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Groupoids of manifolds with corners and index theory

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Abstract. This article is a survey on the relations between pseudodifferential calculus on manifolds with corners and groupoids, based on a contribution to the conference "Groupoids in Geometry, Analysis and Physics" held in Boulder, USA in 1999.

1. Introduction

In the spirit of noncommutative geometry, a pseudodifferential calculus is a groupoid. This approach is due to A. Connes, who made an extensive use of the holonomy groupoid of a foliation to study its index properties ([**2, 4**], and who gave a proof of Atiyah-Singer's index theorem through the use of groupoids ([**3**]): he showed that in the case of a smooth manifold M, the classical pseudodifferential calculus is given by the groupoid $M \times M$, and the analytic index is related to the tangent groupoid.

By replacing some analysis by some geometry, this approach allows one to use general techniques on groupoids and C^*-algebras in order to get results on pseudodifferential calculus and index theory.

In this survey we will see how Connes' ideas can be extended, especially to the case of manifolds with corners. First, we define the pseudodifferential calculus of a Lie groupoid, so that the construction of a pseudodifferential calculus in a specific context is reduced to the construction of a Lie groupoid. Then, in the context of manifolds with boundary, we associate a Lie groupoid to any such manifold, and compare it with Melrose's b-calculus. Last, we build a Lie groupoid for any manifold with corners, and we finally show how this helps to understand the b-calculus.

2. Pseudodifferential calculus on Lie groupoids

2.1. Pseudodifferential operators. In this section we define algebras of pseudodifferential operators on Lie groupoids, following [**12, 10**]. An equivalent definition has been independantly obtained by V. Nistor, A. Weinstein and P. Xu ([**13**]).

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Let G be a Hausdorff, Lie groupoid with base $G^{(0)}$; we denote by s and r its source and range maps. We will not assume $G^{(0)}$ to be a smooth manifold without boundary; it can be any C^∞ manifold, but we require that each fiber $s^{-1}(y)$ (for any $y \in G^{(0)}$) is a manifold without boundary; these are longitudinally smooth groupoids. In this framework, one can define a notion of pseudodifferential operators through their Schwartz kernels: let $I_c^m(G, G^{(0)}; \Omega^{\frac{1}{2}})$ be the space of compactly supported distributional sections K on G with values in the bundle of half-densities $\Omega^{\frac{1}{2}}$, which are of class C^∞ outside $G^{(0)}$, and given by an oscillatory integral in a neighborhood of $G^{(0)}$:

$$K(\gamma) = (2\pi)^{-n} \int_{\mathcal{A}_{r(\gamma)}(G)^*} e^{i\langle \varphi(\gamma), \xi\rangle} a(\gamma, \xi) \mathrm{d}\xi.$$

Here, a is a polyhomogeneous symbol of order m with values in $\Omega^{\frac{1}{2}}$, $\mathcal{A}(G)$ is the Lie algebroid associated to G, and φ is a phase function with values in $\mathcal{A}(G)$, such that the following diagram commutes:

$$\begin{array}{ccc} O & \xrightarrow{\varphi} & \mathcal{A}(G) \\ {\scriptstyle r}\downarrow & & \downarrow \\ G^{(0)} & \xrightarrow{=} & G^{(0)} \end{array}$$

It is a diffeomorphism between a tubular neighborhood O of $G^{(0)}$ in G and $\mathcal{A}(G)$, such that $\varphi(\gamma) = 0 \iff \gamma \in G^{(0)}$, and its differential is the isomorphism $T_x G^x \simeq N_x G^{(0)}$.

Such a Schwartz kernel defines a pseudodifferential operator by

$$Tf(\gamma) = \int_{G_{s(\gamma)}} K(\gamma' \gamma^{-1}) f(\gamma).$$

We obtain thus an algebra of pseudodifferential operators denoted by $\Psi_c^\infty(G) = \cup_{m \in \mathbb{Z}} \Psi_c^m(G)$.

Remark 2.1. If G is non-Hausdorff, $\Psi_c^\infty(G)$ can also be defined.

Remark 2.2. If $G = M \times M$, with M a smooth manifold without boundary, we recover precisely the definition of pseudodifferential operators of L. Hörmander ([**6**]).

If we restrict to order 0 operators, we can define $\overline{\Psi_0}(G)$ as the norm closure (in the multiplier algebra of $C_c^\infty(G)$) of the involutive algebra $\Psi_c^0(G)$.

Theorem 2.3. *The following sequence is exact:*

$$0 \to C^*(G) \to \overline{\Psi_0}(G) \to C_0(S^* N^G_{G^{(0)}}) \to 0,$$

where $S^* N^G_{G^{(0)}}$ *is the cosphere bundle of the Lie algebroid of* G, $\mathcal{A}(G) = N^G_{G^{(0)}}$.

This extends the classical theorem of manifolds without boundary M: if $G = M \times M$, then $C^*(G) = \mathcal{K}$, and $S^* N^G_{G^{(0)}} = S^* M$. We will thus call it the Atiyah-Singer's exact sequence of G, even though it does not give conditions for an operator to be Fredholm.

2.2. Analytic index and tangent groupoid. As in the classical case, one can define the analytic index as the boundary map of the exact sequence of the theorem above. We prefer to give a somewhat improved definition, which does not involve $\overline{\Psi_0}(G)$.

In [**5**], M. Hilsum and G. Skandalis defined the normal cone deformation of a groupoid, which generalizes Connes' tangent groupoid; for this reason we will call it the tangent groupoid of G. As a set, it is defined as $G_T = G \times]0,1] \cup \mathcal{A}(G) \times \{0\}$. Its groupoid structure is induced by that of G and of $\mathcal{A}(G)$, and it has a Lie groupoid structure so that the following sequence is exact:

$$0 \to C^*(G) \otimes C_0(]0,1]) \to C^*(G_T) \to C_0(\mathcal{A}^*(G)) \to 0.$$

As $K_*(C^*(G) \otimes C_0(]0,1]))$, one gets an isomorphism

$$ev_0 : K_*(C^*(G_T)) \xrightarrow{\sim} K_*(C_0(\mathcal{A}^*(G))$$

. Moreover, the evaluation at 1 induces a morphism

$$ev_1 : K_*(C^*(G_T)) \to K_*(C^*(G))$$

.

Definition 2.4. The analytic index of G is

$$Ind_a = ev_1 \circ ev_0^{-1} : K_*(C_0(\mathcal{A}^*(G)) \to K_*(C^*(G)).$$

Remark 2.5. The analytic index takes its values in $K_*(C^*(G))$, which is not equal to $\mathbb{Z}$ in general. This is not a Fredholm index. A Fredholmness condition is developed in section 5.2

3. Pseudodifferential calculus on manifolds with boundary

3.1. Melrose's b-calculus. In this section, we briefly recall the definition of a particular pseudodifferential calculus on manifolds with boundary, Melrose's b-calculus ([**9**]).

Let X be a manifold with boundary. The b-stretched product of X is a C^∞ manifold defined as the gluing

$$X_b^2 = X^2 \backslash (\partial X)^2 \cup S_+N,$$

where $S_+N = (N_{(\partial X)^2}^{X^2} - \{0\})/\mathbb{R}_+^*$ can be trivialized as $(\partial X)^2 \times [-1,1]$ if one chooses a definition function ρ of ∂X (a differentiable function which is zero only on ∂X and whose differential is non-zero on ∂X). Under this choice, X_b^2 is a submanifold of $X^2 \times [-1,1]$:

$$X_b^2 = \Big\{(x,y,t) \in X \times X \times \mathbb{R}, (1-t)\rho(x) = (1+t)\rho(y)\Big\}.$$

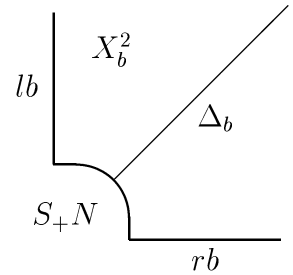
It has three boundary components,

$$lb = \Big\{(x,y,-1) \in \partial X \times X \times \mathbb{R}\Big\} \simeq \partial X \times X, rb = \Big\{(x,y,1) \in X \times \partial X \times \mathbb{R}\Big\} \simeq X \times \partial X,$$

as well as S_+N. It has a "diagonal", denoted by

$$\Delta_b = \Big\{(x,x,0) \in X \times X \times \mathbb{R}\Big\} \simeq X$$

A good picture, due to R. Melrose, is the following, with $X = \mathbb{R}_+$:



This allows R. Melrose to define a b-pseudodifferential operator as an operator whose Schwartz kernel K lives in $I^\infty(X_b^2, \Delta_b; \Omega^{\frac{1}{2}})$, and such that K has a vanishing Taylor series at $lb \cup rb$ (as $lb \cup rb$ is away from Δ_b, K is a C^∞ function on a neighborhood of $lb \cup rb$).

3.2. The groupoid of a manifold with boundary. The definition of the b-calculus is close to that of the pseudodifferential calculus on a groupoid. There are nevertheless two differences: first, X_b^2 is not a Lie groupoid, second, the Schwartz kernels of b-pseudodifferential operators need to fulfill an extra condition, being zero in Taylor series on $lb \cup rb$. But recall for instance that C^∞ functions on the circle, which are zero in Taylor series at 0 are in a one-to-one correspondence with the Schwartz space of $\mathbb{R}$, the C^∞ functions on $\mathbb{R}$ which are rapidly decreasing as well as all their derivatives. We will see that $X_b^2 \backslash (lb \cup rb)$ is precisely a Lie groupoid, introduced in [**11**].

Let X be a manifold with boundary; it is thus the positive part of a manifold without boundary M endowed with a submanifold V of codimension 1, which is transversally oriented and divides M into two parts. Let's then consider the more general case where one is given M and V a submanifold of codimension 1, with no extra condition. Let $NV = TM/TV$ be the normal bundle of V into M. We define the groupoid $\mathbf{G}(V)$ as follows:

$$\mathbf{G}(V) = M\backslash V \times M\backslash V \cup \widetilde{V \times V}$$

where $\widetilde{V \times V}$ is the bundle over $V \times V$ of the isomorphisms of N_yV onto N_xV.

The groupoid structure is that of the couples groupoid on $M\backslash V \times M\backslash V$, and on $\widetilde{V \times V}$ one has

$$r(x,y,\alpha) = x, s(x,y,\alpha) = y, (x,y,\alpha) \circ (y,z,\alpha') = (x,z,\alpha\alpha')$$

.

To define its differentiable groupoid structure, let's take two charts Ω_i ($i = 1,2$) of M which intersect V and such that $V \cap \Omega_i$ is transversally oriented, and $f_i : \Omega_i \to \mathbb{R}$ two definition functions of $V \cap \Omega_i$. By definition, the restriction $\underline{\mathrm{d}f_i}$ of the differential of f_i to $V \cap \Omega_i$ is a non-zero section of the normal bundle of V. Then

$$W_{f_1,f_2} = \{(x,y,\lambda) \in \Omega_1 \times \Omega_2 \times \mathbb{R}^*, f_1(x) = \lambda f_2(y)\}$$

is a submanifold of $\Omega_1 \times \Omega_2 \times \mathbb{R}^*$, and one defines a chart of $\mathbf{G}(V)$ through $\iota_{\Omega_1,\Omega_2}$: $\mathbf{G}(V)_{\Omega_1}^{\Omega_2} \to W_{\varphi_1,\varphi_2}$:

$$\begin{aligned} \iota_{\Omega_1,\Omega_2}(\gamma) &= (r(\gamma), s(\gamma), \tfrac{f_1(r(\gamma))}{f_2(s(\gamma))}) \quad \text{si } \gamma \in \Omega_1 \times \Omega_2, \\ &= (r(\gamma), s(\gamma), \lambda) \quad \text{si } \gamma \in \widetilde{\Omega_1 \times \Omega_2}. \end{aligned}$$

with $\alpha(\underline{\mathrm{d}f_2}) = \lambda \underline{\mathrm{d}f_1}$.

This gives a Lie groupoid structure to $\mathbf{G}(V)$.

We can now define the groupoid of a manifold with corners. Recall that the s-connected component G^c of a groupoid G is the union of the connected components of the units of the s-fibers G_y; it is the smallest open subgroupoid of G which contains $G^{(0)}$.

DEFINITION 3.1. Let X be a manifold with corners, and let M be a manifold without boundary endowed with a transversally oriented submanifold V of codimension 1 which splits M into two parts, such that X is the positive part of M. The groupoid of X is the restriction to X of the s-connected component of $\mathbf{G}(V)$:

$$\Gamma(X) = \mathbf{G}^c(V)_X.$$

REMARK 3.2. This definition does not depend on the choice of (M, V).

The definition above can be simplified if one chooses a definition function ρ of ∂X. Under this choice,

$$\Gamma(X) = \{(x, y, \lambda) \in X \times X \times \mathbb{R}_+^*, \rho(x) = \lambda\rho(y)\} = \overset{\circ}{X} \times \overset{\circ}{X} \cup \partial X \times \partial X \times \mathbb{R}_+^*.$$

Thus a pseudodifferential operator on $\Gamma(X)$ is obtained by gluing a pseudodifferential operator on $\overset{\circ}{X}$ and a $\mathbb{R}_+^*$-invariant pseudodifferential on the cylinder made out of ∂X, as in [**1**].

3.3. b-calculus and pseudodifferential calculus on $\Gamma(X)$. To compare these two calculi, we first compare the spaces on which they define their Schwartz kernels. But the map

$$X \times X \times \mathbb{R}_*^+ \supset \Gamma(X) \ni (x, y, \lambda) \mapsto (x, y, \frac{1-\lambda}{1+\lambda}) \in X_b^2 \subset X \times X \times [-1, 1]$$

defines an embedding of $\Gamma(X)$ into X_b^2, whose complementary is precisely $lb \cup rb$; the units of $\Gamma(X)$ coincide with the diagonal $\Delta - b$. .

As $lb \cup rb$ is disjoint from a neighborhood of Δ_b, one has $I_c^\infty(\Gamma(X)) = I_c^\infty(X_b^2)$. It induces the following result:

THEOREM 3.3. *With the notations above,* $\Psi_c^\infty(\Gamma(X)) = \Psi_{b,c}^\infty(X)$.

Obviously, the only difference between the b-calculus and the pseudodifferential calculus on $\Gamma(X)$ lies at the level of their smoothing operators. To further the comparison, we need to introduce a notion of Schwartz space on $\Gamma(X)$.

DEFINITION 3.4. Let G be a Lie groupoid, and μ a Haar system on G. A function $\phi : G \to \mathbb{R}_+$ will be called a length function with polynomial growth if:

- $\phi(\gamma_1\gamma_2) \leq \phi(\gamma_1) + \phi(\gamma_2)$,
- $\phi(\gamma^{-1}) = \phi(\gamma)$,
- ϕ is proper,
- $\exists c, N, \forall x \in G^{(0)}, \forall r \in \mathbb{R}_+, \mu_x(\varphi^{-1}([0, r]) \leq c(r^N + 1)$.

DEFINITION 3.5. The space of rapidly decreasing functions with respect to ϕ is

$$\mathcal{S}^0 = \{f \in C_0(G, \Omega^{\frac{1}{2}}), \forall P \in \mathbb{C}[X], \sup_{\gamma \in G} |P(\varphi(\gamma))f(\gamma)| < \infty\}.$$

The Schwartz space of G with respect to ϕ is the subspace of $\mathcal{S}^0$ of functions f such that

$$\forall K_1, K_2 \in I_c^\infty(G, G^{(0)}; \Omega^{\frac{1}{2}}), K_1 * f * K_2 \in \mathcal{S}^0.$$

To define a Schwartz space on $\Gamma(X)$, one needs to define a length function with polynomial growth.

Let ρ be a definition function of dx: then

$$\Gamma(X) = \{(x,y,\lambda) \in X \times X \times \mathbb{R}_+^*, \rho(x) = \lambda\rho(y)\} = \overset{\circ}{X} \times \overset{\circ}{X} \cup \partial X \times \partial X \times \mathbb{R}_+^*.$$

Let's define $\phi(x,y,\lambda) = |\log(\lambda)|$. This is a length function with polynomial growth, and the Schwartz space $\mathcal{S}(\Gamma(X))$ induced by ϕ does not depend on ρ.

PROPOSITION 3.6. *The space of smoothing operators of the b-calculus,* Ψ_b^∞ *is a subspace of* $\mathcal{S}(\Gamma(X))$.

The C^∞ functions on X_b^2 which are zero in Taylor series at $lb \cup rb$ are indeed rapidly decreasing as well as their derivatives with respect to $\exp(\phi)$. Thus they are rapidly decreasing as well as their derivatives with respect to ϕ.

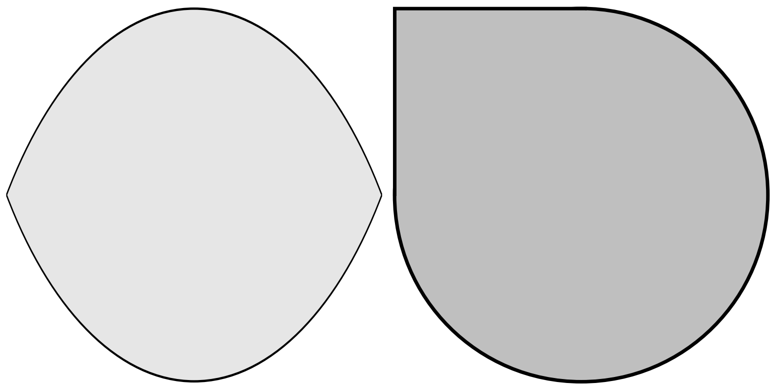
REMARK 3.7. The set of smoothing operators of the b-calculus does not have the properties we expect from a Schwartz-type space, especially it is not stable under holomorphic functionnal calculus.

4. Pseudodifferential calculus on manifolds with corners

4.1. The groupoid of a manifold with embedded corners. In this section we give a few indications to build $\Gamma(X)$ for X a manifold with embedded corners, following [**11**].

Recall that a manifold with corners is a C^∞ manifold modelled on $\mathbb{R}_+^n$. Any point x admits a neighborhood $V \simeq \mathbb{R}_+^k \times U$; k is the codimension of x. An open face $\overset{\circ}{F}$ of codimension k is a connected face of the set of points of codimension k. A face is the closure of an open face. The codimension of X is the maximum of the codimensions of its points.

R. Melrose's definition of manifolds with corners imposes an extra-condition: that the hyperfaces (faces of codimension 1) be embedded, which is equivalent to the existence of a definition function for each hyperface H. For instance, the first manifold below (we call it the two-corners drop) is a manifold with embedded corners, while the second (the one-corner drop) is not, as the boundary meets itself:



As this condition is not necessary to build a pseudodifferential calculus, we will call these the manifolds with embedded corners.

Let X be a manifold with embedded corners. As in the case of manifolds with boundary, we can embed X in a manifold without boundary, M, endowed with a finite set of submanifolds $(V_i)_{i\in I}$, where each V_i is transversally oriented and splits M in two, and such that X is the positive part of $(M,(V_i)_{i\in I})$. By taking fiber products of the groupoids $\mathbf{G}(V_i)$, and then restricting to X, one gets a Lie groupoid denoted by $\Gamma(X)$.

A useful description of $\Gamma(X)$ is the following:

$$\Gamma(X) = \bigcup_{\overset{\circ}{F}} \left\{(x,y,\alpha), x,y \in \overset{\circ}{F}, \alpha : N_y\overset{\circ}{F} \xrightarrow{\sim} N_x\overset{\circ}{F}\right\},$$

where α sends each hyperface onto itself. Thus, if one is given a definition function for each hyperface, the normal bundle of an open face $\overset{\circ}{F}$ is isomorphic to $\mathbb{R}_+^{codim(\overset{\circ}{F})}$, so that one has the trivilization

$$\Gamma(X) \simeq \bigcup_{\overset{\circ}{F}} \overset{\circ}{F} \times \overset{\circ}{F} \times \mathbb{R}_+^{*\,codim(\overset{\circ}{F})}.$$

4.2. The groupoid of a manifold with corners. In this section, X is a manifold with corners, but not necessarily a manifold with embedded corners: a hyperface can intersect itself, and the normal bundles are no longer trivial. For this reason, the definition above has to be extended to take into account all the isomorphisms of the normal bundle, as it does not have necessarily any meaning to send a hyperface onto itself.

For instance, let X be the Klein bottle whose section is a one-corner drop D: $X = D\times[0,1]/\sim$, with $(d,0)\sim(s(d),1)$, where s is the symetry across the diagonal of the drop. This is a codimension-2 manifold with corners, with one hyperface. On a neighborhood of a point of codimension 2, the normal bundle can be locally trivialized, but it is not possible to put an order on the local hyperfaces: if one follows the corner all around, the local hyperfaces are exchanged one with another.

The groupoid of a manifold with corners X, introduced in [**10**], is

$$\mathcal{G}(X) = \left\{(x,y,\alpha), x,y \in X, \mathrm{codim}(x) = \mathrm{codim}(y), \alpha : N_yF(y) \xrightarrow{\sim} N_xF(x)\right\}.$$

In this definition, x and y need not be in the same face, and α is an isomorphism of quadrants; let's further its decription.

Let x and y be two points of same codimension, A and A' be the sets of local hyperfaces at x and y. The normal bundles can be trivialized:

$$N_yF(y) \simeq \mathbb{R}_+^{A'}, N_xF(x) \simeq \mathbb{R}_+^{A},$$

so that an isomorphism $\alpha : N_yF(y) \to N_xF(x)$ is given by a monomial matrix with strictly positive values, *i.e.* by the product of a diagonal matrix with strictly positive values and of the matrix of a bijection $\sigma : A' \to A$.

As before, the groupoid structure is given by

$$s(x,y,\alpha) = y, r(x,y,\alpha) = x, (x,y,\alpha)(y,z,\alpha') = (x,z,\alpha\alpha').$$

To define the C^∞ manifold structure, consider two charts of X, Ω and Ω', of the same codimension. Let A (*resp.* A') be the set of local hyperfaces of Ω (*resp.* Ω'). We have

$$\Omega \xrightarrow{\sim} U \times \mathbb{R}_+^{A}, \Omega' \xrightarrow{\sim} U' \times \mathbb{R}_+^{A'}.$$

Let $\sigma : A' \to A$ be a bijection. One can then define an injective map

$$\begin{array}{rcl}\psi_\sigma : U \times U' \times \mathbb{R}_+^{A'} \times \mathbb{R}_+^{*A'} & \to & \mathcal{G}(X) \\ (u,u',t,\lambda) & \mapsto & (\pi^{-1}(u,\sigma(\lambda t)), {\pi'}^{-1}(u',t), \alpha_{\sigma,t,\lambda})\end{array}$$

where $\alpha_{\sigma,t,\lambda}$ is an isomorphism of the normal bundle defined as follows. Let $B'_t = \{i \in A', t_i = 0\}$, and $B_t = \sigma(B)$, be the sets of local hyperfaces of ${\pi'}^{-1}(u',t)$ and $\pi^{-1}(u,\sigma(\lambda t))$. Then $\alpha_{\sigma,t,\lambda}$ is defined through the trivialisations induced by π and π':

$$\begin{array}{ccc}\alpha_{\sigma,t,\lambda} : N_{{\pi'}^{-1}(u',t)}F({\pi'}^{-1}(u',t)) & \xrightarrow{\simeq} & N_{\pi^{-1}(u,\sigma(\lambda t))}F(\pi^{-1}(u,\sigma(\lambda t))) \\ \simeq\downarrow & & \simeq\downarrow \\ \mathbb{R}_+^{B'} & \xrightarrow{m_{\sigma,t,\lambda}} & \mathbb{R}_+^{B}\end{array}$$

where $m_{\sigma,t,\lambda}$ is the product of the matrix of $\sigma_{|B'}$ by the diagonal matrix composed of the λ_i for $i \in B'$.

This map ψ_σ makes $U \times U' \times \mathbb{R}_+^{A'} \times \mathbb{R}_+^{*A'}$ into a chart of $\mathcal{G}(X)$, and this defines a C^∞ manifold structure for $\mathcal{G}(X)$. This groupoid is amenable.

Remark 4.1. If X is of codimension greater or equal to 2, then $\mathcal{G}(X)$ is non-Hausdorff.

4.3. Pseudodifferential calculus on manifolds with corners. As $\mathcal{G}(X)$ is a (longitudinally smooth) Lie groupoid, we can define a pseudodifferential calculus on any manifold with corners:

Definition 4.2. Let X be a manifold with corners (where the hyperfaces need not be embedded). The pseudodifferential calculus of X is the pseudodifferential calculus of the Lie groupoid of X, $\mathcal{G}(X)$.

It follows that there is a symbol map, and an analytic index

$$Ind_a : K_*(C_0(T^*X)) \to K_*(C^*(\mathcal{G}(X))).$$

Theorem 4.3. *The analytic index of a manifold with corners X takes its values in a group which depends only on the codimension of X.*

Proof. Let X and X' be two manifolds with corners of the same codimension. Let $x \in X$: there exists also $x'_i X'$ such that $\mathrm{codim}(x) = \mathrm{codim}(x')$. This implies that X' intersects any orbit of $\mathcal{G}(X \cup X')$, thus $\mathcal{G}(X \cup X')_X^X = \mathcal{G}(X)$ is an open, full subgroupoid of $\mathcal{G}(X \cup X')$. Equivalently, $\mathcal{G}(X')$ is an open, full subgroupoid of $\mathcal{G}(X \cup X')$, so that $\mathcal{G}(X) \sim \mathcal{G}(X')$. We proved in [**10**] that it implies that $K_*(C^*(\mathcal{G}(X))) = K_*(C^*(\mathcal{G}(X')))$, even if $\mathcal{G}(X)$ is non-Hausdorff. ☐

In collaboration with P.Y. Le Gall, we computed in [**7**] the K-theory groups of $\mathcal{G}(X)$ for any codimension:

Theorem 4.4. *For any $n \in \mathbb{N}$, one has the following equality:*

$$0 \to K_*(C^*(\mathcal{G}_n)) \to K_*(C_0({\mathbb{R}_+^*}^n) \rtimes \mathfrak{S}_n) \to K_{1-*}(C^*(\mathcal{G}_{n-1})) \to 0,$$

and

$$K_0(C_0({\mathbb{R}_+^*}^n) \rtimes \mathfrak{S}_n) = \mathbb{Z}^{p_n}, K_1(C_0({\mathbb{R}_+^*}^n) \rtimes \mathfrak{S}_n) = \mathbb{Z}^{i_n},$$

where p_n is the number of partitions of n of the shape $n = \lambda_1 + \cdots + \lambda_{2k}$ with $1 \leq \lambda_1 < \cdots < \lambda_{2k}$, and i_n is the number of partitions of n of the shape $n = \lambda_1 + \cdots + \lambda_{2k+1}$ with $1 \leq \lambda_1 < \cdots < \lambda_{2k+1}$.

4.4. The reduced groupoid of a manifold with corners. The definition of $\mathcal{G}(X)$, as we mentioned before, is very general. In some situations, a smaller groupoid can be built, as we did for manifolds with embedded corners. More generally, it is always possible to consider the s-connected component of $\mathcal{G}(X)$; in the case of manifolds with embedded corners, it coincides with the groupoid defined in the previous section, so we will denote it by $\Gamma(X)$.

To describe $\Gamma(X)$, let's consider for each open face $\overset{\circ}{F}$ the finite covering $E_{\overset{\circ}{F}}$ whose fiber at a point x is the set of vertices of $N_x\overset{\circ}{F}$. Let $Gal(E_{\overset{\circ}{F}})$ be its Galois covering:

$$Gal(E_{\overset{\circ}{F}}) = \Big\{(x, y, \sigma), x, y \in \overset{\circ}{F}, \sigma : E_{\overset{\circ}{F}y} \xrightarrow{\sim} E_{\overset{\circ}{F}x}\Big\}.$$

Let also

$$Hol(E_{\overset{\circ}{F}}) = Gal(E_{\overset{\circ}{F}})^c$$

be the s-connected component of $Gal(E_{\overset{\circ}{F}})$.

Proposition 4.5. *Let X be a manifold with corners. Then $\Gamma(X)_x^x$ is the semi-direct product of the holonomy group $Hol(E_{\overset{\circ}{F}})_x^x$ and of ${\mathbb{R}_+^*}^A$ where A is the fiber of $E_{\overset{\circ}{F}}$ at x. Moreover,*

$$\Gamma(X) = \bigcup_{\overset{\circ}{F}} \Gamma_{iso}^c(X) \rtimes Hol(E_{\overset{\circ}{F}}),$$

where $\Gamma_{iso}^c(X)$ is the s-connected component of the isotropy subgroupoid of $\Gamma(X)$. The groupoid $\Gamma(X)$ is Hausdorff if and only if for any open face $\overset{\circ}{F}$ the covering $E_{\overset{\circ}{F}}$ is trivial.

This result was proved in [**10**].

5. Applications

By introducing the groupoid of a manifold with corners, we can analyze pseudo-differential and index theory problems with the tools of noncommutative geometry. Let's

5.1. Structure of $\overline{\Psi_0}(\Gamma(X))$ and $\overline{\Psi_0}(\mathcal{G}(X))$. To study the algebra of pseudodifferential operators on a manifold with corners, we first state an easy lemma.

Lemma 5.1. *Let G be a locally compact topological groupoid, and*

$$F_1 \subset F_2 \subset \ldots \subset F_n \subset G^{(0)},$$

an increasing sequence of saturated, closed parts of $G^{(0)}$. Then $C^*(G)$ has a composition series

$$C^*(G) \supset I_1 \supset \ldots \supset I_n,$$

where $I_k = C^*(G_{G^{(0)}\setminus F_k})$. If, moreover, G is amenable, $I_k/I_{k+1} \simeq C^*(G_{F_{k+1}\setminus F_k})$.

Let X be a manifold with embedded corners. We can apply this lemma to the sequence $F_k = \cup_{dim F' < k} F'$. It gives the following result, which was originally proved by R. Melrose and V. Nistor ([**8**]) with analytic methods:

Corollary 5.2. *Let X be a manifold with embedded corners. Then $\overline{\Psi_0}(\Gamma(X))$ admits a composition series*

$$\overline{\Psi_0}(\Gamma(X)) \supset \mathcal{I}_0 \supset \mathcal{I}_1 \supset \ldots \supset \mathcal{I}_n$$

where n is the dimension de X, $\mathcal{I}_0 = C^*(\Gamma(X))$, et $\mathcal{I}_l$ is the closure of the ideal of b-pseudodifferential operators of strictly negative order whose indicial parts are zero on each face $F' \subset F$ of dimension strictement lower than l.

The subquotients are determined by the natural isomorphisms $\sigma : \overline{\Psi_0}(\Gamma(X))/\mathcal{I}_0 \overset{\sim}{\to} C(S^*X)$, and by

$$\mathcal{I}_l/\mathcal{I}_{l+1} \simeq \bigoplus_{\dim F'=l} C_0(\mathbb{R}^{n-l}, \mathcal{K}_{F'}), 0 \leq l \leq n.$$

Now if X is a general manifold with corners, we have a similar result by applying the lemma to the sequence $F_j = \{x \in X, \dim(x) < n + j - p\}$ where n is the dimension of X and p is its codimension.

Corollary 5.3. *Let X be a manifold with corners of dimension n and of codimension p. Then there is a composition series*

$$\overline{\Psi_0}(\mathcal{G}(X)) \supset \mathcal{I}_0 \supset \mathcal{I}_1 \supset \ldots \supset \mathcal{I}_p$$

where $\mathcal{I}_0 = C^*(\mathcal{G}(X))$, and $I_k = C^*(\mathcal{G}(X)_{X \setminus F_k})$. Moreover, $\mathcal{I}_l/\mathcal{I}_{l+1}$ is Morita equivalent to $C^*({\mathbb{R}_+^*}^{n-j} \rtimes \mathfrak{S}_{n-j})$.

5.2. Ellipticity conditions. Another general and easy result gives the ellipticity conditions for pseudodifferential operators on manifolds with corners.

Let G be an amenable, Lie groupoid, and $F \subset G^{(0)}$ a saturated, closed part. Then the following diagram is commutative:

$$\begin{array}{ccccccccc}
 & & 0 & & & & & & \\
 & & \downarrow & & & & & & \\
 & & C^*(G_{G^{(0)}\setminus F}) & & & & & & \\
 & & \downarrow & & & & & & \\
0 & \to & C^*(G) & \to & \overline{\Psi_0}(G) & \xrightarrow{\sigma} & C_0(S^*G^{(0)}) & \to & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle In_F} & & \downarrow & & \\
0 & \to & C^*(G_F) & \to & \overline{\Psi_0}(G_F) & \longrightarrow & C_0(S^*G^{(0)}_{|F}) & \to & 0 \\
 & & \downarrow & & & & \downarrow & & \\
 & & 0 & & & & 0 & &
\end{array}$$

thus $\overline{\Psi_0}(G) \to \overline{\Psi_0}(G_F)$ is onto, for $C^*(G) \to C^*(G_F)$ and $C^*(S^*G^{(0)}) \to C^*(S^*G^{(0)}_{|F})$ are onto.

Proposition 5.4. *$C^*(G_{G^{(0)}\setminus F}) = Ker\, \sigma \cap Ker\, In_F = Ker\, (\sigma \oplus In_F)$.*

Moreover, the image of $\sigma \oplus In_F$ is the fiber product of $\overline{\Psi_0}(G_F)$ and of $C^*(S^*G^{(0)})$ above $C^*(S^*G^{(0)}_{|F})$.

Corollary 5.5. *Let X be a manifold with corners, and G be an open subgroupoid of $\mathcal{G}(X)$ containing $\mathcal{G}(X)^{(0)}$ (for instance, $G = \mathcal{G}(X)$, or $G = \Gamma(X)$). Then the following sequence is exact:*

$$0 \to \mathcal{K} \to \overline{\Psi_0}(G) \to C_0(S^*X) \times_f \overline{\Psi_0}(G_{\partial X}) \to 0,$$

where the last algebra is a fiber product above $C_0(S^*X_{|\partial X})$.

This results implies that a pseudodifferential operator on a manifold with corners is Fredholm if and only if its symbol is invertible, as well as its restriction to the boundary. This generalizes the result of Atiyah-Patodi-Singer, and gives a more general and easier proof to a result of R. Melrose and V. Nistor ([**8**]), obtained in the context of manifolds with embedded corners.

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