Lecture 10: Link with F-KPP equation 21/01/2026

III.5) The F-KPP equation

III.5.1) The link between BBM and F-KPP equation

We write $f(s) = \mathbb{E}[s^L]$, $s \in [0,1]$, for the generating function of L.

Theorem (McKean 1975): Assume $\mathbb{E}[L] < \infty$. Let $g : \mathbb{R} \to [0,1]$ be a measurable function.

For $x \in \mathbb{R}$ and $t \geq 0$, set $u(t,x) = \mathbb{E}\left[\prod_{v \in \mathcal{N}_t} g(X_v(t) + x)\right] \in [0,1]$. Then:

- u is continuous on $(0,\infty) \times \mathbb{R}$ and weakly continuous at $t = 0$ (ie for any $\varphi \in \mathcal{C}_c^\infty(\mathbb{R})$, $\int_{\mathbb{R}} \varphi(x) u(t,x)\, dx \xrightarrow[t \to 0]{} \int_{\mathbb{R}} \varphi(x) u(0,x)\, dx$),
- $\frac{\partial u}{\partial t}$ and $\frac{\partial^2 u}{\partial x^2}$ exist on $(0,\infty) \times \mathbb{R}$,
- u is solution to $\begin{cases} \frac{\partial u}{\partial t} = \frac{1}{2} \frac{\partial^2 u}{\partial x^2} + f(u) - u & \text{on } (0,\infty) \times \mathbb{R} \\ u(0,x) = g(x) \end{cases}$

prove it!

Remark: If g is continuous, then u is continuous on $[0,\infty) \times \mathbb{R}$ (which is a more classical sense for a strong solution to the PDE above)

Remark: The F-KPP equation (introduced by Fisher 1937 and Kolmogorov-Petrovskii-Piskunov 1937) is usually the one obtained by replacing u by $v = 1 - u$ which gives $\frac{\partial v}{\partial t} = \frac{1}{2} \frac{\partial^2 v}{\partial x^2} + (1-v) - f(1-v)$.

Example: Choosing the initial condition $g(x) = \mathbb{1}_{x \geq 0}$, we get

$$u(t,x) = \mathbb{E}\left[\prod_{v \in \mathcal{N}_t} \mathbb{1}_{X_v(t) + x \geq 0}\right] = \mathbb{P}(\forall v \in \mathcal{N}_t,\ X_v(t) \geq -x)$$

$$= \mathbb{P}\Big(\underbrace{\min_{v \in \mathcal{N}_t} X_v(t)}_{\text{has the same law as } -\max X_v(t)} \geq -x\Big) = \mathbb{P}\left(-\max_{v \in \mathcal{N}_t} X_v(t) \geq -x\right) = \mathbb{P}(M_t \leq x).$$

So $u(t,\cdot)$ is the cumulative distribution function of M_t!

Understanding the asymptotic behavior of M_t or of $u(t,\cdot)$ is equivalent.

Before proving the theorem we need some facts on Brownian motion.

For $h : \mathbb{R} \longrightarrow \mathbb{R}$ measurable and bounded and $t \geq 0$, we set

$$(P_t h)(x) = \mathbb{E}_x\left[h(B_t)\right].$$

Exercise 1: For $s, t \geq 0$, prove that $P_{t+s} = P_t P_s$.

Hint: Markov property!

In other words, $(P_t)_{t\geq 0}$ is a semi-group. It is called the semi-group associated to Brownian motion. We will use the following results

Exercise 2:

1. Let $h : \mathbb{R} \to \mathbb{R}$ be measurable and bounded.

1.a. Prove that $(t, x) \in (0, \infty) \times \mathbb{R} \longmapsto (P_t h)(x)$ is continuous.

1.b. Prove that, for any $t > 0$, $P_t h$ is $\mathcal{C}^2$, $\|(P_t h)'\|_\infty \leq \sqrt{\frac{2}{\pi t}}\, \|h\|_\infty$ and $\|(P_t h)''\|_\infty \leq 2 \frac{\|h\|_\infty}{t}$.

1.c. Assume now that h is differentiable with a bounded derivative.

Prove that, for any $t > 0$, $\|(P_t h)'\|_\infty \leq \|h'\|_\infty$ and $\|(P_t h)''\|_\infty \leq \sqrt{\frac{2}{\pi t}}\, \|h'\|_\infty$.

2. Prove that if $h : \mathbb{R} \to \mathbb{R}$ is $\mathcal{C}^2$ with h, h', h'' bounded, then

- $\forall t > 0$, $\left\| \frac{P_t h - h}{t} \right\|_\infty \leq \frac{1}{2} \|h''\|_\infty$
- $\forall x \in \mathbb{R}$, $\lim_{t \to 0} \left(\frac{P_t h - h}{t}\right)(x) = \frac{1}{2} h''(x)$.

Hint (elementary approach): Show $(P_t h - h)(x) = \int_{\mathbb{R}} \left(h(x + \sqrt{t} z) - h(x)\right) \frac{e^{-z^2/2}}{\sqrt{2\pi}}\, dz$ and Taylor expand $h(x + \sqrt{t} z)$.

Hint (stochastic calculus approach): With Itô formula, show $P_t h(x) = h(x) + \frac{1}{2}\int_0^t \mathbb{E}_x\left[h''(B_s)\right] ds$

3. Prove that, if $h : \mathbb{R} \to \mathbb{R}$ is measurable and bounded, then, for any $x \in \mathbb{R}$, $t \longmapsto (P_t h)(x)$ is differentiable at any $t > 0$ with derivative $\frac{1}{2}(P_t h)''(x)$.

Hint: Combine questions 1.b and 2 with Exercise 1.

Remark: The operator $\lim_{t \to 0} \frac{P_t - \mathrm{Id}}{t}$, which is here $\frac{1}{2}\frac{\partial^2}{\partial x^2}$, is called the infinitesimal generator.

⚠ The precise definition of the infinitesimal generator includes the domain (more subtle!).

Proof: We use the "backward approach" seen at the beginning of the class.

Let $t>0$, $x\in\mathbb{R}$.

$$u(t,x) = \underbrace{\mathbb{E}\Big[\mathbb{1}_{T_1\le t}\prod_{v\in\mathcal{N}_t} g(X_v(t)+x)\Big]} + \underbrace{\mathbb{P}(T_1>t)}_{=e^{-t}}\ \underbrace{\mathbb{E}\big[g(B_t+x)\big]}_{=(P_tg)(x)}$$

$$= \mathbb{E}\Big[\mathbb{1}_{T_1\le t}\prod_{i=1}^{L_1}\prod_{\substack{v\in\mathcal{N}_t\\ v\ge i}} g(X_v(t)-X_\emptyset(T_1)+x+X_\emptyset(T_1))\Big]$$

$$= \mathbb{E}\Big[\mathbb{1}_{T_1\le t}\, u(t-T_1, x+X_\emptyset(T_1))^{L_1}\Big]$$ by the branching property at time T_1

$$= \mathbb{E}\Big[\mathbb{1}_{T_1\le t}\, f(u(t-T_1, x+Y_\emptyset(T_1)))\Big]$$ integrating first with respect to L_1

$$= \int_0^t e^{-s}\,\mathbb{E}\big[f(u(t-s,x+Y_\emptyset(s)))\big]\,ds$$ integrating w.r.t. T_1 ($Y_\emptyset$ is indep. of T_1)

so
$$u(t,x) = \int_0^t e^{-s} P_s\big(f(u(t-s,\cdot))\big)(x)\,ds + e^{-t}(P_tg)(x)$$

$\downarrow s=t-r$

$$= \int_0^t e^{r-t} P_{t-r}\big(f(u(r,\cdot))\big)(x)\,dr + e^{-t}(P_tg)(x).$$

We now use this formula to deduce regularity of u and then differentiate it with $\frac{\partial}{\partial t}$ and $\frac{\partial^2}{\partial x^2}$ to get the desired PDE:

- u is continuous on $(0,\infty)\times\mathbb{R}$: it follows from Ex 2 Q 1.a. that $(t,x)\mapsto(P_tg)(x)$ and $(t,x)\in(r,\infty)\times\mathbb{R}\mapsto P_{t-r}\big(f(u(r,\cdot))\big)(x)$ are continuous and the continuity of u follows by dominating $|P_{t-r}(f(u(r,\cdot)))(x)|\le 1$ in the integral.

- u is weakly continuous at $t=0$: Note that $u(0,x)=g(x)$ by definition. Using again $|P_{t-r}(f(u(r,\cdot)))(x)|\le 1$, we get $u(t,x) = o(1) + e^{-t}\,\mathbb{E}[g(B_t+x)]$ as $t\to 0$, uniformly in x. So, for $\varphi\in\mathcal{C}_c(\mathbb{R})$,

$$\int_{\mathbb{R}} u(t,x)\varphi(x)\,dx = o(1) + e^{-t}\,\mathbb{E}\Big[\int_{\mathbb{R}} g(B_t+x)\varphi(x)\,dx\Big]$$

by Fubini, using that g,φ are bounded and φ has compact support

$$= o(1) + e^{-t}\,\mathbb{E}\Big[\int_{\mathbb{R}} g(y)\varphi(B_t+y)\,dy\Big]$$ change of variable

$$\xrightarrow[t\to 0]{} \int_{\mathbb{R}} g(y)\varphi(y)\,dy = \int_{\mathbb{R}} u(0,y)\varphi(y)\,dy$$

by dominated convergence using $\varphi(B_t+y)\xrightarrow[t\to 0]{}\varphi(y)$ by continuity of φ and dominating $|g(y)\varphi(B_t+y)|\le \|g\|_\infty\|\varphi\|_\infty \mathbb{1}_{B_t+y\in[-\Pi,\Pi]}$ for Π such that $\operatorname{supp}\varphi\subset[-\Pi,\Pi]$

• $x \in \mathbb{R} \longmapsto u(t,x)$ is differentiable for $t>0$: $f(u(r,\cdot))$ is bounded by 1 so by Ex 2 Q1.b, $P_{t-r}(f(u(r,\cdot)))$ is differentiable and $\left|[P_{t-r}(f(u(r,\cdot)))]'(x)\right| \leq \frac{1}{\sqrt{t-r}}$.

This provides the domination to justify that

$$\frac{\partial}{\partial x}\int_0^t e^{r-t} P_{t-r}(f(u(r,\cdot)))(x)\,dr = \int_0^t e^{r-t}[P_{t-r}(f(u(r,\cdot)))]'(x)\,dr.$$

and it bounded by $\int_0^t e^{r-t}\frac{dr}{\sqrt{t-r}} \leq \int_0^\infty e^{-s}\frac{ds}{\sqrt{s}} =: C$.

By Ex 2 Q1.b again, $P_t g$ is differentiable and its derivative is bounded by $\frac{1}{\sqrt{t}}$.

It follows that $u(t,\cdot)$ is differentiable with derivative bounded by $C+\frac{1}{\sqrt{t}}$.

• $x \in \mathbb{R} \longmapsto u(t,x)$ is twice differentiable for $t>0$: We now know that $f(u(r,\cdot))$ is differentiable with derivative bounded by $\mathbb{E}[L](C+\frac{1}{\sqrt{r}})$ (note that $f'(s) \leq \mathbb{E}[L]$ for $s \in [0,1]$).

So we can apply Ex 2 Q1.c to get that $P_{t-r}(f(u(r,\cdot)))$ is twice differentiable and $\left|[P_{t-r}(f(u(r,\cdot)))]''(x)\right| \leq \frac{1}{\sqrt{t-r}}\mathbb{E}[L](C+\frac{1}{\sqrt{r}})$ which is integrable in r.

Finally, applying Ex 2 Q1.b again to $P_t g$, we conclude that $u(t,\cdot)$ is twice differentiable and $\frac{\partial^2 u}{\partial x^2}(t,x) = \int_0^t e^{r-t}[P_{t-r}(f(u(r,\cdot)))]''(x)\,dr + e^{-t}(P_t g)''(x)$.

• $t>0 \longmapsto u(t,x)$ is differentiable for $x \in \mathbb{R}$: First note that

① By Ex 2 Q3, for any $r \geq 0$, $t \in (r,\infty) \longmapsto P_{t-r}(f(u(r,\cdot)))(x)$ is differentiable and its derivative at t equals $\frac{1}{2}[P_{t-r}(f(u(r,\cdot)))]''(x)$ which is bounded by $\frac{\mathbb{E}[L]}{\sqrt{t-r}}(C+\frac{1}{\sqrt{r}})$. So letting $\varphi(t,r) = e^{r-t}P_{t-r}(f(u(r,\cdot)))(x)$, we have $\left|\frac{\partial}{\partial t}\varphi(t,r)\right| \leq 1 + \frac{\mathbb{E}[L]}{\sqrt{t-r}}(C+\frac{1}{\sqrt{r}})$ for any $t>r$.

② φ is continuous on $\{(t,r) : t \geq r > 0\}$ because

$$P_{t-r}(f(u(r,\cdot)))(x) = \mathbb{E}[f(u(r,x+B_{t-r}))] \text{ and } u \text{ is continuous on } (0,\infty)\times\mathbb{R}.$$

These two facts are enough to conclude that $t>0 \longmapsto \int_0^t \varphi(t,r)\,dr$ is differentiable with derivative $\varphi(t,t) + \int_0^t \frac{\partial}{\partial t}\varphi(t,r)\,dr$. (Exercise 3: show this.)

Moreover, by Ex 2 Q3, $t>0 \longmapsto (P_t g)(x)$ is differentiable with derivative $\frac{1}{2}(P_t g)''(x)$.

It follows that $u(\cdot,x)$ is differentiable and

$$\frac{\partial u}{\partial t}(t,x) = f(u(t,x)) + \int_0^t \left(-e^{r-t} P_{t-r}(f(u(r,\cdot)))(x) + e^{r-t}\frac{1}{2}\left[P_{t-r}(f(u(r,\cdot)))\right]''(x)\right) dr$$
$$- e^{-t}(P_t g)(x) + e^{-t}\frac{1}{2}(P_t g)''(x)$$
$$= f(u(t,x)) - u(t,x) + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}(t,x). \qquad \blacksquare$$

Solution to exercise 3: For $\varepsilon, t > 0$ we have

$$\frac{1}{\varepsilon}\left(\int_0^{t+\varepsilon} \varphi(t+\varepsilon, r)\,dr - \int_0^t \varphi(t,r)\,dr\right) = \underbrace{\int_0^t \frac{\varphi(t+\varepsilon,r)-\varphi(t,r)}{\varepsilon}\,dr}_{(\mathrm{I})} + \underbrace{\frac{1}{\varepsilon}\int_t^{t+\varepsilon} \varphi(t+\varepsilon,r)\,dr}_{(\mathrm{II})}$$

For (I): By ①, $\frac{\varphi(t+\varepsilon,r)-\varphi(t,r)}{\varepsilon} \xrightarrow[\varepsilon\to 0]{} \frac{\partial}{\partial t}\varphi(t,r)$ for any $r\in(0,t)$

and domination by $\sup_{s\in[t,t+\varepsilon]} \left|\frac{\partial}{\partial s}\varphi(s,r)\right| \leq 1 + \frac{E[L]}{\sqrt{t-r}}\left(C + \frac{1}{\sqrt{r}}\right)$ integrable

so $(\mathrm{I}) \xrightarrow[\varepsilon\to 0]{} \int_0^t \frac{\partial}{\partial t}\varphi(t,r)\,dr$.

For (II): Changing variables, $(\mathrm{II}) = \int_0^1 \underbrace{\varphi(t+\varepsilon, t+u\varepsilon)}_{\xrightarrow[\varepsilon\to 0]{} \varphi(t,t) \text{ by } ②}\,du$

and domination by 1,

so $(\mathrm{II}) \xrightarrow[\varepsilon\to 0]{} \int_0^1 \varphi(t,t)\,du = \varphi(t,t)$.

Exercise 4: Assume $E[L]<\infty$. Let $g:\mathbb{R}\to\mathbb{R}$ be a measurable bounded function.

For $x\in\mathbb{R}$ and $t\geq 0$, set $u(t,x) = E\left[\sum_{v\in\mathcal{N}_t} g(X_v(t)+x)\right]$.

1. Prove that for any $x\in\mathbb{R}$ and $t>0$,

$$u(t,x) = E[L]\cdot\int_0^t e^{r-t} P_{t-r}(u(r,\cdot))(x)\,dr + e^{-t}(P_t g)(x)$$

2. Prove that:

- u is continuous on $(0,\infty)\times\mathbb{R}$ and weakly continuous at $t=0$
- $\frac{\partial u}{\partial t}$ and $\frac{\partial^2 u}{\partial x^2}$ exist on $(0,\infty)\times\mathbb{R}$,
- u is solution to $\begin{cases} \frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + mu & \text{on } (0,\infty)\times\mathbb{R} \\ u(0,x) = g(x) \end{cases}$ (recall $m = E[L]-1$)

Hint: For the regularity properties, you do not need to do everything over again, note that one can apply the argument in the proof above to $f(s)=s$.