K-THEORY OF THE INDICIAL ALGEBRA OF A MANIFOLD WITH CORNERS

PIERRE-YVES LE GALL AND BERTRAND MONTHUBERT

Abstract. We compute the K-theory groups of the C^*-algebra of the groupoid of a manifold with corners, in which the analytic index takes its values.

1. Introduction

In [6], the second author associated a Lie groupoid $\mathcal{G}(X)$ to any manifold with corners X, following general ideas of Alain Connes ([2]). This definition allows one to define a pseudodifferential calculus for any manifold with corners, which is the pseudodifferential calculus of $\mathcal{G}(X)$ (*cf.* [8, 7, 10]), and an analytic index $ind_a : K^*(TX) \to K_*(C^*(\mathcal{G}(X)))$. The latter K-theory group fulfills an interesting property : if X and X' are two manifolds with corners of same maximal codimension, then $K_*(C^*(\mathcal{G}(X))) \simeq K_*(C^*(\mathcal{G}(X')))$. It implies that the K-theory group in which the analytic index takes its values is universal among manifolds of the same codimension, a fact which is well known for manifolds without boundary, *i.e.* codimension 0 manifolds with corners.

In this article we compute these indicial K-theory groups. In section 2 we recall the main definitions, like that of $\mathcal{G}(X)$. In section 3 we show that there is a recurrence relation between the K-theory groups of $\mathcal{G}(X)$ and those of $C_0(\mathbb{R}^n) \rtimes \mathfrak{S}_n$. Using Baum-Connes' Chern character, we compute (following ideas of Kroum Tzanev) the latter groups in section 4. This gives us the K-theory groups of the C^*-algebra of the groupoid of a manifold with corners in section 5.

It's a pleasure for us to thank Max Karoubi, André Legrand, Georges Skandalis and Kroum Tzanev for many enlightening discussions.

2. Main definitions

In this section, we recall some definitions and properties of [6]. We call codimension p manifold with corners a C^∞ manifold modelled on $\mathbb{R}^n_+$, such that each point has a neighborhood diffeomorphic to some $\mathbb{R}^k_+ \times \mathbb{R}^{n-k}$, with $k \leq p$, the point corresponding to the origin. We do not assume that each codimension 1 submanifold is embedded, unlike in the definition of R. Melrose ([5]).

Let X be a manifold with corners, and x a point of X; we denote by $F(x)$ the connected component of the set of points having the same codimension as x which contains x, and by $N_xF(x) = T_xX/T_xF(x)$ the normal space to the boundary at x. We can now define $\mathcal{G}(X)$:

$$\mathcal{G}(X)=\Big\{(x,y,\alpha), x,y\in X, \mathrm{codim}(x)=\mathrm{codim}(y), \alpha : N_yF(y) \overset{\sim}{\to} N_xF(x)\Big\}.$$

In this definition, α is an affine isomorphism of quadrants; locally one can trivialize $N_yF(y) \simeq \mathbb{R}_+^k$ and $N_xF(x) \simeq \mathbb{R}_+^k$, and α is given by a monomial matrix (it has one, and only one non-zero element on each line and each column, and this element is positive).

The groupoid structure of $\mathcal{G}(X)$ is given by $r, s : \mathcal{G}(X) \to X$ with $r(x,y,\alpha) = x$, and $s(x,y,\alpha) = y$; the composition law is induced by the composition of the isomorphisms, in the sense that $(x,y,\alpha)(y,z,\beta) = (x,z,\alpha\beta)$. We endow it with a C^∞ manifold structure as follows.

Let Ω and Ω' be two charts of X of same codimension. Thus $\Omega \overset{\pi}{\simeq} U \times \mathbb{R}_+^A$ and $\Omega' \overset{\pi'}{\simeq} U' \times \mathbb{R}_+^{A'}$, where A and A' are the sets of local hyperfaces contained in Ω and Ω'; they have same cardinal. Let $\sigma : A' \to A$ be a bijection.

This allows one to define a map

$$\begin{array}{rcl} \psi_\sigma : U \times U' \times \mathbb{R}_+^{A'} \times \mathbb{R}_+^{*\,A'} & \to & \mathcal{G}(X) \\ (u,u',t,\lambda) & \mapsto & (\pi^{-1}(u,\sigma(\lambda t)), {\pi'}^{-1}(u',t), \alpha_{\sigma,t,\lambda}) \end{array}$$

where $\alpha_{\sigma,t,\lambda}$ is an isomorphism of the normal bundle defined as follows. Let $B'_t = \{i \in A', t_i = 0\}$, and $B_t = \sigma(B)$. Let $m_{\sigma,t,\lambda}$ be the product of the matrix of $\sigma_{|B'}$ by the diagonal matrix composed of the λ_i for $i \in B'$. Then $\alpha_{\sigma,t,\lambda}$ is given through the trivialisations induced by π and π':

$$\begin{array}{ccc} \alpha_{\sigma,t,\lambda} : N_{\pi'^{-1}(u',t)}F({\pi'}^{-1}(u',t)) & \xrightarrow{\simeq} & N_{\pi^{-1}(u,\sigma(\lambda t))}F(\pi^{-1}(u,\sigma(\lambda t))) \\ \simeq\big\downarrow & & \simeq\big\downarrow \\ \mathbb{R}_+^{B'} & \xrightarrow{m_{\sigma,t,\lambda}} & \mathbb{R}_+^{B} \end{array}$$

This map is injective, and one thus obtains a C^∞ structure on $\mathcal{G}(X)$, which turns it into a Lie groupoid, whose fibers are submanifolds without boundary; it is amenable. Though this groupoid is not necessarily Hausdorff, one can build a pseudodifferential calculus associated to it, and an analytic index map ([8, 6])

$$ind_a : K^*(TX) \to K_*(C^*(\mathcal{G}(X))).$$

Actually, $\mathcal{G}(X)$ is universal among the manifolds with corners of same codimension in the following sense: if X and X' have the same codimension, then $\mathcal{G}(X)$ is equivalent to $\mathcal{G}(X')$, (because X and X' are open and closed subsets of $X \cup X'$ that meet each orbit of $\mathcal{G}(X \cup X')$ so

$\mathcal{G}(X \cup X')_{|X} = \mathcal{G}(X)$ and $\mathcal{G}(X \cup X')_{|X'} = \mathcal{G}(X')$ are both equivalent to $\mathcal{G}(X \cup X')$). This implies ([9]) that $C^*(\mathcal{G}(X))$ is Morita equivalent to $C^*(\mathcal{G}(X'))$, and finally $K_*(C^*(\mathcal{G}(X)))$ is canonically isomorphic to $K_*(C^*(\mathcal{G}(X')))$.

To compute the groups in which the analytic index takes its values for a codimension n manifold with corners, it is thus enough to compute $K_*(C^*(\mathcal{G}_n))$, where $\mathcal{G}_n = \mathcal{G}(\mathbb{R}_+^n)$.

Let's give an explicit description of the stucture of $\mathcal{G}_n$: Let's recall that an isomorphism of $\mathbb{R}_+^n$ is given by a matrix which on each row and each column has exactly one nonzero coefficient, and this coefficient is strictly positive; equivalently it is the product of a diagonal matrix with striclty positive coefficients by the matrix of a permutation on the vectors of the basis. The group of such isomorphisms is thus isomorphic to the semi-direct product $\mathbb{R}_+^{*n} \rtimes \mathfrak{S}_n$ where $\mathfrak{S}_n$ acts on $\mathbb{R}_+^{*n}$ by permutation. For every $\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{R}_+^{*n}$ and $\sigma \in \mathfrak{S}_n$, one will denote by $\alpha_{\sigma,\lambda}$ the isomorphism of $\mathbb{R}_+^n$ given by the product of $\operatorname{diag}(\lambda_1, \cdots, \lambda_n)$ and the matrix of σ.

If $t = (t_1, \cdots, t_n) \in \mathbb{R}_+^n$, then $N_t = \{x = (x_1, \cdots, x_n) \in \mathbb{R}_+^n ; x_i = 0 \text{ if } t_i \neq 0\}$. So the isomorphism $\alpha_{\sigma,\lambda}$ restricts to an isomorphism from N_t onto $N_{\sigma(\lambda t)}$. The local charts are thus defined by the maps $\psi_\sigma : (t, \lambda) \mapsto (\sigma(\lambda t), t, (\alpha_{\sigma,\lambda})_{|N_t})$ from $\mathbb{R}_+^n \times \mathbb{R}_+^{*n}$ onto $\mathcal{G}(\mathbb{R}_+^n)$.

3. A recurrence relation

Let $\mathcal{G}_n'$ be the restriction of $\mathcal{G}_n = \mathcal{G}(\mathbb{R}_+^n)$ to the set Y_n of points of dimension 0 and 1 in $\mathbb{R}_+^n$. The partitions $\mathbb{R}_+^n = \{0\} \cup X'$ and $Y_n = \{0\} \cup Y_n'$ give rise to the following commutative diagram whose lines are exact sequences:

$$\begin{array}{ccccccccc}
0 \to & C^*(Y_n' \times Y_n' \times \mathbb{R}_+^{*n-1} \rtimes \mathfrak{S}_{n-1}) & \xrightarrow{j_n} & C^*(\mathcal{G}_n') & \xrightarrow{q_n} & C^*(\mathbb{R}_+^{*n} \rtimes \mathfrak{S}_n) & \to 0 \\
& \uparrow{\scriptstyle p_{n-1}} & & \uparrow{\scriptstyle p_n'} & & \uparrow{\scriptstyle =} & \\
0 \to & C^*(\mathcal{G}_{n|X'}) & \xrightarrow{i_n} & C^*(\mathcal{G}_n) & \xrightarrow{p_n} & C^*(\mathbb{R}_+^{*n} \rtimes \mathfrak{S}_n) & \to 0
\end{array}$$

(since $C^*(\mathcal{G}_{n|\{0\}}) = C^*(\mathbb{R}_+^{*n} \rtimes \mathfrak{S}_n)$).

As X' is open in $\mathbb{R}_+^n$, one has $\mathcal{G}_{n|X'} = \mathcal{G}(X')$ which is Morita-equivalent to $\mathcal{G}_{n-1}$ since X' is a manifold with corners of codimension $n-1$. From now on, we will write $\mathcal{G}_{n-1}$ instead of $\mathcal{G}_{n|X'}$ in the diagrams.

These exact sequences induce 6-terms exact sequences in K-theory, whose boundary maps will be denoted by ∂_n and δ_n respectively.

Theorem 3.1. *For any $n \in \mathbb{N}$, the K-theory map $i_{n*} = 0$, and the following sequence is exact for ($n \geq 2$):*

$$0 \longrightarrow K_*(C^*(\mathcal{G}_n)) \xrightarrow{p_{n*}} K_*(C^*(\mathbb{R}_+^{*n} \rtimes \mathfrak{S}_n)) \xrightarrow{\delta_{n*}} K_{1-*}(C^*(\mathcal{G}_{n-1})) \longrightarrow 0$$

Proof. We will show this by induction. First when $n = 1$ one has the following exact sequence (where $\mathbb{R}_+^* \times \mathbb{R}_+^*$ denotes the groupoid of pairs):

$$0 \longrightarrow C^*(\mathbb{R}_+^* \times \mathbb{R}_+^*) \longrightarrow C^*(\mathcal{G}_1) \longrightarrow C^*(\mathbb{R}_+^*) \longrightarrow 0$$

whose (non trivial) boundary map is the Bott isomorphism (*cf.* [6]). So $K_*(C^*(\mathcal{G}_1)) = 0$ and, as a consequence, $i_{1*} = 0$ and $i_{2*} = 0$. Let's assume that $i_{k*} = 0$ for $k \leq n-1$. As the following diagram commutes, one gets $i'_{n*} = 0$ and p'_{n*} is injective:

$$\begin{tikzcd}
 & 0 & & & \\
0 \arrow[r] & C^*(Y'_n \times Y'_n \times {\mathbb{R}_+^*}^{n-1} \rtimes \mathfrak{S}_{n-1}) \arrow[u] \arrow[r, "j_n"] & C^*(\mathcal{G}'_n) \arrow[r, "q_n"] & C^*({\mathbb{R}_+^*}^n \rtimes \mathfrak{S}_n) \arrow[r] & 0 \\
0 \arrow[r] & C^*(\mathcal{G}_{n-1}) \arrow[u, "p_{n-1}"] \arrow[r, "i_n"] & C^*(\mathcal{G}_n) \arrow[u, "p'_n"] \arrow[r, "p_n"] & C^*({\mathbb{R}_+^*}^n \rtimes \mathfrak{S}_n) \arrow[u, "="] \arrow[r] & 0 \\
 & C^*(\mathcal{G}_{n-2}) \arrow[u, "i_{n-1}"] \arrow[r, phantom, "="] & C^*(\mathcal{G}_{n-2}) \arrow[u, "i'_n"] & & \\
 & 0 \arrow[u] & 0 \arrow[u] & &
\end{tikzcd}$$

As $p'_{n*} \circ i_{n*} = j_{n*} \circ p_{n-1*}$ one obtains $i_{n*} = 0$ if and only if $\mathrm{Im}(p_{n-1*}) \subset$ Ker (j_{n*}). But by exactness Ker $(j_{n*}) = \mathrm{Im}(\partial_n)$.

Since by assumption $i_{n-2*} = 0$, p_{n-2*} is also injective and from the diagram below

$$\begin{tikzcd}
0 \arrow[r] & C^*(Y'_{n-1} \times Y'_{n-1} \times {\mathbb{R}_+^*}^{n-2} \rtimes \mathfrak{S}_{n-2}) \arrow[r, "j_{n-1}"] & C^*(\mathcal{G}'_{n-1}) \arrow[r, "q_{n-1}"] & C^*({\mathbb{R}_+^*}^{n-1} \rtimes \mathfrak{S}_{n-1}) \arrow[r] & 0 \\
0 \arrow[r] & C^*(\mathcal{G}_{n-2}) \arrow[u, "p_{n-2}"] \arrow[r, "i_{n-1}"] & C^*(\mathcal{G}_{n-1}) \arrow[u, "p'_{n-1}"] \arrow[r, "p_{n-1}"] & C^*({\mathbb{R}_+^*}^{n-1} \rtimes \mathfrak{S}_{n-1}) \arrow[u, "="] \arrow[r] & 0
\end{tikzcd}$$

one gets that $\mathrm{Im}(p_{n-1*}) =$ Ker $(\delta_{n-1}) =$ Ker (∂_{n-1}). It remains to show that Ker $\partial_{n-1} \subset$ Im ∂_n, which is an easy consequence of the following lemma.

Lemma 3.1. *There exists a map* $k_n : C^*({\mathbb{R}_+^*}^n \rtimes \mathfrak{S}_{n-1}) \to C^*({\mathbb{R}_+^*}^n \rtimes \mathfrak{S}_n)$ *such that*

$$\partial_n \circ k_{n*} = \tau_n + k_{n-1*} \circ \tau_{n-1}^{-1} \circ \partial_{n-1} \circ \tau_n,$$

where $\tau_n : K_*(C^*({\mathbb{R}_+^*}^n \rtimes \mathfrak{S}_{n-1})) \to K_{1-*}(C^*({\mathbb{R}_+^*}^{n-1} \rtimes \mathfrak{S}_{n-1}))$ *is the Bott isomorphism.*

To define the map k_n, we build a variant of $\mathcal{G}'_n$, where we do not admit all isomorphisms of the normal bundle of $\mathbb{R}^n_+$, but only those that fix the first axis. Thus an action of $\mathfrak{S}_{n-1}$ appears, instead of $\mathfrak{S}_n$ in the case of $\mathcal{G}'_n$.

Let $\sigma \in \mathfrak{S}_n$, and ψ_σ defined as above. Let's identify the subgroup of $\mathfrak{S}_n$ of permutations for which 1 is invariant with $\mathfrak{S}_{n-1}$.

Let $\Delta_1 = \{x \in Y_n, x_1 \neq 0 \text{ ou } x = 0\}$ and $\Delta' = \{x \in Y_n, x_1 = 0\}$. Let's define

$$H'_n = \bigcup_{\sigma \in \mathfrak{S}_{n-1}} \psi_\sigma(\Delta_1 \times {\mathbb{R}^*_+}^n),$$

$$H''_n = \bigcup_{\sigma \in \mathfrak{S}_{n-1}} \psi_\sigma(\Delta' \times {\mathbb{R}^*_+}^n),$$

$$H_n = H'_n \cup H''_n = \bigcup_{\sigma \in \mathfrak{S}_{n-1}} \psi_\sigma(Y_n \times {\mathbb{R}^*_+}^n).$$

By definition of $\mathcal{G}'_n$, H_n is an open sub-groupoid of $\mathcal{G}'_n$, and one gets a morphism from $C^*(H_n)$ to $C^*(\mathcal{G}'_n)$. Moreover, H'_n (resp. H''_n) is its restriction to Δ_1 (resp. Δ'), which is saturated.

More precisely, $H'_n = \mathcal{G}(\mathbb{R}_+) \times {\mathbb{R}^*_+}^{n-1} \rtimes \mathfrak{S}_{n-1}$. Indeed, $\psi_\sigma(t, \lambda)$ is given by $(\lambda_1 t_1, t_1, \alpha_{\sigma,t,\lambda})$ if $t_1 \neq 0$ and by $(0, 0, \lambda_1, \alpha_{\sigma,1,\lambda})$ if $t_1 = 0$.

One thus obtains the following exact sequence, induced by that of $\mathcal{G}(\mathbb{R}_+)$:

$$0 \to \mathcal{K} \otimes C^*({\mathbb{R}^*_+}^{n-1} \rtimes \mathfrak{S}_{n-1}) \to C^*(H'_n) \to C^*(\mathbb{R}^*_+ \times {\mathbb{R}^*_+}^{n-1} \rtimes \mathfrak{S}_{n-1}) \to 0.$$

The boundary map in K-theory induced by this exact sequence is consequently the Bott isomorphism

$$\tau_n : K_*(C^*(\mathbb{R}^*_+ \times {\mathbb{R}^*_+}^{n-1} \rtimes \mathfrak{S}_{n-1})) \to K_{1-*}(C^*({\mathbb{R}^*_+}^{n-1} \rtimes \mathfrak{S}_{n-1})).$$

In addition, $H''_n = \mathbb{R}^*_+ \times \mathcal{G}'_{n-1}$. Let $t \in \Delta'$, and $\lambda \in {\mathbb{R}^*_+}^n : t = (0, t')$, and $\lambda = (\lambda_1, \lambda')$. The element $\psi_\sigma(t, \lambda)$ is given by $(\lambda' t', t', \alpha_{\sigma,t,\lambda})$. But the normal space to the boundary at t necessarily factors through Δ_1, and Δ_1 is invariant by σ, thus $\alpha_{\sigma,t,\lambda}$ is given by λ_1 and by $\alpha_{\sigma,t',\lambda'}$. Finally, $\psi_\sigma(t, \lambda)$ is given by $(\lambda_1, \psi_\sigma(t', \lambda'))$.

One gets the following exact sequence, from that of $\mathcal{G}'_{n-1}$:

$$0 \to \mathcal{K} \otimes C^*({\mathbb{R}^*_+}^{n-1} \rtimes \mathfrak{S}_{n-2}) \to C^*(\mathbb{R}^*_+ \times \mathcal{G}'_{n-1}) \to C^*({\mathbb{R}^*_+}^{n} \rtimes \mathfrak{S}_{n-1}) \to 0,$$

whose boundary map in K-theory is $\tau_{n-1}^{-1} \circ \partial_{n-1} \circ \tau_n$.

This induces the following diagram of exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & C^*(Y'_n{\times}Y'_n{\times}{\mathbb{R}_+^*}^{n-1}{\rtimes}\mathfrak{S}_{n-1}) \arrow[r, "j_n"] & C^*(\mathcal{G}'_n) \arrow[r, "q_n"] & C_0({\mathbb{R}_+^*}^n){\rtimes}\mathfrak{S}_n \arrow[r] & 0 \\
0 \arrow[r] & \begin{array}{c} C^*(\mathring{\Delta}_1{\times}\mathring{\Delta}_1{\times}{\mathbb{R}_+^*}^{n-1}{\rtimes}\mathfrak{S}_{n-1}) \oplus \\ C^*(Y'_{n-1}{\times}Y'_{n-1}{\times}{\mathbb{R}_+^*}^{n-1}{\rtimes}\mathfrak{S}_{n-2}) \end{array} \arrow[u] \arrow[r] & C^*(H_n) \arrow[u] \arrow[r] & C^*({\mathbb{R}_+^*}^n{\rtimes}\mathfrak{S}_{n-1}) \arrow[u, "k_n"'] \arrow[r] & 0
\end{tikzcd}$$

At the K-theory level, one has the equality:

$$\partial_n \circ k_{n*} = \tau_n + k_{n-1*} \circ \tau_{n-1}^{-1} \circ \partial_{n-1} \circ \tau_n,$$

which implies (composing by ${\tau_n}^{-1}$) that Ker $\partial_{n-1} \subset$ Im ∂_n.

□

4. K-theory of $C_0(\mathbb{R}^n) \rtimes \mathfrak{S}_n$

In this section we aim to compute the K-theory groups $K_*(C_0(\mathbb{R}^n) \rtimes \mathfrak{S}_n)$ (where $\mathfrak{S}_n$ acts on $\mathbb{R}^n$ by the standard representation), since by Fourier transform, $C^*({\mathbb{R}_+^*}^n \rtimes \mathfrak{S}_n) = C^*({\mathbb{R}_+^*}^n) \rtimes \mathfrak{S}_n = C_0(\mathbb{R}^n) \rtimes \mathfrak{S}_n$. Let us mention that lemma 4.1 and proposition 4.1 were explained to us by Kroum Tzanev.

We try to address this problem by using the equivariant cohomology of P. Baum and A. Connes ([1]), whose definition we recall in the case of a finite group. The K-theory groups of $C_0({\mathbb{R}_+^*}^n){\rtimes}\mathfrak{S}_n$ can be deduced from this using Chern's isomorphism for discrete groups ([1], theorem 1.19).

Let Γ be a finite group acting by homeomorphisms on a topological space X. We write $\hat{X} = \{(x,g) \in X \times \Gamma\,;\, xg = x\}$, on which Γ acts by $(x,g')g = (xg, g^{-1}g'g)$.

Definition 4.1. One defines ([1] definition 1.5) $H^0(X,\Gamma)$ and $H^1(X,\Gamma)$ by

$$H^0(X,\Gamma) = \prod_{j\in\mathbb{N}} H_c^{2j}(\hat{X}/\Gamma;\mathbb{C}) \quad \text{and} \quad H^1(X,\Gamma) = \prod_{j\in\mathbb{N}} H_c^{2j+1}(\hat{X}/\Gamma;\mathbb{C})$$

(with $H_c^*(\hat{X}/\Gamma;\mathbb{C})$ the compactly supported Čech cohomology groups).

Lemma 4.1. *Let $<\Gamma>$ be the set of conjugation classes of Γ and $g_1, \cdots, g_p$ be a system of representants. We denote by C_{g_i} the centralizer of g_i in Γ and $X^{g_i} = \{x \in X\,;\, xg_i = x\}$. Then:*

$$H^*(X,\Gamma) = \bigoplus_{g_i\in<\Gamma>} H_c^*(X^{g_i}/C_{g_i};\mathbb{C})\,.$$

Proof. This lemma is induced by the homeomorphism between $\hat{X}/\Gamma$ and the disjoint union $\displaystyle\bigcup_{g_i\in<\Gamma>} X^{g_i}/C_{g_i}$.

We define, moreover, for any $1 \leq i \leq p \,:\, X_{g_i} = \bigcup_{g\in\Gamma} X^{gg_ig^{-1}}$. We get a decomposition of $\hat{X}$ as a disjoint union $\bigcup_{g_i\in<\Gamma>} X_{g_i}$ of subsets which are invariant under the action of Γ. The natural map $X^{g_i} \subset X_{g_i} \to X_{g_i}/\Gamma$ induces a homeomorphism from X^{g_i}/C_{g_i} to X_{g_i}/Γ.

As $C_{g_i} \subset \Gamma$ this application is well defined.

Moreover $X^{g_i} \subset X_{g_i} \to X_{g_i}/\Gamma$ is surjective: if $z' \in X_{g_i}$, $\exists h \in \Gamma, z'h \in X^{hg_ih^{-1}}$. Thus $z'hg_i = z'h$, and $z'h \in X^{g_i}$.

Last, it is injective because if (y, g_i) and $(z, g_i) \in X_{g_i}$ and $(y, g_i)g = (z, g_i)$ then $g^{-1}g_ig = g_i$ *i.e.* $g \in C_{g_i}$.

□

Let p_n be the number of partitions of n of the shape $n = \lambda_1 + \cdots + \lambda_{2k}$ with $1 \leq \lambda_1 < \cdots < \lambda_{2k}$ and i_n the number of partitions of n of the shape $n = \lambda_1 + \cdots + \lambda_{2k+1}$ with $1 \leq \lambda_1 < \cdots < \lambda_{2k+1}$.

The main result of this section is:

Proposition 4.1. *$H^0(\mathbb{R}^n, \mathfrak{S}_n) = \mathbb{C}^{p_n}$ and $H^1(\mathbb{R}^n, \mathfrak{S}_n) = \mathbb{C}^{i_n}$.*

Proof. We recall first ([3]) that the conjugation classes of $\mathfrak{S}_n$ are in one-to-one correspondence with Young tableaux.

Let $\sigma \in \mathfrak{S}_n$ and T be the Young tableau which represents the conjugation class of σ. We denote by $\lambda_1 < \lambda_2 < \cdots < \lambda_m$ the lengths of the lines of T and by n_k the number of lines of length λ_k in T. The total number of lines in T is then $\ell = n_1 + \cdots + n_m$ and $n = n_1\lambda_1 + \cdots + n_m\lambda_m$.

We will prove that in this case

$$(\mathbb{R}^n)^\sigma / C_\sigma \cong \mathbb{R}^{n_1}/\mathfrak{S}_{n_1} \times \cdots \times \mathbb{R}^{n_m}/\mathfrak{S}_{n_m}$$

Recall that if we consider $\mathfrak{S}_n$ as acting on a set of n boxes, the Young tableau of σ is obtained by sorting the orbits of its action by decreasing length, which corresponds to the decomposition of σ into a product of disjoint circular permutations. With our notation λ_1 is the length of the shortest orbit (or cycle) and there are n_1 orbits of this length.

One checks also that $C_\sigma \cong \prod_{k=1}^m (\mathbb{Z}/\lambda_k\mathbb{Z})^{n_k} \rtimes \mathfrak{S}_{n_k}$ where each $(\mathbb{Z}/\lambda_k\mathbb{Z})$ acts by circular permutation on one of the n_k orbits of length λ_k and $\mathfrak{S}_{n_k}$ acts by interchanging the n_k orbits of length λ_k (this is done by choosing in non canonical manner some origin in every λ_k-orbit).

But an element of $\mathbb{R}^n$ is σ-invariant if its coordinates belonging to the same orbit are equal. Thus the action of $(\mathbb{Z}/\lambda_k\mathbb{Z})^{n_k}$ on $(\mathbb{R}^n)^\sigma$ is trivial; $(\mathbb{R}^n)^\sigma$ identifies with $\mathbb{R}^\ell$ and the action of C_σ factors through the natural action of $\mathfrak{S}_{n_1} \times \cdots \times \mathfrak{S}_{n_m} \subset \mathfrak{S}_\ell$ on $\mathbb{R}^\ell$. One thus obtains that $(\mathbb{R}^n)^\sigma / C_\sigma$ is homeomorphic to $\mathbb{R}^l/\mathfrak{S}_{n_1} \times \cdots \times \mathfrak{S}_{n_m}$ and consequently that $(\mathbb{R}^n)^\sigma / C_\sigma \cong \mathbb{R}^{n_1}/\mathfrak{S}_{n_1} \times \cdots \times \mathbb{R}^{n_m}/\mathfrak{S}_{n_m}$.

An element of $\mathbb{R}^{n_k}/\mathfrak{S}_{n_k}$ is determined by an ordered n_k-tuple of reals $x_1 \leq \cdots \leq x_{n_k}$ and the application

$$(x_1 \leq \cdots \leq x_{n_k}) \mapsto (x_1, x_2 - x_1, \cdots, x_{n_k} - x_{n_k - 1})$$

gives an homeomorphism between $\mathbb{R}^{n_k}/\mathfrak{S}_{n_k}$ and $\mathbb{R}\times (\mathbb{R}_+)^{n_k - 1}$. So the cohomology of $\mathbb{R}^{n_k}/\mathfrak{S}_{n_k}$ is zero as soon as $n_k \neq 1$ and finally by Künneth's formula one gets $H_c^*\left((\mathbb{R}^n)^\sigma / C_\sigma\right) = 0$ except if $n_1 = \cdots = n_m = 1$. In this case $(\mathbb{R}^n)^\sigma / C_\sigma \cong \mathbb{R}^\ell$, so $H_c^*\left((\mathbb{R}^n)^\sigma / C_\sigma\right)$ is null in any degree distinct of ℓ, with $H_c^\ell\left((\mathbb{R}^n)^\sigma / C_\sigma\right) = \mathbb{C}$. The result comes out by applying lemma 4.1. □

Corollary 4.1. *$K_0(C_0(\mathbb{R}^n) \rtimes \mathfrak{S}_n) = \mathbb{Z}^{p_n}$ and $K_1(C_0(\mathbb{R}^n) \rtimes \mathfrak{S}_n) = \mathbb{Z}^{i_n}$.*

Proof. By a result of Karoubi ([4]) $K_*(C_0(\mathbb{R}^n) \rtimes \mathfrak{S}_n) = K^*_{\mathfrak{S}_n}(\mathbb{R}^n)$ is finitely generated and torsion free, so the result is an obvious consequence of the Chern isomorphism for discrete groups [1], theorem 1.19: $K^*_{\mathfrak{S}_n}(\mathbb{R}^n) \otimes \mathbb{C} = H^*(\mathbb{R}^n, \mathfrak{S}_n)$. □

Let $q_n = i_n + p_n$. The generating function of the sequence $(q_n)_{n\in\mathbb{N}}$ is $\sum_{n=0}^{\infty} q_n x^k = \prod_{n=1}^{\infty}(1+x^n)$, and admits the following asymptotic equivalent: $\dfrac{e^{\pi\sqrt{\frac{n}{3}}}}{4.3^{\frac{1}{4}} n^{\frac{3}{4}}}$. If $q'_n = i_n - p_n$, the generating function of the sequence $(q'_n)_{n\in\mathbb{N}}$ is $\sum_{n=0}^{\infty} q'_n x^k = \prod_{n=1}^{\infty}(1-x^n) = \sum_{k=0}^{\infty}(-1)^k x^{\frac{3k^2+k}{2}}$.

5. K-theory of $C^*(\mathcal{G}(X))$

Proposition 5.1. *$K_0(C^*(\mathcal{G}_n)) = \mathbb{Z}^{\ell_n}$ and $K_1(C^*(\mathcal{G}_n)) = \mathbb{Z}^{m_n}$, where ℓ_n and m_n satisfy the following equalities, with i_n and p_n defined as above:*

$$\ell_n = p_n - m_{n-1}, m_n = i_n - \ell_{n-1}.$$

Proof. This is a direct consequence of proposition 3.1. □

Let's give a few examples of $K_*(C^*(\mathcal{G}_n))$:

$$
\begin{array}{ll}
K_0(C^*(\mathcal{G}_0)) = \mathbb{Z}^1 & K_1(C^*(\mathcal{G}_0)) = \mathbb{Z}^0 \\
K_0(C^*(\mathcal{G}_1)) = \mathbb{Z}^0 & K_1(C^*(\mathcal{G}_1)) = \mathbb{Z}^0 \\
K_0(C^*(\mathcal{G}_2)) = \mathbb{Z}^0 & K_1(C^*(\mathcal{G}_2)) = \mathbb{Z}^1 \\
K_0(C^*(\mathcal{G}_3)) = \mathbb{Z}^0 & K_1(C^*(\mathcal{G}_3)) = \mathbb{Z}^1 \\
K_0(C^*(\mathcal{G}_4)) = \mathbb{Z}^0 & K_1(C^*(\mathcal{G}_4)) = \mathbb{Z}^1 \\
K_0(C^*(\mathcal{G}_5)) = \mathbb{Z}^1 & K_1(C^*(\mathcal{G}_5)) = \mathbb{Z}^1 \\
K_0(C^*(\mathcal{G}_6)) = \mathbb{Z}^1 & K_1(C^*(\mathcal{G}_6)) = \mathbb{Z}^1 \\
K_0(C^*(\mathcal{G}_7)) = \mathbb{Z}^2 & K_1(C^*(\mathcal{G}_7)) = \mathbb{Z}^1 \\
K_0(C^*(\mathcal{G}_8)) = \mathbb{Z}^2 & K_1(C^*(\mathcal{G}_8)) = \mathbb{Z}^1 \\
K_0(C^*(\mathcal{G}_9)) = \mathbb{Z}^3 & K_1(C^*(\mathcal{G}_9)) = \mathbb{Z}^2 \\
K_0(C^*(\mathcal{G}_{10})) = \mathbb{Z}^3 & K_1(C^*(\mathcal{G}_{10})) = \mathbb{Z}^2 \\
K_0(C^*(\mathcal{G}_{20})) = \mathbb{Z}^{17} & K_1(C^*(\mathcal{G}_{20})) = \mathbb{Z}^{18} \\
K_0(C^*(\mathcal{G}_{50})) = \mathbb{Z}^{966} & K_1(C^*(\mathcal{G}_{50})) = \mathbb{Z}^{967} \\
K_0(C^*(\mathcal{G}_{100})) = \mathbb{Z}^{115826} & K_1(C^*(\mathcal{G}_{100})) = \mathbb{Z}^{115825} \\
K_0(C^*(\mathcal{G}_{200})) = \mathbb{Z}^{125443897} & K_1(C^*(\mathcal{G}_{200})) = \mathbb{Z}^{125443898} \\
K_0(C^*(\mathcal{G}_{300})) = \mathbb{Z}^{29434172078} & K_1(C^*(\mathcal{G}_{300})) = \mathbb{Z}^{29434172078}
\end{array}
$$

References

1. Paul Baum and Alain Connes, *Chern character for discrete groups*, A fête of topology, Academic Press, Boston, MA, 1988, pp. 163–232.
2. Alain Connes, *Noncommutative geometry*, Academic Press, Inc., 1994.
3. William Fulton and Joe Harris, *Representation theory*, Springer-Verlag, New York, 1991, A first course, Readings in Mathematics.
4. Max Karoubi, *k-théorie équivariante d'espaces vectoriels et d'espaces projectifs réels*, Preprint.
5. Richard B. Melrose, *The Atiyah-Patodi-Singer index theorem*, A K Peters Ltd., Wellesley, MA, 1993.
6. Bertrand Monthubert, *Groupoïdes et calcul pseudo-différentiel sur les variétés à coins*, Ph.D. thesis, Université Paris 7, 1998.
7. ———, *Pseudodifferential calculus on manifolds with corners and groupoids*, Proc. Amer. Math. Soc. **127** (1999), no. 10, 2871–2881.
8. Bertrand Monthubert and François Pierrot, *Indice analytique et groupoïdes de Lie*, C. R. Acad. Sci. Paris Sér. I Math. **325** (1997), no. 2, 193–198.
9. Paul Muhly, Jean Renault, and Dana Williams, *Equivalence and isomorphism for groupoid C *-algebras.*, J. Operator Theory **17** (1987), no. 1, 3–22 (English).
10. Victor Nistor, Alan Weinstein, and Ping Xu, *Pseudodifferential operators on differential groupoids*, Pacific J. Math. **189** (1999), no. 1, 117–152.

Laboratoire Emile Picard, Université Paul Sabatier (UFR MIG), 118 route de Narbonne, F-31062 Toulouse CEDEX 4

E-mail address: `legall@picard.ups-tlse.fr`

E-mail address: `monthube@picard.ups-tlse.fr`