Progress in Mathematics, Vol. 342, 157–172

A Categorical Invariant for Geometrically Rational Surfaces with a Conic Bundle Structure

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Abstract. We define a categorical birational invariant for minimal geometrically rational surfaces with a conic bundle structure over a perfect field via components of a natural semiorthogonal decomposition. Together with the similar known result on del Pezzo surfaces, this provides a categorical birational invariant for geometrically rational surfaces.

Mathematics Subject Classification (2010). 14E08, 14F08, 14D10, 14J26.

Keywords. Conic bundles, rational surfaces, derived categories, semiorthogonal decompositions.

1. Introduction

In recent years the study of the derived category of an algebraic variety has been widely developed. It is clear now that semiorthogonal decompositions can provide a useful tool in order to detect the geometrical structure of a variety. Particularly relevant and interesting is the research of a birational invariant to be used, for example, to study the rationality of the variety.

In this context, the first author and M. Bolognesi [6] introduced the concept of categorical representability and formulated the following question: is a rational variety always categorically representable in codimension 2? Analogously, is it possible to characterize obstruction to rationality via natural components of some semiorthogonal decomposition which cannot be realized in codimension 2? Over the complex field, for example, if we consider a V_{14} Fano threefold X, its derived category admits a semiorthogonal decomposition with only one nontrivial component $\mathcal{A}_X$. For a smooth cubic threefold Y we can also find a decomposition with only one nontrivial component $\mathcal{A}_Y$, and Kuznetsov showed that $\mathcal{A}_X$ is equivalent to $\mathcal{A}_Y$ if Y is the unique cubic threefold birational to X [13]. This suggests that one could consider $\mathcal{A}_X$ to be a birational invariant.

In the case of complex conic bundles $\pi : X \to S$ over a minimal rational surface the situation is quite well known. A necessary condition for rationality of X is that the intermediate Jacobian $J(X)$, as principally polarized Abelian variety, is the direct sum of the intermediate Jacobian of smooth projective curves. It follows for example that smooth cubic threefolds are not rational [7]. From a categorical point of view it is possible to characterize the rationality of the conic bundle from the semiorthogonal decomposition of the derived category. By Kuznetsov [12] we have

$$\mathsf{D}^{\mathrm{b}}(X) = \langle \Phi \mathsf{D}^{\mathrm{b}}(S, \mathcal{B}), \pi^* \mathsf{D}^{\mathrm{b}}(S) \rangle, \tag{1.1}$$

where $\mathcal{B}$ is the sheaf of even parts of Clifford algebras associated to the quadratic form defining the fibration and $\Phi : \mathsf{D}^{\mathrm{b}}(S, \mathcal{B}) \to \mathsf{D}^{\mathrm{b}}(X)$ is a fully faithful functor from the derived category of $\mathcal{B}$-algebras over S. If S is rational the only nontrivial part for this semiorthogonal decomposition must then be contained in the component $\mathsf{D}^{\mathrm{b}}(S, \mathcal{B})$. If S is minimal, the first author and Bolognesi proved that X is rational if and only if $\mathsf{D}^{\mathrm{b}}(S, \mathcal{B})$ has a decomposition whose components are derived categories of smooth curves or exceptional objects [5].

All those results hold on the complex field $\mathbb{C}$, but we want to study the problem over an arbitrary perfect field k. Auel and the first author worked out the case of del Pezzo surfaces [2][1]. Given a minimal del Pezzo surface S of degree d and Picard rank 1, a natural subcategory $\mathcal{A}_S \subset \mathsf{D}^{\mathrm{b}}(S)$ is defined by the right orthogonal complement to the structure sheaf. In [2], a category GK_S is defined, roughly speaking, as the product of all components of $\mathcal{A}_S$ which are not representable in dimension 0, and it is a birational invariant. Such Griffiths–Kuznetsov component, where it is defined, is then the suitable birational invariant to detect the rationality of the given variety. Our aim is to extend this approach to the other class of geometrically rational minimal surfaces, that is, conic bundles.

The precise definition of such an invariant is given in Definition 12. Roughly speaking, we define the Griffiths–Kuznetsov component to be the direct sum of subcategories of $\mathsf{D}^{\mathrm{b}}(S)$ which are not representable in dimension 0. However, unlike in the case of del Pezzo surfaces, there is no, to the best of the authors' knowledge, argument to prove that the (natural) decomposition we choose to define GK_S is unique up to mutations. This motivates the involved case-by-case definition, and a fundamental part of this work is to prove that GK_S is indeed well defined. Our main result is the following.

Theorem 1. *Let k be a perfect field and S be a geometrically rational surface birational to a conic bundle over k. The Griffiths–Kuznetsov component GK_S is well defined and is a birational invariant.*

Recall the classification of minimal geometrically rational surfaces over an arbitrary field (see, e.g., [9]): minimal conic bundles are one of the two possible

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[1] The results in [2] are claimed to hold over general fields, but perfection is required, as we will show in Remark 4, to ensure that every birational map can be factored into Sarkisov links as in [10].

classes of such surfaces, namely the ones with Picard rank two, the other being del Pezzo surfaces with Picard rank one. Combining Theorem 1 with the results from [2], we obtain the following result.

Theorem 2. *Let S be a geometrically rational surface over a perfect field k. Then the Griffiths–Kuznetsov component GK_S is well defined and it is a birational invariant.*

Notations

Functors of geometric origin between derived categories will be denoted underived (i.e., f^* instead of Lf^* for the pull-back via a morphism). Given a k-algebra A, the notation $\mathsf{D}^{\mathrm{b}}(k, A)$ stands for the k-linear bounded derived category of coherent A-modules.

2. Basics on geometrically rational surfaces

In this section we will introduce some useful and known results. Let k be a perfect field and $\overline{k}$ an algebraic closure. Let us consider S, a smooth projective geometrically integral surface over k. We say that S is geometrically rational if $\overline{S} := S \times_k \overline{k}$ is $\overline{k}$-rational. A field extension l of k is a splitting field for S if $S \times_k l$ is birational to $\mathbb{P}^2_l$ through a sequence of monoidal transformations centered at closed l-points.

A smooth projective surface S is minimal over k if every birational morphism $\phi : S \to Y$, defined over k, to a smooth variety Y is an isomorphism. If k is algebraically closed, the only minimal rational surfaces are the projective plane and projective bundles over $\mathbb{P}^1$. Over a general field, we have the following classification (see, e.g., [9]).

Proposition 3. *Let S be a minimal geometrically rational surface over k. Then S is one of the following:*

(i) *$S = \mathbb{P}^2_k$ is the projective plane, so* $\mathrm{Pic}(S) = \mathbb{Z}$, *generated by the hyperplane section $\mathcal{O}(1)$;*

(ii) *$S \subset \mathbb{P}^3_k$ is a smooth quadric and* $\mathrm{Pic}(S) = \mathbb{Z}$, *generated by the hyperplane section $\mathcal{O}(1)$;*

(iii) *S is a del Pezzo surface with* $\mathrm{Pic}(S) = \mathbb{Z}$, *generated by the canonical class ω_S;*

(iv) *S is a conic bundle $\pi : S \to C$ over a geometrically rational curve, with* $\mathrm{Pic}(S) \simeq \mathbb{Z} \oplus \mathbb{Z}$.

2.1. Elementary links

We recall some elements of the Sarkisov program describing the factorization of a birational map between minimal rational surfaces in elementary links [10]. Let S be a minimal geometrically rational surface with an extremal contraction $\pi : S \to Y$. Then either Y is a point and S is a minimal surface with Picard rank 1 or Y is a curve and S is a conic bundle with Picard rank 2. If $\pi : S \to Y$ and $\pi' : S' \to Y'$

are two such extremal contractions (but S and S' not necessarily minimal), an elementary link is a birational map $\phi : S \dashrightarrow S'$ of one of the following types:

Type I) There is a commutative diagram

$$\begin{tikzcd}
S & S' \arrow[l, "\sigma"'] \arrow[d] \\
Y \arrow[from=u] & Y' \arrow[l, "\psi"']
\end{tikzcd}$$

where $\phi = \sigma^{-1}$, the morphism $\sigma : S' \to S$ is a Mori divisorial elementary contraction and $\psi : Y' \to Y$ is a morphism. In this case, $Y = \mathrm{Spec}(k)$, $\rho(S) = 1$, S is a minimal del Pezzo surface, and $S' \to Y'$ is a conic bundle over a geometrically rational curve.

Type II) There is a commutative diagram

$$\begin{tikzcd}
S \arrow[d] & X \arrow[l, "\sigma"'] \arrow[r, "\tau"] & S' \arrow[d] \\
Y & & Y' \arrow[ll, "\cong"']
\end{tikzcd}$$

where $\phi = \tau \circ \sigma^{-1}$, and $\sigma : X \to S$ and $\tau : X \to S'$ are Mori divisorial elementary contractions. In this case, S and S' have the same Picard number, and are hence either both minimal del Pezzo surfaces (and Y is a point), or both conic bundles of Picard rank two (and Y is a geometrically rational curve).

Type III) There is a commutative diagram

$$\begin{tikzcd}
S \arrow[r, "\sigma"] \arrow[d] & S' \arrow[d] \\
Y \arrow[r, "\psi"] & Y'
\end{tikzcd}$$

where $\phi = \sigma$, $\sigma : S \to S'$ is a Mori divisorial elementary contraction, $S \to Y$ is a conic bundle, Y is a geometrically rational curve, S' is minimal del Pezzo surface, $Y' = \mathrm{Spec}(k)$ and $\psi : Y \to Y'$ is the structural morphism. Links of type III can be seen as inverses of links of type I.

Type IV) There is a commutative diagram

$$\begin{tikzcd}
S \arrow[rr, dashed, "\phi"] \arrow[d] & & S' \arrow[d] \\
Y \arrow[dr, "\psi"] & & Y' \arrow[dl, "\psi'"'] \\
& \mathrm{Spec}(k) &
\end{tikzcd}$$

where $S \simeq S'$ are isomorphic minimal conic bundles, Y and Y' are geometrically rational curves and ψ and ψ' are the structural morphisms. Then both S and S' are conic bundles and the link amounts to a change of conic bundle structure on S.

Any birational map $\phi : S \dashrightarrow S'$ between minimal geometrically rational surfaces can be factored through elementary links, and Iskovskikh gives the complete list of all possible such links [10]. We note that the Picard rank is invariant under links of type II and IV, while it changes under links of type I and III. Moreover, if we suppose that S is not rational the list of links of type I (and hence of their inverses of type III) is very limited: either S is of degree 8 with a point of degree 2, and S' is of degree 6 and the curve Y' can be rational (according to S being a quadric or not), or S is of degree 4, has a rational point and S' is of degree 3 and Y' is a rational curve.

Remark 4. If k is not perfect, then a birational map may not be decomposable in a finite sequence of elementary links centered at closed points. An example of such map was given in [14, Rmk. 1.3]: if $k = (\mathbb{Z}/2\mathbb{Z})[t]$, the birational map ϕ of $\mathbb{P}^2_k$ given by

$$[x_0 : x_1 : x_2] \dashrightarrow [x_0x_2 : x_1x_2 : x_0^2 + tx_1^2]$$

has $[\sqrt{t} : 1 : 0]$ as a base point, and such a point is never defined over a separable field extension of k.

3. Basics on derived categories

3.1. Categorical representability

Using semiorthogonal decompositions, one can define a notion of *categorical representability* for triangulated categories. In the case of smooth projective varieties, this is inspired by the classical notions of representability of cycles, see [6]. We refrain here from recalling standard notions of semiorthogonal decompositions, exceptional objects, and mutations. The interested reader can refer to [1]. Let us just recall a nonstandard definition of exceptional object.

Definition 5. Let A be a division (not necessarily central) simple k-algebra (i.e., the center of A could be a field extension of k), and $\mathcal{A}$ a k-linear triangulated category. An object V of $\mathcal{A}$ is called *A-exceptional* if

$$\mathrm{Hom}(V, V) = A \quad \text{and} \quad \mathrm{Hom}(V, V[r]) = 0 \quad \text{for} \quad r \neq 0.$$

An exceptional object in the classical sense of the term [8, Def. 3.2] is a k-exceptional object. By *exceptional* object, we mean an A-exceptional object for some division k-algebra A.

Remark 6. Note that, if E is an A-exceptional object of $\mathcal{A}$, then the (full triangulated) category $\langle E \rangle$ generated by E is equivalent to $\mathsf{D}^{\mathrm{b}}(k, A)$, the category of bounded complexes of coherent A-algebras.

Example 7. Let A be a central simple algebra over k and $X := SB(A)$ the Severi-Brauer variety associated to it, and let $n = \dim X$. The Quillen vector bundle V is a rank $n+1$ indecomposable vector bundle whose base change to a splitting field is $\mathcal{O}(1)^{\oplus n+1}$, and is in particular an A-exceptional object [15].

Definition 8. A triangulated category $\mathcal{A}$ is *representable in dimension m* if it admits a semiorthogonal decomposition

$$\mathcal{A} = \langle \mathcal{A}_1, \ldots, \mathcal{A}_r \rangle,$$

and for each $i = 1, \ldots, r$ there exists a smooth projective k-variety Y_i with $\dim Y_i \leq m$, such that $\mathcal{A}_i$ is equivalent to an admissible subcategory of $\mathsf{D}^{\mathrm{b}}(Y_i)$.

The above definition is motivated by the following question:

Question 9. *Let X be a smooth projective k-variety of dimension n. Does X rational imply $\mathsf{D}^{\mathrm{b}}(X)$ categorically representable in dimension $n-2$?*

In this work, we consider the above question for surfaces, and we are hence interested in characterizing categories which are representable in dimension 0. These were fully described in [2].

Lemma 10. *A triangulated category $\mathcal{A}$ is representable in dimension 0 if and only if there exists a semiorthogonal decomposition*

$$\mathcal{A} = \langle \mathcal{A}_1, \ldots, \mathcal{A}_r \rangle,$$

such that for each i, there is a k-linear equivalence $\mathcal{A}_i \simeq \mathsf{D}^{\mathrm{b}}(K_i)$ for a separable field extension K_i/k.

3.2. Conic bundles

We recall a natural semiorthogonal decomposition of the derived category of a surface with the structure of a conic bundle, following the work of Kuznetsov [12] and its generalization to general fields [3]. Let S be a surface over the field k with a structure of conic bundle $\pi : S \to C$ over a geometrically rational curve. Such conic bundle is associated to a quadratic form $q : E \to L$ on a locally free $\mathcal{O}_C$-module E of rank 3. Denote by $\mathcal{O}_{S/C}(1)$ the restriction to S of the line bundle $\mathcal{O}_{\mathbb{P}E/C}(1)$, and let $\mathcal{B}$ be the even Clifford algebra associated to the form q, which is a locally free $\mathcal{O}_C$-algebra, well defined up to isomorphism.

Under these conditions, we have that $\pi^* : \mathsf{D}^{\mathrm{b}}(C) \to \mathsf{D}^{\mathrm{b}}(S)$ is fully faithful, and there exist a fully faithful functor $\Phi : \mathsf{D}^{\mathrm{b}}(C, \mathcal{B}) \to \mathsf{D}^{\mathrm{b}}(S)$ such that

$$\mathsf{D}^{\mathrm{b}}(S) = \langle \pi^* \mathsf{D}^{\mathrm{b}}(C), \Phi \mathsf{D}^{\mathrm{b}}(C, \mathcal{B}) \rangle.$$

Notice that one can choose fully faithful functors from $\mathsf{D}^{\mathrm{b}}(C, \mathcal{B})$ to $\mathsf{D}^{\mathrm{b}}(S)$ to realize either the left (as in the introduction) or the right (as above) orthogonal complement to $\pi^* \mathsf{D}^{\mathrm{b}}(C)$. The two functors are different (the differ by a twist by the canonical bundle), but they anyway give equivalent subcategories of $\mathsf{D}^{\mathrm{b}}(S)$. Here and further we choose the functor giving the above semiorthogonal decomposition for practical computational reasons that will be clear in the proofs.

Moreover, since C is a geometrically rational curve, there is a central simple k-algebra A (trivial if and only if $C \simeq \mathbb{P}^1_k$) such that $C = SB(A)$. In particular, there is an A-exceptional object V, which is either $\mathcal{O}(1)$ if $C \simeq \mathbb{P}^1_k$ or the Quillen bundle V as in example 7 if C is not rational, such that (as proved in [4])

$$\mathsf{D}^{\mathrm{b}}(C) = \langle \mathcal{O}_C, V \rangle = \langle V^*, \mathcal{O}_C \rangle.$$

It follows that we can refine the semiorthogonal decomposition of S (by abuse of notation, we set $V := \pi^* V$) as follows:

$$\mathsf{D}^{\mathrm{b}}(S) = \langle \mathcal{O}_S, V, \Phi \mathsf{D}^{\mathrm{b}}(C, \mathcal{B}) \rangle. \tag{3.1}$$

Now, let $\overline{\pi} : \overline{S} \to \mathbb{P}^1_{\overline{k}}$ be the base change of the conic bundle to the algebraic closure. Such a conic bundle is not necessarily a Hirzebruch surface and can indeed be not minimal, and have a finite number, say r, of singular fibers which are given by two lines meeting in a point. We can pick one line in each such fiber and denote such set of lines by $F_1, \ldots, F_r$. The Picard rank of $\overline{S}$ is then $2 + r$, and there is a semiorthogonal decomposition obtained by considering the $\overline{k}$-minimal model $\overline{S} \to S_0$, which is a Hirzebruch surface:

$$\mathsf{D}^{\mathrm{b}}(\overline{S}) = \langle \mathcal{O}_{\overline{S}}, \mathcal{O}_{\overline{S}}(F), \mathcal{O}_{\overline{S}}(\Sigma), \mathcal{O}_{\overline{S}}(\Sigma + F), \mathcal{O}_{F_1}, \ldots, \mathcal{O}_{F_r} \rangle, \tag{3.2}$$

where F is the general fiber of $\overline{\pi}$, and Σ is a general section of $\overline{\pi}$.

We finally notice that the base change of the semiorthogonal decomposition (3.1) is exactly the semiorthogonal decomposition (3.2): indeed, either C is rational and we already have $V = \mathcal{O}_S(F)$, or C is not rational, V has rank 2 and we have $\overline{V} = \mathcal{O}_{\overline{S}}(F)^{\oplus 2}$. The latter generates the same category as $\mathcal{O}_{\overline{S}}(F)$ since we are considering thick subcategories.

It follows that the base change of $\Phi \mathsf{D}^{\mathrm{b}}(C, \mathcal{B})$ to $\overline{S}$ is the subcategory

$$\langle \mathcal{O}_{\overline{S}}, \mathcal{O}_{\overline{S}}(F) \rangle^{\perp} = \langle \mathcal{O}_{\overline{S}}(\Sigma), \mathcal{O}_{\overline{S}}(\Sigma + F), \mathcal{O}_{F_1}, \ldots, \mathcal{O}_{F_r} \rangle. \tag{3.3}$$

4. Links of type I/III and the definition of the Griffiths–Kuznetsov component

We are going to construct a birational invariant for geometrically rational surfaces with a conic bundle structure $\pi : S \to C$ as the collection of subcategories in the semiorthogonal decomposition (3.1) which are not representable in dimension 0. Such an invariant will match the one constructed in [2] in the case where S is birational to a minimal del Pezzo surface. Based on results of Karpov and Nogin [11], the authors have shown uniqueness of semiorthogonal decompositions for such del Pezzo surfaces. We are unfortunately not able to show this result for conic bundles, and the definition has to be more *ad hoc.* We start by recalling [2, Def. 1].

Definition 11. Let S be a minimal del Pezzo surface over k, and $\mathcal{A}_S = \langle \mathcal{O}_S \rangle^{\perp}$. We define the *Griffiths–Kuznetsov component* GK_S of S as follows: if $\mathcal{A}_S$ is representable in dimension 0, set $\mathsf{GK}_S = 0$. If not, GK_S is either the product of all indecomposable components of $\mathcal{A}_S$ of the form $\mathsf{D}^{\mathrm{b}}(l,\alpha)$ with l/k a field extension and $\alpha \in \mathrm{Br}(l)$ nontrivial or, if $\mathcal{A}_S$ does not admit such subcategories, $\mathsf{GK}_S = \mathcal{A}_S$.

If S is not minimal, then we set $\mathsf{GK}_S = \mathsf{GK}_{S'}$ for a minimal model $S \to S'$.

Note that the term component is slightly abused here, since there are cases where GK_S is not a component of $\mathsf{D}^{\mathrm{b}}(S)$ but rather the direct sum of some components. Since we can operate mutations on semiorthogonal decompositions (and we will indeed do so to prove the main theorem), we cannot in general give any canonical gluing of components contributing to GK_S. The same abuse of terminology will appear in Definition 12.

One of the main results of [2] is that GK_S is well defined, unless S is non-rational and has either degree 8 and a point of degree 2, or degree 4 and a rational point. In this case, S is birational to a minimal conic bundle (by blowing up the given point), so these cases will be included in Definition 12. Let us analyze them first.

Let S' be a minimal non-rational del Pezzo surface of degree 8 with a point of degree 2. Then (see, e.g., [2, §7]), S' is an involution surface in a Severi-Brauer threefold $SB(B)$, and there is an associated even Clifford algebra $\mathcal{C}$, which is a simple algebra whose center is a degree two field extension of k, and a semiorthogonal decomposition

$$\mathsf{D}^{\mathrm{b}}(S') = \langle \mathsf{D}^{\mathrm{b}}(k), \mathsf{D}^{\mathrm{b}}(k,B), \mathsf{D}^{\mathrm{b}}(k,\mathcal{C})\rangle,$$

where the first category is generated by $\mathcal{O}_{S'}$ and the second one either by $\mathcal{O}_{S'}(1)$ (in the case $B=0$ and S' is a quadric) or by the restriction of the Quillen bundle of $SB(B)$ to S' (in the case where S' is not a quadric). It follows that the Griffiths–Kuznetsov component for S' should be $\mathsf{GK}_{S'} := \mathsf{D}^{\mathrm{b}}(k,\mathcal{C})$ if S' is a quadric and $\mathsf{GK}_{S'} := \mathsf{D}^{\mathrm{b}}(k,\mathcal{C}) \oplus \mathsf{D}^{\mathrm{b}}(k,B)$ if S' is not a quadric. In this case, such a category is not shown to be a birational invariant in [2], and this is due to the existence of a link of type I, from which follows that the birational class of S' contains minimal conic bundles of degree 6. Indeed, the blow-up of a degree 2 point $S \to S'$ is a conic bundle $\pi : S \to C$, with C either rational if S' is a quadric or non-rational if S' is not a quadric.

In [2, §B], it is proved that the component we want to construct is indeed related to the standard semiorthogonal decomposition of the conic bundle as follows: writing $C = SB(A)$ we have that A and B are Morita-equivalent (note that B has order dividing 2 since it has an involution defining S'), and that there is a degree 2 extension l/k and a semiorthogonal decomposition:

$$\mathsf{D}^{\mathrm{b}}(C,\mathcal{B}) = \langle \mathsf{D}^{\mathrm{b}}(l), \mathsf{D}^{\mathrm{b}}(k,\mathcal{C})\rangle,$$

where we keep the previous notations for the Clifford algebra associated to the conic bundle $S \to C$. It follows that the components which are (potentially) not representable in dimension 0 in the standard decomposition (3.1) are exactly $\mathsf{D}^{\mathrm{b}}(k,\mathcal{C})$ and $\mathsf{D}^{\mathrm{b}}(k,B)$ as suggested by Definition 11 for the del Pezzo surface S' of degree 8.

Let S' be a minimal del Pezzo surface of degree 4 with a rational point. Note that S' is not rational. In particular, there is a semiorthogonal decomposition

$$\mathsf{D}^{\mathrm{b}}(S') = \langle \mathcal{O}_{S'}, \mathcal{A}_{S'} \rangle,$$

and $\mathcal{A}_{S'}$ (or rather, its non-zero-dimensional component) is expected to be the good candidate for the birational invariant we are looking for. This case neither was treated in [2], since, again, the existence of a link of type I implies that the birational class of S' contains minimal conic bundles of degree 3. Indeed, the blow-up of a rational point $S \to S'$ is a conic bundle $\pi : S \to \mathbb{P}^1$.

In [2, A.2] (see also [3]), it is proved that the component we want to construct is indeed related to the standard semiorthogonal decomposition of the conic bundle since there is an equivalence $\mathsf{D}^{\mathrm{b}}(\mathbb{P}^1, \mathcal{B}) \simeq \mathcal{A}_{S'}$, where we keep the previous notations for the Clifford algebra associated to the conic bundle $S \to \mathbb{P}^1$. It follows that the components which are (potentially) not representable in dimension 0 in the standard decomposition (3.1) are the ones suggested by Definition 11 for the del Pezzo surface S' of degree 4.

Definition 12. Let S be a surface admitting a structure of conic bundle $\pi : S \to C$ over a geometrically rational smooth curve C. If S is minimal, the Griffiths–Kuznetsov component GK_S of S is defined as follows:

(i) if S is rational, $\mathsf{GK}_S = 0$;
(ii) if $S = C_1 \times C_2$ where C_i is a geometrically rational curve with associated Azumaya algebra A_i, then GK_S is the sum of those between $\mathsf{D}^{\mathrm{b}}(k, A_1)$, $\mathsf{D}^{\mathrm{b}}(k, A_2)$ and $\mathsf{D}^{\mathrm{b}}(k, A_1 \otimes A_2)$ which are not equivalent to $\mathsf{D}^{\mathrm{b}}(k)$ (equivalently, the algebra is not Brauer-trivial);
(iii) if $C \simeq \mathbb{P}^1$ and S is birational to a non-rational quadric with associated even Clifford algebra $\mathcal{C}$, then $\mathsf{GK}_S = \mathsf{D}^{\mathrm{b}}(k, \mathcal{C})$;
(iv) if $C \simeq \mathbb{P}^1$, and S is neither rational nor birational to a quadric, $\mathsf{GK}_S = \mathsf{D}^{\mathrm{b}}(\mathbb{P}^1, \mathcal{B})$;
(v) if C is irrational with associated Azumaya algebra A, and S is birational to a quadric with associated even Clifford algebra $\mathcal{C}$, then $\mathsf{GK}_S = \mathsf{D}^{\mathrm{b}}(k, A) \oplus \mathsf{D}^{\mathrm{b}}(k, \mathcal{C})$;
(vi) if C is irrational with associated Azumaya algebra A, and S is not birational to a quadric, then $\mathsf{GK}_S = \mathsf{D}^{\mathrm{b}}(k, A) \oplus \mathsf{D}^{\mathrm{b}}(C, \mathcal{B})$;

If S is not minimal, the Griffiths–Kuznetsov component is $\mathsf{GK}_S := \mathsf{GK}_{S'}$ for a minimal model $S \to S'$.

The rest of the paper is dedicated to the proof of Theorem 1. Note that we can restrict to minimal models. If $\pi : S \to C$ and $\pi' : S' \to C'$ be are minimal conic bundle structures and $\phi : S \dashrightarrow S'$ is a birational morphism, then ϕ can be decomposed in a finite number of links. By the discussion above on links of type I/III, we are left to prove the invariance under links of type II (Theorem 13) and type IV (Corollary 16), in the non rational cases.

5. Links of type II

Links of type II between conic bundles are the most common birational transformation. They are given by an elementary transformation along a closed fiber of the conic bundle structure. Let $\pi : S \to C$ be a conic bundle. To define a link of type II, pick a closed point $x \in S$ of degree d such that the geometric points $\{x_i\}_{i=1}^{d} = x \otimes \overline{k}$ are in general position on $\overline{S}$, and denote by S_x the fiber of π containing x. Then perform the blow-up of S in x followed by the subsequent contraction of the fiber S_x. This gives a conic bundle $\pi' : S' \to C$ and a commutative diagram:

$$\begin{tikzcd}
& Z \arrow[dl, "p"'] \arrow[dr, "q"] & \\
S \arrow[rr, dashed, "\phi"] \arrow[d, "\pi"'] & & S' \arrow[d, "\pi'"] \\
C \arrow[rr, "\mathrm{id}"] & & C
\end{tikzcd}$$

Theorem 13. *In the above setting, we have* $\mathsf{GK}_S \simeq \mathsf{GK}_{S'}$.

Proof. First note that S and S' can be of any type except (i) from Definition 12. However, if the degree K_S^2 (which is constant under the link) is 8, the invariance has been proven in [2, §C]. This rules out all of the case (ii). In the other cases, note that it will be sufficient to show two equivalences: between the categories $\langle V \rangle$ and $\langle V' \rangle$ generated by the pull-back of the Quillen bundle from C (which is redundant if $C \simeq \mathbb{P}^1$, cases (iii) and (iv)), and between the categories $\mathsf{D}^{\mathrm{b}}(C, \mathcal{B})$ and $\mathsf{D}^{\mathrm{b}}(C, \mathcal{B}')$ of sheaves over the Clifford algebras associated to the conic bundle structures. The latter is evident in cases (iv) and (vi), and follows from the discussion preceeding definition 12 for in cases (iii) and (v).

We proceed to show the above equivalences for links of type II. Let E be the exceptional divisor of p and E' the exceptional divisor of q. We denote by f (resp. f') the pullback of the generic fiber of π (resp. π') via p (resp. q) in Z. Recall [10] that

$$\begin{aligned} p^*(-K_S) &= q^*(-K_{S'}) + df - 2E' \\ f &= f' \\ E &= df' - E' = p^*(-K_S) - q^*(-K_{S'}) + E', \end{aligned} \tag{5.1}$$

up to linear equivalence. Since the isomorphism class of C is preserved under this link, so is the algebra A (which is trivial if and only if $C \simeq \mathbb{P}^1$), and therefore the category $\mathsf{D}^{\mathrm{b}}(k, A)$.

It is left to prove that the equivalence class of the category $\mathcal{A}_S := \mathsf{D}^{\mathrm{b}}(C, \mathcal{B})$ is also preserved. Over the algebraic closure, the category $\mathcal{A}_{\overline{S}} \subset \mathsf{D}^{\mathrm{b}}(\overline{S})$ admits the following semiorthogonal decomposition:

$$\mathcal{A}_{\overline{S}} = \langle \mathcal{O}_{\overline{S}}(\Sigma), \mathcal{O}_{\overline{S}}(\Sigma + F), \mathcal{O}_{F_1}, \ldots, \mathcal{O}_{F_r} \rangle, \tag{5.2}$$

as in (3.2). Note that we use here F to denote the class of the general fiber of $\overline{S}$, while over k we used the notation f as in (5.1). This is because, if C is not rational, then $\overline{f} \neq F$, but rather $\overline{f} = 2F$. We have a semiorthogonal decomposition

$$\mathcal{A}_{\overline{S'}} = \langle \mathcal{O}_{\overline{S'}}(\Sigma'), \mathcal{O}_{\overline{S'}}(\Sigma' + F'), \mathcal{O}_{F'_1}, \dots, \mathcal{O}_{F'_r} \rangle. \tag{5.3}$$

Over $\overline{k}$, we have that $\overline{S}$ is the blow up of r points on a Hirzebruch surface $\mathbb{F}_n$. If we denote by $D_1, \dots, D_r$ the exceptional divisors of such blow up, and by $D = \sum_{i=1}^r D_i$, we have that $-K_{\overline{S}} = 2\Sigma + (n+2)F - D$. Similarly, $\overline{S'}$ is the blow-up of r points on a Hirzebruch surface $\mathbb{F}_m$, with exceptional divisors $D'_1, \dots, D'_r$, and we use the notation $D' = \sum_{i=1}^r D'_i$, so that $-K_{\overline{S'}} = 2\Sigma' + (m+2)F' - D'$. By abuse of notations, we will drop p^* (resp. q^*) when dealing with divisors on $\overline{Z}$ that are pull-back of divisors of $\overline{S}$ (resp. $\overline{S}'$). For example, Σ will denote $p^*\Sigma$ and Σ' will denote $q^*\Sigma'$ and so on.

One can see that the map $\overline{\phi}$ is obtained by lifting to $\overline{S}$ the composition of d elementary transformations on $\mathbb{F}_n$, in particular $m = n - d$, along fibers that do not contain the divisors D_i. It follows that the divisors D_i are preserved by $\overline{\phi}$ and so D is sent to D'. We conclude with the following equalities of (rational equivalence classes of) divisors in $\overline{Z}$:

$$D = D', \quad F = F'.$$

Using the above equalities and the above explicit descriptions of the canonical bundles, the first relation in (5.1) gives $\Sigma = \Sigma' - E'$ as (rational equivalence classes of) divisors in $\overline{Z}$.

Then we have the following equivalence of subcategories of $\mathsf{D}^{\mathrm{b}}(\overline{Z})$:

$$p^*\mathcal{A}_{\overline{S}} \otimes \mathcal{O}(E') = q^*\mathcal{A}_{\overline{S'}}.$$

Indeed, first note the singular fibers are preserved under the birational transformation ϕ, and we can choose F'_i such that $p^*F_i = q^*F'_i$ for $i = 1, \dots, r$. Moreover $\mathcal{O}_{q^*F_i}$ does not change under tensoring with $\mathcal{O}(E')$, since the exceptional divisor is not supported on singular fibers. Secondly, using the above relation $\Sigma = \Sigma' - E'$, it is not difficult to see that

$$\langle \mathcal{O}(\Sigma), \mathcal{O}(\Sigma + F) \rangle \otimes \mathcal{O}(E') = \langle \mathcal{O}(\Sigma'), \mathcal{O}(\Sigma' + F') \rangle$$

We can now conclude since the autoequivalence $\otimes\mathcal{O}(E')$ descends to an autoequivalence of $\mathsf{D}^{\mathrm{b}}(Z)$, since E' is defined over k. It follows, that $\mathcal{A}_S$ is equivalent to $\mathcal{A}_{S'}$. □

6. Links of type IV

A link of type IV is a birational self-transformation of a minimal irrational surface S exchanging two conic bundle structures $\pi_i : S \to C_i$, for $i = 1, 2$. In particular, the birational map $\phi : S \dashrightarrow S$ fits a commutative diagram

$$\begin{tikzcd}
S \arrow[rr, dashed, "\phi"] \arrow[d, "\pi_1"'] & & S \arrow[d, "\pi_2"] \\
C_1 \arrow[dr] & & C_2 \arrow[dl] \\
& \mathrm{Spec}(k) &
\end{tikzcd}$$

For $i = 1, 2$, we will denote by V_i the Quillen bundle of C_i and by A_i the associated Azumaya algebra, and by $\mathcal{B}_i$ the even Clifford algebra of the conic bundle π_i and by $\Phi_i : \mathsf{D}^{\mathrm{b}}(C_i, \mathcal{B}_i) \to \mathsf{D}^{\mathrm{b}}(S)$ the functor described in [12] giving the semiorthogonal decomposition (3.1). We will also use the shorter notation $\mathcal{A}_i := \Phi_i \mathsf{D}^{\mathrm{b}}(C_i, \mathcal{B}_i) \subset \mathsf{D}^{\mathrm{b}}(S)$. As proved in [10], the degree K_S^2 of S must be 8, 4, 2 or 1, and the list of birational maps is quite limited. In this case, we need to prove that the Griffiths–Kuznetsov component is well defined, namely that it does not depend on the choice of semiorthogonal decomposition given by the different conic bundle structures. We proceed by a case by case analysis.

Proposition 14. *In the situation above, let S have degree 8 and ϕ a link of type IV. Then GK_S is invariant under ϕ.*

Proof. In this case, $S = C_1 \times C_2$, and we are in case (ii) of Definition 12. The fact that GK_S is well defined is proved in [2, §C]. □

Proposition 15. *In the situation above, let S have degree 4, 2, or 1, and ϕ a link of type IV. Then GK_S is invariant under ϕ.*

Proof. First note that such an S cannot be birational to a quadric: indeed, the only type I/III link relating a non-rational quadric to a conic bundle ends up in a degree 6 conic bundle, and links of type II do not change the degree. Similarly, S is not birational to a del Pezzo of degree four with a rational point, and hence S is not birational to any minimal del Pezzo surface. In particular, we are either in case (iv) (if $C_1 \simeq C_2 \simeq \mathbb{P}^1$) or in case (vi) (if C_1, C_2 are not rational) of Definition 12.

For $i = 1, 2$, consider the semiorthogonal decomposition

$$\mathsf{D}^{\mathrm{b}}(S) = \langle \mathcal{O}_S, V_i, \mathcal{A}_i \rangle, \tag{6.1}$$

and recall that $\langle \overline{V_i} \rangle = \langle \mathcal{O}_{\overline{S}}(F_i) \rangle$, for F_i a geometric fiber of π_i. Indeed, either we are in case *(iv)*, C_i is rational and $V_i = \mathcal{O}_S(F_i)$ is already a line bundle (and does not contribute to the Griffiths–Kuznetsov components), or we are in case *(vi)* C_i is not rational and $\overline{V_i} = \mathcal{O}_{\overline{S}}(F_i)^{\oplus 2}$. Now we proceed by case by case analysis following the possibilities given by [10].

Degree 4. Assume S has degree 4. We have $F_1 = -K_S - F_2$ (see [10, Page 611]), so that $V_1 = V_2^* \otimes \omega_S^*$. It follows that $\langle V_1 \rangle \simeq \langle V_2^* \rangle \simeq \langle V_2 \rangle$, first via the autoequivalence

$\otimes\omega_S^*$ and secondly since A_2^{op} and A_2 are Brauer equivalent. It follows that A_1 and A_2 are Brauer equivalent.

It is left to prove that $\mathcal{A}_1 \simeq \mathcal{A}_2$. To this end, consider the semiorthogonal decompositions:

$$\mathsf{D}^{\mathrm{b}}(S) = \langle V_2^*, \mathcal{O}_S, \mathcal{A}_2\rangle = \langle V_1, \omega_S^*, \mathcal{A}_2 \otimes \omega_S^*\rangle = \langle \mathcal{O}_S, \mathcal{A}_2, V_1\rangle,$$

where the second equality is given by the autoequivalence $\otimes\omega_S^*$ on $\mathsf{D}^{\mathrm{b}}(S)$ and the third is the mutation of $\langle \omega_S^*, \mathcal{A}_2 \otimes \omega_S^*\rangle$ to the left with respect to V_1. We can then mutate $\mathcal{A}_2$ to the right with respect to V_1 and obtain the right orthogonal complement of $\langle \mathcal{O}_S, V_1\rangle$ which is $\mathcal{A}_1$, as in (6.1) for $i = 1$. This last mutation gives then the required equivalence between $\mathcal{A}_2$ and $\mathcal{A}_1$.

Degree 2. Assume S has degree 2. We have $F_1 = -2K_S - F_2$ (see [10, Page 611]), so that $V_1 = V_2^* \otimes (\omega^*)^{\otimes 2}$. It follows that $\langle V_1\rangle \simeq \langle V_2^*\rangle \simeq \langle V_2\rangle$, first via the autoequivalence $\otimes(\omega_S^*)^{\otimes 2}$ and secondly since A_2^{op} and A_2 are Brauer equivalent. It follows that A_1 and A_2 are Brauer equivalent.

It is left to prove that $\mathcal{A}_1 \simeq \mathcal{A}_2$. To this end, consider the semiorthogonal decompositions:

$$\mathsf{D}^{\mathrm{b}}(S) = \langle \mathcal{O}_S, V_1, \mathcal{A}_1\rangle = \langle V_1, \mathcal{A}_1, \omega_S^*\rangle = \langle \mathcal{A}_1', V_1, \omega_S^*\rangle,$$

where the second equality is the mutation of $\mathcal{O}_S$ to the right with respect to its orthogonal complement and the third one is the mutation of $\mathcal{A}_1$ to the left with respect to V_1, so that $\mathcal{A}_1' = {}^{\perp}\langle V_1, \omega_S^*\rangle$ is equivalent to $\mathcal{A}_1$.

Now consider the second conic bundle structure and the semiorthogonal decompositions

$$\begin{aligned}\mathsf{D}^{\mathrm{b}}(S) &= \langle V_2^*, \mathcal{O}_S, \mathcal{A}_2\rangle = \langle V_1, (\omega_S^*)^{\otimes 2}, \mathcal{A}_2 \otimes (\omega_S^*)^{\otimes 2}\rangle \\ &= \langle \omega_S^*, \mathcal{A}_2 \otimes (\omega_S^*), V_1\rangle = \langle \mathcal{A}_2', \omega_S^*, V_1\rangle,\end{aligned}$$

where we first tensor with $(\omega_S^*)^{\otimes 2}$, then mutate $\langle (\omega_S^*)^{\otimes 2}, \mathcal{A}_2 \otimes (\omega_S^*)^{\otimes 2}\rangle$ to the left with respect to its orthogonal complement, then mutate $\mathcal{A}_2 \otimes \omega_S^*$ to the left with respect to ω_S^*. It follows in particular that $\mathcal{A}_2' = {}^{\perp}\langle \omega_S^*, V_1\rangle$ is equivalent to $\mathcal{A}_2$.

Finally, the two semiorthogonal decompositions give the full orthogonality between V_1 and ω_S^*, so that $\langle \omega_S^*, V_1\rangle = \langle V_1, \omega_S^*\rangle$. This implies that $\mathcal{A}_1' = \mathcal{A}_2'$ and the proof follows.

Degree 1. Assume S has degree 1. Then $F_1 = -4K_S - F_2$ and C_i are rational (see [10, Page 611]), so that $V_i = \mathcal{O}_S(F_i)$ are k-exceptional and we are in case *(iv)* of Definition 12. In particular, we only need to prove that the categories $\mathcal{A}_1$ and $\mathcal{A}_2$ are equivalent.

Let us consider the first conic bundle structure and the semiorthogonal decompositions:

$$\begin{aligned}\mathsf{D}^{\mathrm{b}}(S) &= \langle \mathcal{O}(-F_1), \mathcal{O}_S, \mathcal{A}_1\rangle = \langle \mathcal{O}(F_2), (\omega_S^*)^{\otimes 4}, \mathcal{A}_1 \otimes (\omega_S^*)^{\otimes 4}\rangle \\ &= \langle (\omega_S^*)^{\otimes 3}, \mathcal{A}_1 \otimes (\omega_S^*)^{\otimes 3}, \mathcal{O}(F_2)\rangle = \langle (\omega_S^*)^{\otimes 3}, \mathcal{O}(F_2), \mathcal{A}_1'\rangle,\end{aligned}$$

where we first tensor by $(\omega_S^*)^{\otimes 4}$, then mutate $\langle (\omega_S^*)^{\otimes 4}, \mathcal{A}_1 \otimes (\omega_S^*)^{\otimes 4}\rangle$ to the left with respect to its orthogonal complement, then mutate $\mathcal{A}_1 \otimes (\omega_S^*)^{\otimes 3}$ to the right with respect to $\mathcal{O}(F_2)$. The category $\mathcal{A}_1'$ is the result of this last mutation and is then equivalent to $\mathcal{A}_1$.

Now we need to mutate $\mathcal{O}(F_2)$ to the left with respect to $(\omega_S^*)^{\otimes 3}$. To this end, let us calculate :

$$\mathrm{Hom}^i((\omega_S^*)^{\otimes 3}, \mathcal{O}(F_2)) = H^i(S, \mathcal{O}(3K_S + F_2)).$$

First of all, note that $(3K_S+F_2).F_2 < 0$, which implies that $H^0(S, \mathcal{O}(3K_S + F_2)) = 0$. Similarly, by Serre duality we have that ⟸ disp

$$H^2(S, \mathcal{O}(3K_S + F_2)) = H^0(S, \mathcal{O}(2K_S + F_1)) = 0$$

since $(2K_S + F_1).F_1 < 0$.

Finally we are left with $\dim H^1(S, \mathcal{O}(3K_S + F_2)) = -\chi(\mathcal{O}_S, \mathcal{O}(3K_S + F_2))$. The latter can be calculated by Riemann-Roch:

$$\chi(\mathcal{O}_S, \mathcal{O}(3K_S + F_2)) = \frac{1}{2}(3K_S + F_2) \cdot (2K_S + F_2) + 1 = -1,$$

since $K_S \cdot F_2 = -2$ and S has degree 1. It follows that there is a unique extension

$$0 \longrightarrow \mathcal{O}(F_2) \longrightarrow \mathcal{F} \longrightarrow (\omega_S^*)^{\otimes 3} \longrightarrow 0,$$

which has rank 2 and first Chern class $F_2 - 3K_S$. Moreover, $\mathcal{F}$ is the result of the mutation of $\mathcal{O}(F_2)$ to the left with respect to $(\omega_S^*)^{\otimes 3}$, so that we end up with the decomposition

$$\mathsf{D}^{\mathrm{b}}(S) = \langle \mathcal{F}, (\omega_S^*)^{\otimes 3}, \mathcal{A}_1' \rangle. \tag{6.2}$$

Now consider the second conic bundle structure and the semiorthogonal decompositions:

$$\mathsf{D}^{\mathrm{b}}(S) = \langle \mathcal{O}_S, \mathcal{O}(F_2), \mathcal{A}_2 \rangle = \langle \mathcal{O}(F_2), \mathcal{A}_2, \omega_S^* \rangle = \langle \mathcal{O}(F_2), \omega_S^*, \mathcal{A}_2' \rangle,$$

where the first equality is the mutation of $\mathcal{O}_S$ to the right with respect to its right orthogonal, and $\mathcal{A}_2'$ is the mutation of $\mathcal{A}_2$ to the left with respect to ω_S^* and hence equivalent to $\mathcal{A}_2$.

We mutate now $\mathcal{O}(F_2)$ to the right with respect to ω_S^*. A calculation similar to the above one shows that there is exactly one nontrivial extension

$$0 \longrightarrow \omega_S^* \longrightarrow \mathcal{G} \longrightarrow \mathcal{O}(F_2) \longrightarrow 0,$$

which has rank 2 and first Chern class $F_2 - K_S$. Moreover, $\mathcal{G}$ is the result of the mutation of $\mathcal{O}(F_2)$ to the right with respect to ω_S^*, and is an exceptional object. Thanks to Gorodentsev [8], we know that exceptional bundles on S are characterized by their rank and their first Chern class. Note that $\mathcal{F}$ and $\mathcal{G}$ both have rank 2, while the first Chern class of $\mathcal{G}$ is the first Chern class of $\mathcal{F} \otimes \omega_S$. It follows that $\mathcal{G} \simeq \mathcal{F} \otimes \omega_S$ is the mutation of $\mathcal{O}(F_2)$ to the right with respect to ω_S^*. We hence end up with the decompositions

$$\begin{aligned} \mathsf{D}^{\mathrm{b}}(S) &= \langle \omega_S^*, \mathcal{F} \otimes \omega_S, \mathcal{A}_2' \rangle = \langle (\omega_S^*)^{\otimes 2}, \mathcal{F}, \mathcal{A}_2' \otimes \omega_S^* \rangle \\ &= \langle \mathcal{F}, \mathcal{A}_2' \otimes \omega_S^*, (\omega_S^*)^{\otimes 3} \rangle = \langle \mathcal{F}, (\omega_S^*)^{\otimes 3}, \mathcal{A}_2'' \rangle, \end{aligned}$$

where first we tensor by ω_S^*, then mutate $(\omega_S^*)^{\otimes 2}$ to the right with respect to its right orthogonal, and $\mathcal{A}_2''$ is the left mutation of $\mathcal{A}_2' \otimes (\omega_S^*)^{\otimes 2}$ to the right with respect to $(\omega_S^*)^{\otimes 3}$ and is therefore equivalent to $\mathcal{A}_2$. The proof follows then by comparison with 6.2. ☐

Corollary 16. *The Griffiths–Kuznetsov component is invariant under links of type IV and hence well defined for minimal conic bundles.*

Acknowledgment

The authors are grateful to Stéphane Lamy for providing and discussing the example in Remark 4, and to Jérémy Blanc and Michele Bolognesi for fruitful conversations. We thank moreover the anonymous referee for his/her careful work and advices which helped us improve the exposition.

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