such that $1 - w(x) \sim C_w x e^{-\lambda_c x}$ as $x \to \infty$.

Remark:

- In other words, $M_t - m_t$ converges in distribution as $t \to \infty$ towards a r.v. with cumulative distribution function w.
- w is the shape of the travelling wave solution of speed λ_c.

III.7) The derivative martingale

With the intention of giving a probabilistic description of the limiting law of $M_t - m_t$, Lalley and Sellke introduced a new martingale which describes the growth of the BBM in the direction of the maximum (ie with slope λ_c) and which replaces $W_t^{\lambda_c}$ which has a zero limit.

Let $Z_t = \sum_{u \in \mathcal{N}_t} (\lambda_c t - X_u(t)) e^{\lambda_c (X_u(t) - \lambda_c t)}$ for $t \geq 0$.

Proposition: $(Z_t)_{t\geq 0}$ is a $(\mathcal{F}_t)$-martingale.

The process $(Z_t)_{t\geq 0}$ is called the derivative martingale, because it is obtained by $Z_t = -\frac{\partial}{\partial \lambda} W_t^{\lambda} \Big|_{\lambda = \lambda_c}$.

Proof 1 (by direct calculation):

- We first prove $\left((\lambda_c t + L - B_t) e^{\lambda_c B_t - \frac{\lambda_c^2}{2} t}\right)_{t \geq 0}$ is a martingale under $\mathbb{P}_x$: for $s,t \geq 0$,

$$\mathbb{E}_x\left[(\lambda_c(s+t) + L - B_{s+t}) e^{\lambda_c B_{s+t} - \frac{\lambda_c^2}{2}(s+t)} \,\Big|\, \mathcal{F}_s\right]$$

$$= (\lambda_c s + L - B_s) e^{\lambda_c B_s - \frac{\lambda_c^2}{2} s} \mathbb{E}_x\left[e^{\lambda_c (B_{s+t} - B_s) - \frac{\lambda_c^2}{2} t} \,\Big|\, \mathcal{F}_s\right] + \mathbb{E}_x\left[(\lambda_c t - (B_{t+s} - B_s)) e^{\lambda_c (B_{s+t} - B_s) - \frac{\lambda_c^2}{2} t} \,\Big|\, \mathcal{F}_s\right]$$

$B_{s+t} - B_s$ is indep of $\mathcal{F}_s$ and $\stackrel{(d)}{=} B_t$ under $\mathbb{P}$

$$= (\lambda_c s + L - B_s) e^{\lambda_c B_s - \frac{\lambda_c^2}{2} s} \underbrace{\mathbb{E}\left[e^{\lambda_c B_t - \frac{\lambda_c^2}{2} t}\right]}_{= \mathbb{E}[1] = 1} + \underbrace{\mathbb{E}\left[(\lambda_c t - B_t) e^{\lambda_c B_t - \frac{\lambda_c^2}{2} t}\right]}_{= \mathbb{E}[-B_t] = 0}$$

$$= (\lambda_c s + L - B_s) e^{\lambda_c B_s - \frac{\lambda_c^2}{2} s}$$ by Girsanov.

So $\left((\lambda_c t + L - B_t) e^{\lambda_c B_t - \frac{\lambda_c^2}{2} t}\right)_{t \geq 0}$ is a martingale under $\mathbb{P}_x$.

many-to-one and $e^{mt} = e^{\frac{\lambda_c^2}{2} t}$

- In particular, $\mathbb{E}\left[\sum_{u \in \mathcal{N}_t} (\lambda_c t + L - X_u(t)) e^{\lambda_c X_u(t) - \lambda_c^2 t}\right] = \mathbb{E}\left[(\lambda_c t + L - B_t) e^{\lambda_c B_t - \frac{\lambda_c^2}{2} t}\right]$

$= L$ (value at $t = 0$)

• Now we prove $(Z_t)_{t\geq 0}$ is a martingale:

$$\mathbb{E}[Z_{t+s}|\mathcal{F}_s] = \sum_{v\in\mathcal{N}_s} \mathbb{E}\Big[\sum_{\substack{u\in\mathcal{N}_{t+s}\\ u\geq v}} (\lambda_c(t+s) - X_u(t+s))\, e^{\lambda_c X_u(t+s) - \lambda_c^2(t+s)} \Big| \mathcal{F}_s\Big]$$

$$= \sum_{v\in\mathcal{N}_s} e^{\lambda_c X_v(s) - \lambda_c^2 s}\, \mathbb{E}\Big[\sum_{\substack{u\in\mathcal{N}_{t+s}\\ u\geq v}} (\lambda_c t + \underbrace{\lambda_c s - X_v(s)}_{=:\, L_v(s)\ \mathcal{F}_s\text{-measurable}} - [X_u(t+s) - X_v(s)])\, e^{\lambda_c(X_u(t+s)-X_v(s)) - \frac{\lambda_c^2}{2}t} \Big| \mathcal{F}_s\Big]$$

by the branching property, conditionally on $\mathcal{F}_s$, $(X_u(t+s) - X_v(s),\ u\in\mathcal{N}_{t+s},\ u\geq v)$ for $v\in\mathcal{N}_s$ has the same law as $(X_u(t),\ u\in\mathcal{N}_t)$ so by the previous calculation $\mathbb{E}[\cdots|\mathcal{F}_s] = L_v(s)$

$$= \sum_{v\in\mathcal{N}_s} e^{\lambda_c X_v(s) - \lambda_c^2 s} L_v(s) = Z_s .$$

■

Proof 2 (differentiating with respect to λ): Let $0\leq s\leq t$.

We already know that, for any $\lambda\in\mathbb{R}$, $\mathbb{E}[W_t^\lambda|\mathcal{F}_s] = W_s^\lambda$.

So $\frac{\partial}{\partial\lambda} W_s^\lambda = \frac{\partial}{\partial\lambda}\mathbb{E}[W_t^\lambda|\mathcal{F}_s] \overset{?}{=} \mathbb{E}\Big[\frac{\partial}{\partial\lambda} W_t^\lambda\Big|\mathcal{F}_s\Big]$

To justify this we need a domination $\left|\frac{\partial}{\partial\lambda} W_t^\lambda\right| \leq Y_t$ independent of $\lambda\in[-\Pi,\Pi]$, with $\mathbb{E}[Y_t]<\infty$. This is provided by:

$$\left|\frac{\partial}{\partial\lambda} W_t^\lambda\right| = \left|\sum_{u\in\mathcal{N}_t} (X_u(t) - \lambda t)\, e^{\lambda X_u(t) - (\frac{\lambda^2}{2}+m)t}\right| \leq \sum_{u\in\mathcal{N}_t} (|X_u(t)| + \Pi t)\, e^{\Pi|X_u(t)| - mt} =: Y_t$$

$$\mathbb{E}[Y_t] = \mathbb{E}\big[(|B_t| + \Pi t)\, e^{\Pi|B_t|}\big] < \infty .$$

This proves $\left(\frac{\partial}{\partial\lambda} W_t^\lambda\right)_{t\geq 0}$ is a martingale for any $\lambda\in\mathbb{R}$ and so $(Z_t)_{t\geq 0}$ is a martingale.

■

Note that here a.s. convergence is unclear: Z_t is not nonnegative...

Theorem (Lalley-Sellke 1987): $(Z_t)_{t\geq 0}$ converges almost surely to a limit Z_∞.

Moreover $Z_\infty > 0$ a.s. on the survival event.

Note that there is no L^1 convergence: $\mathbb{E}[Z_t] = 0$ but $\mathbb{E}[Z_\infty] > 0$.

Actually, we even have $\mathbb{E}[Z_\infty] = \infty$! (see the exercise for a proof)

The proof of the theorem relies on introducing another martingale, which is a truncated and non-negative version of $(Z_t)_{t\geq 0}$.

Let $L>0$. For $t\geq 0$, let $Z_t^{(L)} = \sum_{u\in\mathcal{N}_t} (\lambda_c t + L - X_u(t))\, e^{\lambda_c(X_u(t)-\lambda_c t)}\, \mathbb{1}_{\max_{s\in[0,t]} X_u(s) - \lambda_c s \leq L}$.

<u>Proposition</u>: $(Z_t^{(L)})_{t\geq 0}$ is a $(\mathcal{F}_t)$-martingale and it converges a.s. to a limit $Z_\infty^{(L)} \geq 0$.

<u>Proof</u>:

- We first show that $\left((\lambda_c t + L - B_t)\, e^{\lambda_c B_t - \frac{\lambda_c^2}{2}t}\, \mathbb{1}_{\max_{s\in[0,t]} B_s - \lambda_c s \leq L}\right)_{t\geq 0}$ is a martingale:

  Write $M_t = (\lambda_c t + L - B_t)\, e^{\lambda_c B_t - \frac{\lambda_c^2}{2}t}$ and $\tau = \inf\{s\geq 0 : B_s = \lambda_c s + L\}$

  Then τ is a stopping time so $(M_{t\wedge\tau})_{t\geq 0}$ is a martingale.

  But $M_{t\wedge\tau} = \underbrace{M_\tau\, \mathbb{1}_{\tau\leq t}}_{\substack{=0 \\ \text{because } \lambda_c\tau + L - B_\tau = 0}} + M_t\, \underbrace{\mathbb{1}_{\tau > t}}_{= \mathbb{1}_{\max_{s\in[0,t]} B_s - \lambda_c s < L} = \mathbb{1}_{\max_{s\in[0,t]} B_s - \lambda_c s \leq L} \text{ a.s.}} = (\lambda_c t + L - B_t)\, e^{\lambda_c B_t - \frac{\lambda_c^2}{2}t}\, \mathbb{1}_{\max_{s\in[0,t]} B_s - \lambda_c s \leq L}$ a.s.

  So $\left((\lambda_c t + L - B_t)\, e^{\lambda_c B_t - \frac{\lambda_c^2}{2}t}\, \mathbb{1}_{\max_{s\in[0,t]} B_s - \lambda_c s \leq L}\right)_{t\geq 0}$ is a martingale.

- Note that, for $t\geq 0$, by the many-to-one

$$\begin{aligned} \mathbb{E}[Z_t^{(L)}] &= e^{mt}\, \mathbb{E}\left[(\lambda_c t + L - B_t)\, e^{\lambda_c(B_t - \lambda_c t)}\, \mathbb{1}_{\max_{s\in[0,t]} B_s - \lambda_c s \leq L}\right] \quad \Big\}\ m = \frac{\lambda_c^2}{2} \\ &= \mathbb{E}\left[(\lambda_c t + L - B_t)\, e^{\lambda_c B_t - \frac{\lambda_c^2}{2}t}\, \mathbb{1}_{\max_{s\in[0,t]} B_s - \lambda_c s \leq L}\right] \\ &= L \quad \text{using that it is a martingale.} \end{aligned}$$

- Now let $s,t\geq 0$,

$$\begin{aligned} Z_{s+t}^{(L)} &= \sum_{u\in\mathcal{N}_{s+t}} (\lambda_c(s+t) + L - X_u(s+t))\, e^{\lambda_c(X_u(s+t) - \lambda_c(s+t))}\, \mathbb{1}_{\max_{r\in[0,s+t]} X_u(r) - \lambda_c r \leq L} \\ &= \sum_{v\in\mathcal{N}_s} e^{\lambda_c(X_v(s) - \lambda_c s)}\, \mathbb{1}_{\max_{r\in[0,s]} X_v(r) - \lambda_c r \leq L} \underbrace{\sum_{\substack{u\in\mathcal{N}_{s+t} \\ u\geq v}} \Big(\underbrace{\lambda_c s + L - X_v(s)}_{=: L_v(s)} + \lambda_c t - (X_u(t+s) - X_v(s))\Big) e^{\lambda_c(X_u(s+t) - X_v(s) - \lambda_c t)} \times \mathbb{1}_{\max_{r\in[0,t]} X_u(r) - X_v(s) - \lambda_c r \leq \underbrace{L + \lambda_c s - X_v(s)}_{= L_v(s)}}}_{\text{by the branching property, given } \mathcal{F}_s \text{, these are independent r.v. with the same distribution as } Z_t^{(L_v(s))}} \end{aligned}$$

So $\mathbb{E}\left[Z_{s+t}^{(L)} \mid \mathcal{F}_s\right] = \sum_{v \in \mathcal{N}_s} e^{\lambda_c (X_v(s) - \lambda_c s)} \mathbb{1}_{\max_{r \in [0,s]} X_v(r) - \lambda_c r \leq L} \underbrace{\mathbb{E}\left[Z_t^{(L_v(s))}\right]}_{= L_v(s) = \lambda_c s + L - X_v(s)} = Z_s^{(L)}$

and $(Z_t^{(L)})_{t \geq 0}$ is a martingale.

- Finally note that it is a nonnegative martingale so it converges a.s. $\blacksquare$