Chapter 3

Selfadjoint operators

Let $\mathcal{H}$ be a Hilbert space.

3.1 Selfadjoint operators

3.1.1 Symmetric operators

Definition 3.1. *Let A be an operator on $\mathcal{H}$. We say that A is symmetric if*

$$\forall \varphi, \psi \in \mathsf{Dom}(A), \quad \langle A\varphi, \psi \rangle_{\mathcal{H}} = \langle \varphi, A\psi \rangle_{\mathcal{H}} .$$

✎ *Ex. 3.1*

Remark 3.2. If A is symmetric then $\langle A\varphi, \varphi \rangle_{\mathcal{H}} \in \mathbb{R}$ for all $\varphi \in \mathsf{Dom}(A)$. The converse is also true, as can be seen from the polarization formula

$$\begin{aligned} \forall \varphi, \psi \in \mathsf{Dom}(A), \quad 4\langle A\varphi, \psi \rangle &= \langle A(\varphi + \psi), \varphi + \psi \rangle - \langle A(\varphi - \psi), \varphi - \psi \rangle \\ &\quad + i\langle A(\varphi + i\psi), \varphi + i\psi \rangle - i\langle A(\varphi - i\psi), \varphi - i\psi \rangle . \end{aligned}$$

Definition 3.3. *Let A be a symmetric operator on $\mathcal{H}$.*

(i) *We say that A is non-negative (and we write $A \geqslant 0$) if $\langle A\varphi, \varphi \rangle_{\mathcal{H}} \geqslant 0$ for all $\varphi \in \mathsf{Dom}(A)$.*

(ii) *We say that A is semi-bounded from below if there exists $\gamma \in \mathbb{R}$ such that $A - \gamma \geqslant 0$ (we can write $A \geqslant \gamma$). Equivalently, $\langle A\varphi, \varphi \rangle_{\mathcal{H}} \geqslant \gamma \|\varphi\|_{\mathcal{H}}^2$ for all $\varphi \in \mathsf{Dom}(A)$. In this case we say that γ is a lower bound for A.*

Proposition 3.4. *Let A be a symmetric and densely defined operator on $\mathcal{H}$. Then A^* is a closed extension of A.*

Proof. Let $\psi \in \mathsf{Dom}(A)$. For all $\varphi \in \mathsf{Dom}(A)$ we have $\langle A\varphi, \psi \rangle = \langle \varphi, A\psi \rangle$ so $\psi \in \mathsf{Dom}(A^*)$ and $A^*\varphi = A\varphi$. This proves that A^* is an extension of A. Moreover A^* is closed by Proposition 1.60. □

Proposition 3.5. *Let A be a symmetric operator on $\mathcal{H}$. The eigenvalues of A (if any) are real, and two eigenvectors of A associated to different eigenvalues are orthogonal.*

Proof. • Let $\lambda \in \mathbb{C}$ and assume that for some $\varphi \in \mathsf{Dom}(A) \setminus \{0\}$ we have $A\varphi = \lambda\varphi$. Then $\lambda \|\varphi\|_{\mathcal{H}}^2 = \langle A\varphi, \varphi \rangle_{\mathcal{H}} \in \mathbb{R}$. This implies that $\lambda \in \mathbb{R}$.
• Now let λ, μ be two distinct eigenvalues of A. Let $\varphi \in \ker(A - \lambda)$ and $\psi \in \ker(A - \mu)$. Then

$$(\mu - \lambda)\langle \psi, \varphi \rangle_{\mathcal{H}} = \langle \mu\psi, \varphi \rangle_{\mathcal{H}} - \langle \psi, \lambda\varphi \rangle_{\mathcal{H}} = \langle A\psi, \varphi \rangle_{\mathcal{H}} - \langle \psi, A\varphi \rangle_{\mathcal{H}} = 0.$$

Since $\mu - \lambda \neq 0$, this implies that $\langle \psi, \varphi \rangle_{\mathcal{H}} = 0$. □

Proposition 3.6. *Let A be a non-negative and densely defined operator. Let $\varphi \in \mathsf{Dom}(A)$ such that $\langle A\varphi, \varphi \rangle_{\mathcal{H}} = 0$. Then $A\varphi = 0$.*

Proof. Since A is non-negative, we can apply the Cauchy-Schwarz inequality to the sesquilinear form $(\zeta,\psi) \mapsto \langle A\zeta,\psi\rangle_{\mathcal{H}}$ on $\mathsf{Dom}(A)$. Then for all $\psi \in \mathsf{Dom}(A)$ we have

$$|\langle A\varphi,\psi\rangle_{\mathcal{H}}| \leqslant |\langle A\varphi,\varphi\rangle_{\mathcal{H}}|\,|\langle A\psi,\psi\rangle_{\mathcal{H}}| = 0.$$

Since $\mathsf{Dom}(A)$ is dense in $\mathcal{H}$, this proves that $A\varphi = 0$. □

Proposition 3.7. *Let A be a symmetric operator on $\mathcal{H}$.*

(i) *For $z \in \mathbb{C}\backslash\mathbb{R}$ and $\varphi \in \mathsf{Dom}(A)$ we have*

$$\|(A-z)\varphi\|_{\mathcal{H}} \geqslant |\mathrm{Im}(z)|\,\|\varphi\|_{\mathcal{H}}\,.$$

(ii) *Assume moreover that $A \geqslant \gamma$ for some $\gamma \in \mathbb{R}$. Then for $\lambda < \gamma$ and $\varphi \in \mathsf{Dom}(A)$ we have*

$$\|(A-z)\varphi\|_{\mathcal{H}} \geqslant (\gamma-\lambda)\,\|\varphi\|_{\mathcal{H}}\,.$$

Proof. Let $\varphi \in \mathsf{Dom}(A)$.

- Let $z \in \mathbb{C}\backslash\mathbb{R}$, $\lambda = \mathrm{Re}(z)$ and $\varepsilon = \mathrm{Im}(z)$. We have

$$\|(A-z)\varphi\|^2 = \|(A-\lambda)\varphi\|^2 + \varepsilon^2\,\|\varphi\|^2 + 2\,\mathrm{Re}\,\langle (A-\lambda)\varphi, -i\varepsilon\varphi\rangle\,.$$

Since

$$\langle (A-\lambda)\varphi, -i\varepsilon\varphi\rangle = i\varepsilon\,\langle A\varphi,\varphi\rangle - i\varepsilon\lambda\,\|\varphi\|^2 \in i\mathbb{R},$$

this gives

$$\|(A-z)\varphi\|^2 = \|(A-\lambda)\varphi\|^2 + \varepsilon^2\,\|\varphi\|^2 \geqslant \varepsilon^2\,\|\varphi\|^2\,.$$ □

- Similarly, if $A-\gamma \geqslant 0$ then for $\lambda \in]-\infty,\gamma[$ we have

$$\begin{aligned}\|(A-\lambda)\varphi\|_{\mathcal{H}}^2 &= \|(A-\gamma)\varphi\|_{\mathcal{H}}^2 + (\gamma-\lambda)^2\,\|\varphi\|_{\mathcal{H}}^2 + 2(\gamma-\lambda)\,\mathrm{Re}\,\langle (A-\gamma)\varphi,\varphi\rangle_{\mathcal{H}}\\ &\geqslant (\gamma-\lambda)^2\,\|\varphi\|_{\mathcal{H}}^2\,,\end{aligned}$$

and the second statement follows.

3.1.2 Selfadjoint operators

Definition 3.8. *An operator A on $\mathcal{H}$ is said to be selfadjoint if it is densely defined and $A^* = A$.* *Ex. 3.2-3.3*

Definition 3.9. *An operator A on $\mathcal{H}$ is said to be skew-adjoint if it is densely defined and $A^* = -A$.*

Remark 3.10. An operator A is skew-adjoint if and only if iA is selfadjoint. Thus we only discuss the properties of selfadjoint operators, and we can deduce similar properties for skew-adjoint operators.

Remark 3.11. A bounded and symmetric operator is selfadjoint.

Example 3.12.

- The Laplacian $H = -\Delta$ on $L^2(\mathbb{R}^d)$ (with domain $\mathsf{Dom}(H) = H^2(\mathbb{R}^d)$) is selfadjoint. The Laplacian $H_0 = -\Delta$ with domain $C_0^\infty(\mathbb{R}^d)$ is symmetric but not selfadjoint (in particular $C_0^\infty(\mathbb{R}^d) \subsetneq H^2(\mathbb{R}^d) \subset \mathsf{Dom}(H_0^*)$). However, H_0 has a selfadjoint extension (H).

Example 3.13. The Dirichlet and Neumann Laplacians on $]0,1[$ (introduced in Section 1.3.3) are selfadjoint.

Example 3.14. The harmonic oscillator introduced in Section 2.1.2 is selfadjoint.

Remark 3.15. A selfadjoint operator is closed by Proposition 1.59.

Proposition 3.16. *Let A be a selfadjoint operator on $\mathcal{H}$. Then*

$$\overline{\mathsf{Ran}(A)} = \ker(A)^\perp.$$

Definition 3.17. *Let A be a selfadjoint operator on $\mathcal{H}$. Let F be a subspace of $\mathcal{H}$. We say that F is reducing for the operator A (or that it reduces A) if it is closed and the orthogonal projection Π on F satisfies $\Pi A \subset A\Pi$.*

Remark 3.18. If F reduces A, then so does $\mathsf{F}^\perp$.

Proposition 3.19. *Let A be a selfadjoint operator on $\mathcal{H}$. Let F be a reducing subspace for A. Then the restriction A_F of A on F is a selfadjoint operator on F.*

Proof. For all $\varphi, \psi \in \mathsf{Dom}(A_\mathsf{F}) = \mathsf{Dom}(A) \cap \mathsf{F}$ we have

$$\langle A_\mathsf{F}\varphi, \psi\rangle = \langle A\varphi, \psi\rangle = \langle \varphi, A\psi\rangle = \langle \varphi, A_\mathsf{F}\psi\rangle,$$

so A_F is symmetric. Let $\psi \in \mathsf{Dom}(A_\mathsf{F}^*) \subset \mathsf{F}$. Let $\varphi \in \mathsf{Dom}(A)$. We write $\varphi = \varphi_\mathsf{F} + \varphi^\perp$ with $\varphi_\mathsf{F} \in \mathsf{Dom}(A) \cap \mathsf{F} = \mathsf{Dom}(A_\mathsf{F})$ and $\varphi^\perp \in \mathsf{Dom}(A) \cap \mathsf{F}^\perp$. Since $A\varphi^\perp \in \mathsf{F}^\perp \subset \langle\psi\rangle^\perp$, we have

$$\langle A\varphi, \psi\rangle_\mathcal{H} = \langle A_\mathsf{F}\varphi_\mathsf{F}, \psi\rangle_\mathsf{F} = \langle \varphi_F, A_\mathsf{F}^*\psi\rangle_\mathsf{F} = \langle \varphi, A_\mathsf{F}^*\psi\rangle_\mathcal{H}.$$

This proves that $\psi \in \mathsf{Dom}(A^*) = \mathsf{Dom}(A)$, so $\psi \in \mathsf{Dom}(A) \cap \mathsf{F} = \mathsf{Dom}(A_\mathsf{F})$, and $A_\mathsf{F}^*\psi = A^*\psi = A\psi = A_\mathsf{F}\psi$. This proves that $A_\mathsf{F}^* \subset A_\mathsf{F}$, and finally A_F is selfadjoint by Proposition 3.4. ☐

Proposition 3.20. *Let A be a selfadjoint operator on $\mathcal{H}$. Then $\ker(A)$ is reducing for A.*

Proof. Since A is closed, $\ker(A)$ is closed in $\mathcal{H}$. Let Π be the orthogonal projection on $\ker(A)$. For all $\varphi \in \mathsf{Dom}(A)$ we have $A\varphi \in \mathsf{Ran}(A) \subset \ker(A)^\perp$, so $\Pi A\varphi = 0$. On the other hand we have $\Pi\varphi \in \ker(A) \subset \mathsf{Dom}(A)$ and $A\Pi\varphi = 0$. This proves that $\Pi A \subset A\Pi$. ☐

Proposition 3.21. *If A and B are two selfadjoint operators on $\mathcal{H}$ such that $A \subset B$ then $A = B$.*

Proof. We have $A \subset B = B^* \subset A^* = A$, so $A = B$. ☐

3.1.3 A criterion for self-adjointness

Proposition 3.22. *Let A be a symmetric and densely defined operator on $\mathcal{H}$. Let $z \in \mathbb{C}\backslash\mathbb{R}$. Then the following assertions are equivalent.*

(i) *A is self-adjoint.*

(ii) *A is closed and $z, \overline{z} \in \rho(A)$.*

(iii) *A is closed and $\ker(A^* - z) = \ker(A^* - \overline{z}) = \{0\}$.*

(iv) *$\mathsf{Ran}(A - z) = \mathsf{Ran}(A - \overline{z}) = \mathcal{H}$.*

Proof. • $(i) \Rightarrow (iii)$. Assume that A is self-adjoint. In particular, A is closed. Moreover, $\ker(A^* - z) = \ker(A - z) = \{0\}$ by Proposition 3.7. Similarly, $\ker(A^* - \overline{z}) = \{0\}$.
• $(iii) \Rightarrow (iv)$. By Proposition 1.58 we have $\overline{\mathsf{Ran}(A - z)} = \ker(A^* - \overline{z})^\perp = \{0\}$, so $\mathsf{Ran}(A - z)$ is dense in $\mathcal{H}$. On the other hand, $(A - z)$ has closed range by Propositions 3.7 and 1.36, so $\mathsf{Ran}(A - z) = \mathcal{H}$. Similarly, $\mathsf{Ran}(A - \overline{z}) = \mathcal{H}$.
• $(iv) \Rightarrow (i)$. We already know by Proposition 3.4 that A^* is an extension of A. Let $\varphi \in \mathsf{Dom}(A^*)$. Since $(A - z)$ is surjective, there exists $\psi \in \mathsf{Dom}(A)$ such that $(A^* - z)\varphi = (A - z)\psi = (A^* - z)\psi$. By Proposition 1.58 we have $\ker(A^* - z) = \mathsf{Ran}(A - \overline{z})^\perp = \{0\}$, so $\varphi = \psi \in \mathsf{Dom}(A)$. This proves that $\mathsf{Dom}(A) = \mathsf{Dom}(A^*)$, and hence $A = A^*$.
• $(ii) \Rightarrow (iv)$ is clear.
• $(iii) - (iv) \Rightarrow (ii)$. A is closed by (iii). By Proposition 3.7 we already know that $A - z$ is injective. It is surjective by (iv) so it is bijective and $z \in \rho(A)$. Similarly, $\overline{z} \in \rho(A)$. ☐ ✎ Ex. 3.4, 3.5

The proof of the implication $(iv) \implies (i)$ also holds if $z \in \mathbb{R}$. This gives the following sufficient condition.

Corollary 3.23. *Let A be a symmetric and densely defined operator on $\mathcal{H}$. Assume that there exists $\lambda \in \mathbb{R}$ such that $\mathsf{Ran}(A - \lambda) = \mathcal{H}$. Then A is selfadjoint.*

3.1.4 Essentially selfadjoint operators

We have seen that if A is symmetric then $A \subset A^*$. It may happen that A is not selfadjoint because we have chosen the domain too small. Given a symmetric operator, the question is then wether it has a selfadjoint extension.

We know from Proposition 3.4 that a densely defined and symmetric operator is always closable, so the first try is to look at its closure.

Definition 3.24. *Let A be a densely defined symmetric operator on $\mathcal{H}$. We say that A is essentially selfadjoint if its closure $\overline{A}$ is selfadjoint.*

Proposition 3.25. *Let A be a densely defined symmetric operator on $\mathcal{H}$. Then A is essentially selfadjoint if and only if $\overline{A} = A^*$.*

Proof. • By Proposition 1.60 we have $\overline{A}^* = A^*$. If A is essentially selfadjoint, we also have $\overline{A}^* = \overline{A}$, and hence $\overline{A} = A^*$.
• Conversely, assume that $\overline{A} = A^*$. By Proposition 1.60 again we have $A^{**} = \overline{A}$, so $\overline{A}^* = A^{**} = \overline{A}$. □

We will see below that a symmetric operator may have many selfadjoint extensions. However, when it is essentially selfadjoint, the extension is unique.

Proposition 3.26. *Let A be a densely defined symmetric operator on $\mathcal{H}$. If A is essentially selfadjoint then $\overline{A}$ is the unique selfadjoint extension of A.*

Proof. Let B be a selfadjoint extension of A. Since it is a closed extension of A, it is an extension of $\overline{A}$. Since B and $\overline{A}$ are selfadjoint, we have $B = \overline{A}$ by Proposition 3.21. □

Proposition 3.27. *Let A be a densely defined symmetric operator on $\mathcal{H}$. Let $z \in \mathbb{C}\backslash\mathbb{R}$. The following assertions are equivalent.*

(i) *A is essentially selfadjoint ;*

(ii) $\ker(A^* - z) = \ker(A^* - \overline{z}) = \{0\}$ *;*

(iii) $\overline{\mathsf{Ran}(A - z)} = \overline{\mathsf{Ran}(A - \overline{z})} = \mathcal{H}$*.*

Proof. • Assume that A is essentially selfadjoint. In particular, A is closable and $\overline{A}^* = A^*$ by Proposition 1.60. By Proposition 3.22 applied to the selfadjoint operator $\overline{A}$, we have $\ker(A^* - z) = \ker(A^* - \overline{z}) = \{0\}$.
• Conversely, assume that (ii) holds. Since $\overline{A}^* \subset A^*$ we have $\ker(\overline{A}^* - z) = \ker(\overline{A}^* - \overline{z}) = \{0\}$. By Proposition 3.22, $\overline{A}$ is selfadjoint.
• Finally (ii) and (iii) are equivalent by Proposition 1.58. □

3.1.5 Examples of closed symmetric operators which are not essentially selfadjoint

We consider on $L^2(0,1)$ the operator A which acts as

$$A = i\frac{\mathrm{d}}{\mathrm{d}x}$$

on the domain

$$\mathsf{Dom}(A) = H_0^1(0,1).$$

Then A is closed (by Example 1.34) and symmetric: for $u, v \in H_0^1(0,1)$ we have by the Green formula

$$\begin{aligned}\langle Au, v\rangle_{L^2(0,1)} &= i\int_0^1 u'(x)\overline{v(x)}\,\mathrm{d}x = i\big(u(1)\overline{v(1)} - u(0)\overline{v(0)}\big) - i\int_0^1 u(x)\overline{v'(x)}\,\mathrm{d}x \\ &= \langle u, Av\rangle_{L^2(0,1)}\,.\end{aligned}$$

Notice that for the boundary terms it was not necessary that both u and v vanish.

Now we compute A^*. Let $v \in \mathsf{Dom}(A^*)$. We have $v \in L^2(0,1)$ and for all $\phi \in C_0^\infty(]0,1[)$ we have

$$\int_{\mathbb{R}} i\phi'(x)\overline{v(x)}\,\mathrm{d}x = \langle A\phi, v\rangle_{L^2(0,1)} = \langle \phi, A^*v\rangle_{L^2(0,1)} = \int_{\mathbb{R}} \phi(x)\overline{(A^*v)(x)}\,\mathrm{d}x.$$

This prove that in the sense of distributions we have $v' \in L^2(0,1)$ and

$$A^*v = iv'.$$

Conversely, if $v \in H^1(0,1)$ then the same computation as above shows that

$$\forall u \in \mathsf{Dom}(A), \quad \left\langle iu', v\right\rangle_{L^2(0,1)} = \left\langle u, iv'\right\rangle_{L^2(0,1)},$$

so $v \in \mathsf{Dom}(A^*)$ (and we recover $A^*v = iv'$). This proves that $\mathsf{Dom}(A^*) = H^1(0,1) \neq \mathsf{Dom}(A)$. Thus A is not selfadjoint.

Notice that for $z \in \mathbb{C}$ the function $x \mapsto e^{-izx}$ belongs to $\ker(A^* - z)$. In particular, $\ker(A^* - z) \neq \{0\}$. By Proposition 3.22, this confirms that A cannot be selfadjoint. It is not even essentially selfadjoint. Moreover, for $z \in \mathbb{C}$ we have by Proposition 1.58

$$\overline{\mathsf{Ran}(A - z)} = \ker(A^* - \overline{z})^\perp \neq \mathcal{H}.$$

This proves that $\sigma(A) = \mathbb{C}$.

Now the question is: does A have a selfadjoint extension ? The answer is: yes, many ! Assume that $\tilde{A}$ is a selfadjoint extension of A. Then $\tilde{A} = \tilde{A}^* \subset A^*$. Let $v \in \mathsf{Dom}(\tilde{A})\backslash\mathsf{Dom}(A)$. For all $u \in \mathsf{Dom}(\tilde{A})$ we have

$$0 = \left\langle \tilde{A}u, v\right\rangle - \left\langle u, \tilde{A}v\right\rangle = i\big(u(1)\overline{v}(1) - u(0)\overline{v}(0)\big).$$

Assume that $v(1) = 0$. Since v is not in $\mathsf{Dom}(A)$ we have $v(0) \neq 0$. Then for all $u \in \mathsf{Dom}(\tilde{A})$ we have $u(0) = 0$. This gives a contradiction since $v(0) \neq 0$. This proves that $v(1) \neq 0$. We set $\alpha = \overline{v}(0)/\overline{v}(1)$. Then for all $u \in \mathsf{Dom}(\tilde{A})$ we have

$$u(1) = \alpha u(0).$$

In particular we have $v(1) = \alpha v(0)$. Since by definition we have $v(0) = \overline{\alpha} v(1)$, this proves that $|\alpha| = 1$. This proves that there exists $\alpha \in \mathbb{U}$ such that $\mathsf{Dom}(\tilde{A}) \subset D_\alpha$, where we have set

$$D_\alpha = \left\{u \in H^1(0,1) \,:\, u(1) = \alpha u(0)\right\}.$$

For $\alpha \in \mathbb{U}$ we denote by A_α the operator defined by $A_\alpha u = iu'$ for u in $\mathsf{Dom}(A_\alpha) = D_\alpha$. In particular, A_α is an extension of A and A^* is an extension of A_α for all α.

We check that A_α is selfadjoint. For $u, v \in \mathsf{Dom}(A_\alpha)$ we have

$$\begin{aligned}
\langle A_\alpha u, v\rangle &= i\int_{\mathbb{R}} u'(x)\overline{v}(x)\,\mathrm{d}x \\
&= iu(1)\overline{v}(1) - iu(0)\overline{v}(0) - i\int_{\mathbb{R}} u(x)\overline{v}'(x)\,\mathrm{d}x \\
&= i(|\alpha|^2 - 1)u(0)\overline{v}(0) - i\int_{\mathbb{R}} u(x)\overline{v}'(x)\,\mathrm{d}x \\
&= \langle u, A_\alpha v\rangle.
\end{aligned}$$

Then A_α is symmetric, and hence A_α^* is an extension of A_α. Now let $v \in \mathsf{Dom}(A_\alpha^*)$. The same computation with $u \in C_0^\infty(]0,1[)$ shows that $v \in H^1(0,1)$ and $A_\alpha^* v = iv'$. Then for all $u \in \mathsf{Dom}(A_\alpha)$ we have

$$0 = \langle A_\alpha u, v\rangle - \langle u, A_\alpha^* v\rangle = -iu(0)(\overline{\alpha}v(1) - v(0)).$$

This proves that $v(1) = \alpha v(0)$, so $v \in \mathsf{Dom}(A_\alpha)$, and finally $A_\alpha^* = A_\alpha$.

All this proves that the operators A_α for $\alpha \in \mathbb{U}$ are the selfadjoint extensions of A.

Moreover we have seen that if $\tilde{A}$ is a selfadjoint extension of A then we have $\tilde{A} \subset A_\alpha$ for some $\alpha \in \mathbb{U}$, and hence $\tilde{A} = A_\alpha$. So finally, the operators A_α for $\alpha \in \mathbb{U}$ are exactly the selfadjoint extensions of A.

Example 3.28. We consider the previous example but on $L^2(0,+\infty)$:

$$A = i\frac{d}{dx}, \quad \mathsf{Dom}(A) = H_0^1(0,+\infty).$$

With the same proof we see that A is symmetric with $\mathsf{Dom}(A^*) = H^1(0,+\infty)$. The difference is that in this case A has no selfadjoint extension. Indeed, assume by contradiction that $\tilde{A}$ is a selfadjoint extension of A. We can check by direct computation that $\mathsf{Ran}(A+i) = L^2(0,+\infty)$ (or equivalently, that $\mathrm{Ker}(A^* - i) = \{0\}$), so A has no selfadjoint extension by Exercise 3.4.

3.1.6 Friedrichs extension

We have seen in the previous paragraph that a symmetric operator which is not selfadjoint can have many selfadjoint extensions, and it is also possible that it does not have any.

In this paragraph we consider the case of lower semibounded symmetric operators and choose in an abstract setting a selfadjoint extension. This ensures in particular that such an operator has at least one selfadjoint extension.

We begin with the case where the symmetric operator is lower bounded by a positive constant.

Proposition-Definition 3.29. *Let A be a densely defined symmetric operator on $\mathcal{H}$. Assume that there exists $\alpha > 0$ such that $A \geqslant \alpha$. We consider the quadratic form $\mathsf{q}_{0,A}$ associated to A, defined on $\mathsf{Dom}(A)$ by*

$$\forall \varphi \in \mathsf{Dom}(A), \quad \mathsf{q}_A(\varphi) = \langle A\varphi, \varphi \rangle_{\mathcal{H}}.$$

This defines a norm on $\mathsf{Dom}(A)$ by

$$\forall \varphi \in \mathsf{Dom}(A), \quad \|\varphi\|_{\mathsf{q}_A} = \sqrt{\mathsf{q}_A(\varphi)}.$$

The form domain $\mathcal{V}_A$ of A is then the set of limits in $\mathcal{H}$ of Cauchy sequences in $\mathsf{Dom}(A)$ (endowed with the norm $\|\cdot\|_{\mathsf{q}_A}$). In other words, a vector $\varphi \in \mathcal{H}$ belongs to $\mathcal{V}_A$ if and only if there exists a sequence $(\varphi_n)_{n\in\mathbb{N}}$ in $\mathsf{Dom}(A)$ such that

$$\|\varphi_n - \varphi_m\|_A \xrightarrow[n,m\to\infty]{} 0 \quad \text{and} \quad \|\varphi_n - \varphi\|_{\mathcal{H}} \to 0.$$

Then q_A extends to a continuous and coercive (in the sense of Remark 1.64) quadratic form Q_A on $\mathcal{V}_A$, and $\mathcal{V}_A$ is a Hilbert space for the corresponding norm.

Proof. • The sesquilinear form associated to q_A is given by

$$\forall \varphi, \psi \in \mathsf{Dom}(A), \quad \mathsf{q}_A(\varphi,\psi) = \langle A\varphi, \psi \rangle_{\mathcal{H}}.$$

It is staightforward to check that this is an inner product on $\mathsf{Dom}(A)$, and then $\|\cdot\|_{\mathsf{q}_A}$ is the corresponding norm.

- Let $(\varphi_n)_{n\in\mathbb{N}}$ be a sequence in $\mathsf{Dom}(A)$ such that

$$\mathsf{q}_A(\varphi_n - \varphi_m) \xrightarrow[n,m\to\infty]{} 0 \quad \text{and } \|\varphi_n\|_{\mathcal{H}} \to 0.$$

In particular, $\|\varphi n\|_{\mathsf{q}_A}$ is bounded. Let $\varepsilon > 0$ and $N \in \mathbb{N}$ such that for $n, m \geqslant N$ we have $\mathsf{q}_A(\varphi_n - \varphi_m) \leqslant \varepsilon^2$. Let $n \geqslant N$. For $m \geqslant N$ we have

$$\begin{aligned}\mathsf{q}_A(\varphi_n) &= \mathsf{q}_A(\varphi_n, \varphi_n - \varphi_m) + \langle A\varphi_n, \varphi_m \rangle \\ &\leqslant \mathsf{q}_A(\varphi_n)^{\frac{1}{2}} \mathsf{q}_A(\varphi_n - \varphi_m)^{\frac{1}{2}} + \langle A\varphi_n, \varphi_m \rangle \\ &\leqslant \varepsilon \mathsf{q}_A(\varphi_n)^{\frac{1}{2}} + \langle A\varphi_n, \varphi_m \rangle.\end{aligned}$$

Letting m go to infinity gives $\mathsf{q}_A(\varphi_n) \leqslant \varepsilon \mathsf{q}_A(\varphi_n)^{\frac{1}{2}}$ for all $n \geqslant N$. This proves that

$$\mathsf{q}_A(\varphi_n) \xrightarrow[n\to\infty]{} 0.$$

• The set $\mathcal{V}_A$ is a subspace of $\mathcal{H}$. Let $\varphi, \psi \in \mathcal{V}_A$. Let $(\varphi_n)_{n\in\mathbb{N}}$ in $\mathsf{Dom}(A)$ be a sequence such that $\mathsf{q}_A(\varphi_n - \varphi_m) \to 0$ and $\|\varphi_n - \varphi\|_{\mathcal{H}} \to 0$. We similarly consider a sequence $(\psi_n)_{n\in\mathbb{N}}$ corresponding to ψ. For $n, m \in \mathbb{N}$ we have

$$\begin{aligned}|\mathsf{q}_A(\varphi_n, \psi_n) - \mathsf{q}_A(\varphi_m, \psi_m)| &\leqslant \mathsf{q}_A(\varphi_n, \psi_n - \psi_m) + \mathsf{q}_A(\varphi_n - \varphi_m, \psi_m)\\ &\leqslant \mathsf{q}_A(\varphi_n)^{\frac{1}{2}}\mathsf{q}_A(\psi_n - \psi_m)^{\frac{1}{2}} + \mathsf{q}_A(\varphi_n - \varphi_m)^{\frac{1}{2}}\mathsf{q}_A(\psi_m)^{\frac{1}{2}}\\ &\xrightarrow[n,m\to\infty]{} 0.\end{aligned}$$

This proves that $\mathsf{q}_A(\varphi_n, \psi_n)$ has a limit, which we denote by $Q_A(\varphi, \psi)$. This limit does not depend on the choice of the sequence $(\varphi_n)_{n\in\mathbb{N}}$. Indeed, if $\tilde{\varphi}n$ is another sequence such that $\mathsf{q}_A(\tilde{\varphi}_n - \tilde{\varphi}_m) \to 0$ and $\|\varphi_n - \varphi\|_{\mathcal{H}} \to 0$, then by the second step we have $\mathsf{q}_A(\varphi_n - \tilde{\varphi}_n) \to 0$. Similarly, the definition of $Q_A(\varphi, \psi)$ does not depend on the choice of the sequence $(\psi_n)_{n\in\mathbb{N}}$. For $\varphi \in \mathcal{V}_A$ we set

$$\|\varphi\|_{\mathcal{V}_A} = \sqrt{Q_A(\varphi)}.$$

• We can check that the map $(\varphi, \psi) \mapsto Q_A(\varphi, \psi)$ defines an inner product on $\mathcal{V}_A$. Let $\varphi \in \mathcal{V}_A$ and let $(\varphi_n)_{n\in\mathbb{N}}$ be a sequence in $\mathsf{Dom}(A)$ as above. Let $\varepsilon > 0$ and $N \in \mathbb{N}$ such that $\mathsf{q}_A(\varphi_n - \varphi_m) \leqslant \varepsilon$ for all $n, m \geqslant N$. Let $k \geqslant N$. Then the sequence $(\varphi_k - \varphi_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $(\mathsf{Dom}(A), \|\cdot\|_{\mathsf{q}_A})$ which goes to $\varphi_k - \varphi$ in $\mathcal{H}$, so

$$Q_A(\varphi_k - \varphi) = \lim_{n\to\infty} \mathsf{q}_A(\varphi_k - \varphi_n) \leqslant \varepsilon.$$

This proves that

$$\mathsf{q}_A(\varphi_k - \varphi) \xrightarrow[k\to\infty]{} 0.$$

In particular, $\mathsf{Dom}(A)$ is dense in $(\mathcal{V}_A, \|\cdot\|_{\mathcal{V}_A})$.

• It remains to check that $\mathcal{V}_A$ is complete. Let $(\varphi_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{V}_A$. It is also a Cauchy sequence in $\mathcal{H}$, so it has a limit $\varphi \in \mathcal{H}$. For all $n \in \mathbb{N}$ there exists $\psi_n \in \mathsf{Dom}(A)$ such that $\|\varphi_n - \psi_n\|_{\mathcal{V}_A} \leqslant 2^{-n}$. Then $(\psi_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $(\mathsf{Dom}(A), \|\cdot\|_{\mathsf{q}_A})$. Thus, by definition, $\varphi \in \mathcal{V}_A$, and moreover we have $\|\psi_n - \varphi\|_{\mathcal{V}_A} \to 0$. This gives $\|\varphi_n - \varphi\|_{\mathcal{V}_A} \to 0$ and completes the proof. □

Alternatively, we can define $\mathcal{V}_A$ as the completion of $(\mathsf{Dom}(A), \|\cdot\|_{\mathsf{q}_A})$ and observe that $\mathcal{V}_A$ can be identified with a subspace of $\mathcal{H}$.

Example 3.30. We consider on $L^2(0,1)$ the operator $A = -\partial^2 + 1$ with domain $\mathsf{Dom}(A) = C_0^\infty(]0,1[)$. For all $u \in C_0^\infty(]0,1[)$ we have

$$\mathsf{q}_A(u, u) = \langle -u'', u\rangle_{L^2(0,1)} + \|u\|^2_{L^2(0,1)} = \|u\|^2_{H^1(0,1)}.$$

A function u in $L^2(0,1)$ is the limit in L^2 of a Cauchy sequence in $C_0^\infty(]0,1[)$ for the H^1-norm if and only if u belongs to $H^1(0,1)$. Then the form domain of A is $H^1_0(0,1)$.

Proposition 3.31. *Let $\mathcal{V}$ be a Hilbert space densely and continuously embedded in $\mathcal{H}$. Let q be a continuous sesquilinear form on $\mathcal{V}$ such that, for some $\alpha > 0$,*

$$\forall \varphi \in \mathcal{V}, \quad \mathsf{q}(\varphi, \varphi) \geqslant \alpha \|\varphi\|^2_{\mathcal{V}} \tag{3.1}$$

(q is coercive in the usual sense, see Remark 1.64). Let A be the operator given by the representation theorem (Theorem 1.71). Then A is selfadjoint on $\mathcal{H}$, $A \geqslant \alpha$ and $\mathcal{V}$ is the form domain of A.

Proof. For all $\varphi \in \mathsf{Dom}(A)$ we have $\langle A\varphi, \varphi\rangle = \mathsf{q}(\varphi, \varphi)$ by definition of A. By continuity of q and (3.1), the norm $\varphi \mapsto \sqrt{\mathsf{q}(\varphi, \varphi)}$ is equivalent to the norm $\|\cdot\|_{\mathcal{V}}$. Since $\mathsf{Dom}(A)$ is dense in $\mathcal{V}$ by Theorem 1.71, the closure of $\mathsf{Dom}(A)$ for $\|\cdot\|_{\mathcal{V}}$ is $\mathcal{V}$. Finally, as for Remark 3.2, since the quadratic form takes real values it is symmetric. Then we deduce that A is selfajdjoint by Theorem 1.71. □

Definition 3.32. *Let A be a densely defined symmetric operator. Assume that A is lower bounded by a positive constant and let $\mathcal{V}_A$ be the form domain of A. Then the Friedrichs extension A_F of A is the operator associated to the quadratic form Q_A by the representation theorem (Theorem 1.71).*

Example 3.33. Let A be the operator of Example 3.30. Its Friedrichs extension is the operator $A_F = -\partial^2 + 1$ with domain $\mathsf{Dom}(A_F) = H^2(0,1) \cap H_0^1(0,1)$.

Remark 3.34. We use the notation of Definition 3.32. Let T be a selfadjoint extension of A. Then the form domain $\mathcal{V}_T$ of T contains the form domain $\mathcal{V}_A$ of A_F (which is also the form domain of A). Then, amongst all the selfadjoint extensions of A, A_F has the smallest form domain.

Now we can define the Friedrichs extension of a general lower bounded operator.

Proposition-Definition 3.35. *Let A be a densely defined and lower bounded symmetric operator. Let $\beta \in \mathbb{R}$ such that $A_\beta = A + \beta$ (with domain $\mathsf{Dom}(A_\beta) = \mathsf{Dom}(A)$) is lower bounded by a positive constant.*

(i) *The form domain of A is defined as the form domain of A_β.*

(ii) *The Friedrichs extension of A is $A_F = A_{\beta,F} - \beta$, where $A_{\beta,F}$ is the Friedrichs extension of A_β.*

These definitions do not depend on the choice of β.

Remark 3.36. If A is selfadjoint then $A_F = A$.

Example 3.37. The form domain of the Dirichlet Laplacian on $]0,1[$ (see Example 1.76) is $H_0^1(0,1)$ and the form domain of the Neumann Laplacian (see Example 1.75) is $H^1(0,1)$.

Example 3.38. We consider on $L^2(0,1)$ the operator $A = -\partial^2$ with domain $\mathsf{Dom}(A) = C_0^\infty(]0,1[)$. Then the form domain of A is $H_0^1(0,1)$ and its Friedrichs extension is the Dirichlet Laplacian on $]0,1[$.

3.1.7 Relatively bounded perturbations of self-adjoint operators

Definition 3.39. *Let A and T be operators on E. We say that T is A-bounded if $\mathsf{Dom}(A) \subset \mathsf{Dom}(T)$ and there exist $a, b \geqslant 0$ such that*

$$\forall \varphi \in \mathsf{Dom}(A), \quad \|T\varphi\|_{\mathsf{E}} \leqslant a \|A\varphi\|_{\mathsf{E}} + b \|\varphi\|_{\mathsf{E}} . \tag{3.2}$$

The A-bound of T is the infimum of all $a \geqslant 0$ for which there exists b such that (3.2) holds.

Remark 3.40. T is A-bounded if and only if $\mathsf{Dom}(A) \subset \mathsf{Dom}(T)$ and T is a continuous map from $(\mathsf{Dom}(A), \|\cdot\|_{\mathsf{Dom}(A)})$ to E.

Remark 3.41. If T is bounded then it is A bounded with A-bound 0 (we can take $\alpha = 0$ and $b = \|T\|_{\mathcal{L}(\mathsf{E})}$ in (3.2)).

Remark 3.42. The A-bound of T is defined as the infimum of all possible a in (3.2). This infinimum is not necessarily atteined. In particular, T can be unbounded but A-bounded with A-bound 0. For example, if T is a symmetric operator on $\mathcal{H}$ then T is T^2-bounded with bound 0. Indeed,

$$\mathsf{Dom}(T^2) = \{\varphi \in \mathsf{Dom}(T) \, : \, T\varphi \in \mathsf{Dom}(T)\} \subset \mathsf{Dom}(T)$$

and for $\varepsilon > 0$ and $\varphi \in \mathsf{Dom}(T^2)$ we have

$$0 \leqslant \left\|(\varepsilon^2 T^2 - 1)\varphi\right\|^2 = \varepsilon^4 \left\|T^2\varphi\right\|^2 + \|\varphi\|^2 - 2\varepsilon^2 \|T\varphi\|^2 ,$$

so

$$\|T\varphi\|^2 \leqslant \frac{\varepsilon^2}{2} \left\|T^2\varphi\right\|^2 + \frac{\varepsilon^{-2}}{2} \|\varphi\|^2 \leqslant \frac{1}{4} \left(\varepsilon \left\|T^2\varphi\right\| + \varepsilon^{-1} \|\varphi\|\right)^2 .$$

Thus (3.2) holds with $a = \varepsilon/4$ and $b = 1/(4\varepsilon)$ for all $\varepsilon > 0$ and T is T^2-bounded with T^2-bound 0 (but (3.2) cannot hold with $a = 0$ if T is not bounded).

We give examples of operators which are relatively bounded with respect to the usual Laplacian on $\mathbb{R}^d$. We denote by H_0 the Laplacian $-\Delta$ on $L^2(\mathbb{R}^d)$, with domain $H^2(\mathbb{R}^d)$.

Example 3.43. Let $\beta, V \in L^\infty(\mathbb{R}^d)$ and $j \in [\![1,d]\!]$. Then $\beta(x)\partial_j$ and V are H_0-bounded with H_0-bound equal to 0. Indeed for $u \in H^2(\mathbb{R}^d)$,

$$\|\partial_j u\|^2 = \langle \partial_j u, \partial_j u \rangle = \left\langle -\partial_j^2 u, u \right\rangle \leqslant \langle -\Delta u, u \rangle \leqslant \|H_0 u\| \, \|u\| \leqslant \varepsilon \|H_0 u\|^2 + \frac{\|u\|^2}{4\varepsilon}.$$

Theorem 3.44 (Kato-Rellich)**.** *Let A be a selfadjoint operator on the Hilbert space $\mathcal{H}$. Let T be a symmetric operator on $\mathcal{H}$. Assume that T is A-bounded with bound smaller than 1.*

(i) *The operator $A+T$, defined on the domain $\mathsf{Dom}(A+T) = \mathsf{Dom}(A)$, is selfadjoint.*

(ii) *Let $\mathcal{D} \subset \mathsf{Dom}(A)$. If A is essentially selfadjoint on $\mathcal{D}$, then so is $A+T$.*

Proof. The operator $A+T$ is symmetric as the sum of two symmetric operators. There exist $a \in [0,1[$ and $b \geqslant 0$ such that (3.2) holds. Let $\beta > 0$. We recall that for $\varphi \in \mathsf{Dom}(A)$ we have

$$\|(A - i\beta)\varphi\|^2 = \|A\varphi\|^2 + \beta^2 \|\varphi\|^2,$$

so

$$\|T\varphi\| \leqslant a\|A\varphi\| + b\|\varphi\| \leqslant (a + b\beta^{-1}) \|(A - i\beta)\varphi\|.$$

Let $\psi \in \mathcal{H}$. Applied with $\varphi = (A - i\beta)^{-1}\psi \in \mathsf{Dom}(A)$, this inequality gives

$$\left\|T(A - i\beta)^{-1}\psi\right\| \leqslant (a + b\beta^{-1}) \|\psi\|.$$

Assume that $|\beta| > \frac{b}{1-a}$. Then $S = T(A - i\beta)^{-1}$ is bounded with bound smaller than 1, so $(1+S)$ has a bounded inverse on $\mathcal{H}$. We deduce that

$$\mathsf{Ran}(A + T - i\beta) = \mathsf{Ran}\big((1+S)(A - i\beta)\big) = \mathcal{H}.$$

We similarly prove that $\mathsf{Ran}(A + T + i\beta) = \mathcal{H}$. By Proposition 3.22, this proves that $A+T$ is selfadjoint. □

Proposition 3.45. *Assume that $d \leqslant 3$. Let V be a potential (Borel function) on $\mathbb{R}^d$. We assume that we can write $V = V_2 + V_\infty$ where $V_2 \in L^2(\mathbb{R}^d)$ and $V_\infty \in L^\infty(\mathbb{R}^d)$. Then the Schrödinger operator $H = H_0 + V$ is selfadjoint on $L^2(\mathbb{R}^d)$ with domain $\mathsf{Dom}(H) = \mathsf{Dom}(H_0) = H^2(\mathbb{R}^d)$.*

Proof. Let $u \in H^2(\mathbb{R}^d)$. For $\varepsilon > 0$ we have

$$\begin{aligned}\|u\|_{L^\infty(\mathbb{R}^d)} \leqslant \|\hat{u}\|_{L^1(\mathbb{R}^d)} &\leqslant \left\|(1 + \varepsilon^2 |\xi|^2)^{-1}\right\|_{L^2(\mathbb{R}^d)} \left\|(1 + \varepsilon^2 |\xi|^2)\hat{u}\right\|_{L^2(\mathbb{R}^d)} \\ &\leqslant C_\varepsilon \big(\varepsilon^2 \|\Delta u\|_{L^2(\mathbb{R}^d)} + \|u\|_{L^2(\mathbb{R}^d)}\big),\end{aligned}$$

where

$$C_\varepsilon = \sqrt{\int_{\mathbb{R}^d} \left(1 + \varepsilon^2 |\xi|^2\right)^{-2} \mathrm{d}\xi}.$$

We have $C_\varepsilon = \varepsilon^{-\frac{d}{2}} C_1$, so

$$\begin{aligned}\|Vu\|_{L^2} &\leqslant \|V_2\|_{L^2} \|u\|_{L^\infty} + \|V_\infty\|_{L^\infty} \|u\|_{L^2} \\ &\leqslant \varepsilon^{2-\frac{d}{2}} C_1 \|V_2\| \|\Delta u\|_{L^2} + \big(\varepsilon^{-\frac{d}{2}} C_1 \|V_2\|_{L^2} + \|V_\infty\|_{L^\infty}\big) \|u\|_{L^2}.\end{aligned}$$

Applied with $\varepsilon > 0$ small enough this proves that V is H_0-bounded with H_0-bound smaller than 1. We conclude with Theorem 3.44. □

Remark 3.46. We can prove that the same conclusion holds for $V \in L^p(\mathbb{R}^d) + L^\infty(\mathbb{R}^d)$ for $p \geqslant 2$ if $d = 4$ and $p \in \left[2, \frac{2d}{d-4}\right[$ if $d \geqslant 5$.

Example 3.47. Let $d \leqslant 3$ and $\alpha \in \left[0, \frac{d}{2}\right[$. Then for any $c \in \mathbb{R}$ the operator

$$H = H_0 + \frac{c}{|x|^\alpha}$$

is well-defined and selfadjoint on the domain $\mathsf{Dom}(H) = H^2(\mathbb{R}^d)$.

3.2 Spectrum of selfadjoint operators

3.2.1 Basic properties

Proposition 3.48. *Let A be a selfadjoint operator on $\mathcal{H}$. Then $\sigma(A) \subset \mathbb{R}$, $\sigma(A) \neq \varnothing$ and for $z \in \rho(A)$ we have*

$$\left\|(A-z)^{-1}\right\|_{\mathcal{L}(\mathcal{H})} = \frac{1}{\mathsf{dist}(z,\sigma(A))}. \tag{3.3}$$

Proof. The first statement follows from Proposition 3.22. Let $z \in \rho(A)$. By Proposition 1.61 we have

$$((A-z)^{-1})^* = (A^* - \overline{z})^{-1} = (A-\overline{z})^{-1}.$$

Since $(A-z)^{-1}$ and $(A-\overline{z})^{-1}$ commute, $(A-z)^{-1}$ is a bounded normal operator on $\mathcal{H}$. Then, by Propositions 2.31 and 2.11,

$$\left\|(A-z)^{-1}\right\|_{\mathcal{L}(\mathcal{H})} = \sup_{\mu\in\sigma((A-z)^{-1})} |\mu| = \sup_{\lambda\in\sigma(A)} |\lambda - z|^{-1} = \frac{1}{\inf_{\lambda\in\sigma(A)} |\lambda - z|}.$$

The proposition follows. □

Proposition 3.49. *Let A be a selfadjoint operator on $\mathcal{H}$ and $\lambda \in \mathbb{R}$.*

(i) Let $\varepsilon > 0$. If there exists $\varphi \in \mathsf{Dom}(A)\setminus\{0\}$ such that $\|(A-\lambda)\varphi\|_{\mathcal{H}} \leqslant \varepsilon \|\varphi\|_{\mathcal{H}}$ then $\sigma(A) \cap [\lambda-\varepsilon, \lambda+\varepsilon] \neq \varnothing$.

(ii) $\lambda \in \sigma(A)$ if and only if there exists a sequence $(\varphi_n)_{n\in\mathbb{N}}$ in $\mathsf{Dom}(A)$ such that $\|\varphi_n\|_{\mathcal{H}} = 1$ for all $n \in \mathbb{N}$ and

$$\|(A-\lambda)\varphi_n\|_{\mathcal{H}} \xrightarrow[n\to+\infty]{} 0.$$

Such a sequence is called a Weyl sequence.

Proof. • Assume that $[\rho-\varepsilon, \rho+\varepsilon] \subset \rho(A)$. Since $\rho(A)$ is open there exists $\varepsilon_1 > \varepsilon$ such that $[\rho-\varepsilon_1, \rho+\varepsilon_1] \subset \rho(A)$. By Proposition 3.48 we have $\left\|(A-\lambda)^{-1}\right\|_{\mathcal{L}(\mathcal{H})} \leqslant \varepsilon_1^{-1}$. Then for $\varphi \in \mathsf{Dom}(A)\setminus\{0\}$ we have

$$\|\varphi\| \leqslant \left\|(A-\lambda)^{-1}\right\| \|(A-\lambda)\varphi\| \leqslant \frac{\|(A-\lambda)\varphi\|}{\varepsilon_1},$$

so $\|(A-\lambda)\varphi\| \geqslant \varepsilon_1 \|\varphi\| > \varepsilon \|\varphi\|$. This prove the first statement by contraposition.
• If a Weyl sequence exists then $\lambda \in \sigma(A)$ by Proposition 2.7 (we can also use the first statement). Now assume that there exists $c > 0$ such that

$$\forall \varphi \in \mathsf{Dom}(A), \quad \|(A-\lambda)\varphi\|_{\mathcal{H}} \geqslant c \|\varphi\|_{\mathcal{H}}.$$

Then $A-\lambda$ is injective with closed range by Proposition 2.9. On the other hand, by Proposition 1.58,

$$\overline{\mathsf{Ran}(A-\lambda)} = \ker((A-\lambda)^*)^{\perp} = \ker(A-\lambda)^{\perp} = \mathcal{H}.$$

Ex. 3.7 This proves that $\lambda \in \rho(A)$. □

3.2.2 Discrete and essential spectra

We recall that for a general operator we have defined the discrete spectrum as the set of isolated eigenvalues with finite (algebraic) multiplicities (see Definition 2.45).

Definition 3.50. *Let A be a selfadjoint operator. We define the essential spectrum by*

$$\sigma_{\mathsf{ess}}(A) = \mathbb{C}\setminus\sigma_{\mathsf{disc}}(A).$$

Proposition 3.51. *Let A be a selfadjoint operator on $\mathcal{H}$. Assume that λ is an isolated element of $\sigma(A)$. Let Π_λ be the corresponding Riesz projection. Then Π_λ is the orthogonal projection on $\ker(A-\lambda)$. In particular, λ is an eigenvalue of A and if $\mathsf{dim}(\ker(A-\lambda)) < \infty$, then its geometric and algebraic multiplicities coincide.*

Proof. Let $r > 0$ be so small that $\sigma(A) \cap D(\lambda, 2r) = \{\lambda\}$. We have

$$\Pi_\lambda = -\frac{1}{2i\pi} \int_{\mathcal{C}(\lambda,r)} (A-\zeta)^{-1} \, d\zeta.$$

Then

$$\Pi_\lambda^* = \frac{1}{2i\pi} \int_{\mathcal{C}(\lambda,r)} (A-\bar{\zeta})^{-1} \, d\zeta = \Pi_\lambda,$$

so Π_λ is an orthogonal projection. By Proposition 2.41 we have $\ker(A-\lambda) \subset \mathsf{Ran}(\Pi_\lambda)$.

For $\varphi \in \mathcal{H}$ we have

$$\begin{aligned}(A-\lambda)\Pi\varphi &= -\frac{1}{2i\pi} \int_{\mathcal{C}(\lambda,r)} (A-\lambda)(A-\zeta)^{-1}\varphi \, d\zeta \\ &= -\frac{1}{2i\pi} \int_{\mathcal{C}(\lambda,r)} \big(\varphi + (\zeta-\lambda)(A-\zeta)^{-1}\varphi\big) \, d\zeta.\end{aligned} \tag{3.4}$$

The map $\zeta \mapsto (\zeta-\lambda)(A-\zeta)^{-1}$ is analytic in $D(\lambda,r)\setminus\{\lambda\}$. By (3.3) it is also bounded. Thus it extends to an analytic function on $D(\lambda,r)$ and (3.4) vanishes. This proves that $\mathsf{Ran}(\Pi) \subset \ker(A-\lambda)$, so $\mathsf{Ran}(\Pi) = \ker(A-\lambda)$. Finally, $\mathsf{Ran}(\Pi)$ cannot be $\{0\}$ (since λ belongs to the spectrum of the restriction of A to $\mathsf{Ran}(\Pi)$), so λ is an eigenvalue of A. ☐

Corollary 3.52. *Let A be a selfadjoint operator on $\mathcal{H}$ and let λ be an isolated element of $\sigma(A)$. Let G be a reducing subspace for A and let A_{G} be the restriction of A to G. If $\mathsf{G} \subset \ker(A-\lambda)^\perp$ then $\sigma(A_{\mathsf{G}}) \subset \sigma(A)\setminus\{\lambda\}$.*

Proof. By Proposition 2.18 we have $\sigma(A_{\mathsf{G}}) \subset \sigma(A)$. Moreover, A_{G} is a selfadjoint operator by Proposition 3.19 and λ is not an eigenvalue of A_{G} since $\ker(A_{\mathsf{G}}-\lambda) = \ker(A-\lambda) \cap \mathsf{G} = \{0\}$. By Proposition 3.51, $\lambda \in \rho(A_G)$. ☐

Lemma 3.53. *Let A be a selfadjoint operator on $\mathcal{H}$. Let $\lambda \in \sigma(A)$. Assume that* $\ker(A-\lambda)$ *has finite dimension and that there exists* $c > 0$ *such that*

$$\forall \varphi \in \ker(A-\lambda)^\perp, \quad \|(A-\lambda)\varphi\| \geqslant c\,\|\varphi\|. \tag{3.5}$$

Then λ is isolated in $\sigma(A)$.

Proof. Let $\mathsf{F} = \ker(A-\lambda)$ and $\mathsf{G} = \mathsf{F}^\perp$. Then F and G are closed. Let Π be the orthogonal projection on F. Let A_{F} and A_{G} be the restrictions of A to F and G. We have $\sigma(A_{\mathsf{F}}) = \{\lambda\}$. On the other hand, A_{G} is a selfadjoint operator on G such that $\ker(A_{\mathsf{G}}-\lambda) = \{0\}$. Then $\overline{\mathsf{Ran}(A_{\mathsf{G}}-\lambda)} = \ker(A_{\mathsf{G}}-\lambda)^\perp = \mathsf{G}$. By (3.5) and Proposition 1.36, $\mathsf{Ran}(A_{\mathsf{G}}-\lambda)$ is closed so $\lambda \in \rho(A_{\mathsf{G}})$. Since $\rho(A_G)$ is open, there exists $\varepsilon > 0$ such that $]\lambda-\varepsilon, \lambda+\varepsilon[\subset \rho(A_{\mathsf{G}})$. Then, by Proposition 2.18, $]\lambda-\varepsilon,\lambda+\varepsilon[\setminus\{\lambda\} \subset \rho(A_{\mathsf{F}}) \cap \rho(A_{\mathsf{G}}) = \rho(A)$. ☐

Proposition 3.54 (Weyl Criterion)**.** *Let A be a selfadjoint operator on $\mathcal{H}$ and $\lambda \in \mathbb{R}$. The following assertions are equivalent.*

(i) $\lambda \in \sigma_{\mathsf{ess}}(A)$.

(ii) *There exists a sequence $(\varphi_n)_{n\in\mathbb{N}}$ in $\mathsf{Dom}(A)$ such that $\|\varphi_n\|_{\mathcal{H}} = 1$ for all $n \in \mathbb{N}$, φ_n goes weakly to 0 and $\|(A-\lambda)\varphi_n\|_{\mathcal{H}} \to 0$ as $n \to \infty$.*

(iii) *There exists a sequence $(\varphi_n)_{n\in\mathbb{N}}$ in $\mathsf{Dom}(A)$ such that $\|\varphi_n\|_{\mathcal{H}} = 1$ for all $n \in \mathbb{N}$, $(\varphi_n)_{n\in\mathbb{N}}$ has no convergent subsequence in $\mathcal{H}$ and $\|(A-\lambda)\varphi_n\|_{\mathcal{H}} \to 0$ as $n \to \infty$.*

✎ *Ex.* 3.8

Proof. We set $\mathsf{F} = \ker(A-\lambda)$ and $\mathsf{G} = \ker(A-\lambda)^\perp$. We denote by A_{G} the restriction of A to G.

- Assume that $\lambda \in \sigma_{\mathsf{ess}}(A)$. If $\mathsf{dim}(\mathsf{F}) = \infty$ then we can construct an orthonormal sequence $(\varphi_n)_{n\in\mathbb{N}}$ in F, and (ii) is satisfied. Now assume that $\mathsf{dim}(\mathsf{F}) < \infty$. By Lemma 3.53, (3.5) cannot hold, so there exists a normalized sequence $(\varphi_n)_{n\in\mathbb{N}}$ in G such that $\|(A-\lambda)\varphi_n\| \to 0$

as $n \to \infty$. For $\psi \in \mathsf{F}$ we have $\langle \psi, \varphi_n \rangle = 0$ for all $n \in \mathbb{N}$. It remains to prove that $\langle \psi, \varphi_n \rangle \to 0$ for all $\psi \in \mathsf{G}$. It is enough to prove this for ψ in a dense subset of G. We have

$$\overline{\mathsf{Ran}(A_{\mathsf{G}} - \lambda)}^{\perp} = \ker(A_{\mathsf{G}} - \lambda) = \{0\},$$

so it is enough to consider $\psi \in \mathsf{Ran}(A_{\mathsf{G}} - \lambda)$. In this case we consider $\zeta \in \mathsf{Dom}(A_{\mathsf{G}})$ such that $\psi = (A_{\mathsf{G}} - \lambda)\zeta$ and write

$$\langle \psi, \varphi_n \rangle = \langle (A_{\mathsf{G}} - \lambda)\zeta, \varphi_n \rangle = \langle \zeta, (A_{\mathsf{G}} - \lambda)\varphi_n \rangle \xrightarrow[n \to +\infty]{} 0.$$

This proves that $\varphi_n \rightharpoonup 0$ as $n \to \infty$. Thus (i) implies (ii).

- A normalized sequence which goes weakly to 0 cannot have a convergent subsequence, so (ii) $\implies$ (iii).
- Assume that there exists a sequence $(\varphi_n)_{n\in\mathbb{N}}$ as in (iii). By Proposition 3.49 we have $\lambda \in \sigma(A)$. Assume by contradiction that $\lambda \in \sigma_{\mathsf{disc}}(A)$. For $n \in \mathbb{N}$ we write $\varphi_n = \psi_n + \psi_n^{\perp}$ where $\psi_n \in F$ and $\psi_n^{\perp} \in \mathsf{G} \cap \mathsf{Dom}(A)$. We have

$$(A_{\mathsf{G}} - \lambda)\psi_n^{\perp} = (A - \lambda)\psi_n^{\perp} = (A - \lambda)\varphi_n \xrightarrow[n \to \infty]{} 0.$$

Since $\lambda \in \rho(A_{\mathsf{G}})$ by Corollary 3.52, we deduce that $\psi_n^{\perp} \to 0$ as $n \to \infty$. In particular, $\|\varphi_n - \psi_n\|_{\mathcal{H}} \to 0$. But the sequence $(\psi_n)_{n\in\mathbb{N}}$ is in F which has finite dimension, so it has a convergent subsequence. This gives a contradiction and proves that $\lambda \in \sigma_{\mathsf{ess}}(A)$. Then (iii) implies (i), and the proof is complete. □

Proposition 3.55. *Let A be a selfadjoint operator on $\mathcal{H}$ and $\lambda \in \sigma_{\mathsf{ess}}(A)$. Let $N \in \mathbb{N}^*$ and $\varepsilon > 0$. There exists an orthonormal family $(\varphi_n)_{1 \leqslant n \leqslant N}$ such that*

$$\forall n \in [\![1, N]\!], \quad \|(A - \lambda)\varphi_n\|_{\mathcal{H}} \leqslant \varepsilon.$$

Proof. • If λ is isolated, it is an eigenvalue of infinite multiplicity, so we can consider an orthonormal family $(\varphi_n)_{1 \leqslant n \leqslant N}$ in $\ker(A - \lambda)$.

- Now assume that λ is not isolated. We fix distinct elements $\lambda_1, \dots, \lambda_N$ of $\sigma(A)$ such that, for all $n \in [\![1, N]\!]$,

$$|\lambda_n - \lambda| \leqslant \frac{\varepsilon}{2}. \tag{3.6}$$

Let $\eta \in]0, 1]$. Let $n \in [\![1, N]\!]$. By Proposition 3.54 we can consider $\psi_n \in \mathsf{Dom}(A)$ such that $\|\psi_n\|_{\mathcal{H}} = 1$ and

$$\|(A - \lambda_n)\psi_n\|_{\mathcal{H}} \leqslant \eta.$$

We set $\tilde{\varphi}_1 = \psi_1$ and for $n \in [\![2, N]\!]$ we define by induction

$$\tilde{\varphi}_n = \psi_n - \sum_{k=1}^{n-1} \langle \tilde{\varphi}_k, \psi_n \rangle_{\mathcal{H}} \, \tilde{\varphi}_k.$$

- We prove by induction on $n \in [\![1, N]\!]$ that there exists a constant $C_n > 0$ independant of $\eta \in]0, 1]$ such that

$$\|(A - \lambda_n)\tilde{\varphi}_n\| \leqslant C_n \eta \quad \text{and} \quad \big| \|\tilde{\varphi}_n\| - 1 \big| \leqslant C_n \eta. \tag{3.7}$$

This is clear for $n = 1$. Now assume that this holds up to order $n - 1$ for some $n \in [\![2, N]\!]$. For $k \in [\![1, n - 1]\!]$ we have

$$(\lambda_n - \lambda_k) \langle \tilde{\varphi}_k, \psi_n \rangle = \langle (A - \lambda_k)\tilde{\varphi}_k, \psi_n \rangle - \langle \tilde{\varphi}_k, (A - \lambda_n)\psi_n \rangle,$$

so, for some $\tilde{C}_{k,n} > 0$,

$$|\langle \tilde{\varphi}_k, \psi_n \rangle| \leqslant \frac{C_k \eta + (1 + C_k \eta)\eta}{|\lambda_k - \lambda_n|} \leqslant \tilde{C}_{k,n} \eta.$$

Then

$$\big| \|\tilde{\varphi}_n\| - 1 \big| \leqslant \|\tilde{\varphi}_n - \psi_n\| \leqslant \sum_{k=1}^{n-1} |\langle \tilde{\varphi}_k, \psi_n \rangle| \, \|\tilde{\varphi}_k\|$$

and

$$\|(A-\lambda_n)\tilde{\varphi}_n\| \leqslant \|(A-\lambda_n)\psi_n\| + \sum_{k=1}^{n-1} |\langle \tilde{\varphi}_k, \psi_n\rangle| \left(\|(A-\lambda_k)\tilde{\varphi}_k\| + |\lambda_k - \lambda_n| \, \|\tilde{\varphi}_k\| \right).$$

We deduce (3.7). If η is chosen small enough then for $n \in [\![1, N]\!]$ we can set

$$\varphi_n = \frac{\tilde{\varphi}_n}{\|\tilde{\varphi}_n\|}.$$

Then there exists $C > 0$ such that for $n \in [\![1, N]\!]$ and $\eta \in]0,1]$ we have

$$\|(A-\lambda_n)\varphi_n\| \leqslant C\eta.$$

It remains to chose η smaller that $\varepsilon/(2C)$ and conclude with (3.6). □

3.2.3 Min-max principle

We consider on $\mathcal{H}$ a self-adjoint operator A bounded from below.

Proposition 3.56. *We have*

$$\min(\sigma(A)) = \inf_{\varphi \in \mathsf{Dom}(A)\setminus\{0\}} \frac{\langle A\varphi, \varphi\rangle_{\mathcal{H}}}{\|\varphi\|_{\mathcal{H}}^2}. \tag{3.8}$$

Proof. We denote by μ_1 the right-hand side of (3.8).

• Let $\lambda \in \sigma(A)$. By the Weyl criterion (Proposition 3.54) there exists a sequence (φ_n) such that $\|\varphi_n\| = 1$ for all n and $\|(A-\lambda)\varphi_n\| \to 0$. This implies in particular

$$\mu_1 \leqslant \langle A\varphi_n, \varphi_n\rangle \xrightarrow[n\to\infty]{} \lambda,$$

so $\mu_1 \leqslant \min(\sigma(A))$.

• Now assume by contradiction that $\mu_1 \in \rho(A)$. We set $R = (A-\mu_1)^{-1}$. For $\eta, \psi \in \mathcal{H}$ we set

$$\mathsf{q}(\eta, \psi) = \langle R\eta, \psi\rangle_{\mathcal{H}}.$$

This defines a continuous sesquilinear form q on $\mathcal{H}$. For $\eta \in \mathcal{H}$ and $\psi = R\eta \in \mathsf{Dom}(A)$ we have

$$\mathsf{q}(\eta, \eta) = \langle \psi, (A-\mu_1)\psi\rangle \geqslant 0,$$

so q is a non-negative. Let (ψ_n) be a sequence in $\mathsf{Dom}(A)$ such that $\|\psi_n\|_{\mathcal{H}} = 1$ for all $n \in \mathbb{N}$ and

$$\langle A\psi_n, \psi_n\rangle \xrightarrow[n\to+\infty]{} \mu_1.$$

For $n \in \mathbb{N}$ we set $\eta_n = (A-\mu_1)\psi_n$. Then by the Cauchy-Schwarz inequality we have

$$\begin{aligned} 1 = \|\psi_n\|_{\mathcal{H}}^2 &= \mathsf{q}(\eta_n, \psi_n) \\ &\leqslant \mathsf{q}(\eta_n, \eta_n)^{\frac{1}{2}} \mathsf{q}(\psi_n, \psi_n)^{\frac{1}{2}} \\ &= \langle \psi_n, (A-\mu_1)\psi_n\rangle^{\frac{1}{2}} \langle R\psi_n, \psi_n\rangle^{\frac{1}{2}} \\ &\xrightarrow[n\to+\infty]{} 0. \end{aligned}$$

This gives a contradiction and proves that $\mu_1 \in \sigma(A)$, and in particular $\mu_1 \geqslant \min(\sigma(A))$. The conclusion follows. □

Theorem 3.57 (Min-max Theorem)**.** *Let A be a lower-bounded self-adjoint operator on $\mathcal{H}$. We denote by $(\lambda_k)_{k\in\mathbb{N}^*, k\leqslant N}$ with $N \in \mathbb{N} \cup \{\infty\}$ the non-decreasing sequence of eigenvalues (counted with multiplicities) smaller than $\inf \sigma_{\mathsf{ess}}(A)$. For $n \in \mathbb{N}^*$ (with $n \leqslant \mathsf{dim}(\mathcal{H})$ if $\mathcal{H}$ is of finite dimension) we have*

$$\inf_{\substack{\mathsf{F}\subset\mathsf{Dom}(A)\\ \mathsf{dim}(\mathsf{F})=n}} \sup_{\varphi\in\mathsf{F}\setminus\{0\}} \frac{\langle A\varphi, \varphi\rangle_{\mathcal{H}}}{\|\varphi\|_{\mathcal{H}}^2} = \begin{cases} \lambda_n & \text{if } n \leqslant N, \\ \inf \sigma_{\mathsf{ess}}(A) & \text{if } n > N. \end{cases}$$

Proof. For $n \in \mathbb{N}^*$ we set

$$\mu_n = \inf_{\substack{\mathsf{F} \subset \mathsf{Dom}(A) \\ \dim(\mathsf{F}) = n}} \sup_{\varphi \in \mathsf{F} \setminus \{0\}} \frac{\langle A\varphi, \varphi \rangle_{\mathcal{H}}}{\|\varphi\|_{\mathcal{H}}^2}.$$

- We set $\mathcal{N} = [\![1, N]\!]$ if $N \in \mathbb{N}$ and $\mathcal{N} = \mathbb{N}$ if $N = +\infty$. We consider an orthonormal family $(\varphi_k)_{k \in \mathcal{N}}$ such that $\varphi_k \in \mathsf{Dom}(A)$ and $A\varphi_k = \lambda_k \varphi_k$ for all $k \in \mathcal{N}$. For $n \in \mathcal{N}$ we set $\mathsf{F}_n = \mathsf{span}(\varphi_1, \dots, \varphi_n)$. We also set $\eta = \inf \sigma_{\mathsf{ess}}(A)$.
- Let $n \in \mathcal{N}$. Let $\varphi \in \mathsf{F}_n$ such that $\|\varphi\| = 1$. We can write $\varphi = \sum_{k=1}^n \alpha_k \varphi_k$ with $\sum_{k=1}^n |\alpha_k|^2 = 1$. Then we have

$$\langle A\varphi, \varphi \rangle = \sum_{k=1}^n |\alpha_k|^2 \lambda_k \leqslant \lambda_n,$$

so $\mu_n \leqslant \lambda_n$.
- By Corollary 3.52, the restriction of A to $F_{n-1}^\perp$ is selfadjoint and its spectrum is included in $[\lambda_n, +\infty[$. Let F be a subspace of $\mathsf{Dom}(A)$ of dimension n. There exists $\varphi \in \mathsf{F} \cap \mathsf{F}_{n-1}^\perp$ with $\|\varphi\| = 1$. For such a φ we have $\langle A\varphi, \varphi \rangle \geqslant \lambda_n$ by Proposition 3.56. This proves that $\mu_n \geqslant \lambda_n$. Then $\mu_n = \lambda_n$ and the infimum is a minimum.
- Now assume that N is finite and consider $n > N$. As in the previous step, we see that $\mu_n \geqslant \eta$. Then let $\varepsilon > 0$. Since $\eta \in \sigma_{\mathsf{ess}}(A)$ there exists by Proposition 3.55 an orthonormal family $(\psi_k)_{1 \leqslant k \leqslant n}$ of vectors in $\mathsf{Dom}(A)$ such that

$$\forall k \in [\![1, n]\!], \quad \|\psi_k\|_{\mathcal{H}} = 1 \quad \text{and} \quad \|(A - \eta)\psi_k\|_{\mathcal{H}} \leqslant \frac{\varepsilon}{\sqrt{n}}.$$

Let $\psi \in \mathsf{F} = \mathsf{span}(\psi_1, \dots, \psi_n)$ such that $\|\psi\| = 1$. We write $\psi = \sum_{k=1}^n \alpha_k \psi_k$ with $\sum_{k=1}^n |\alpha_k|^2 = 1$. Then we have

$$\begin{aligned} \langle A\psi, \psi \rangle &\leqslant \eta + \|(A - \eta)\psi\| \\ &\leqslant \eta + \sum_{k=1}^n |\alpha_k| \, \|(A - \eta)\psi_k\|_{\mathcal{H}} \\ &\leqslant \eta + \left(\sum_{k=1}^n \|(A - \eta)\psi_k\|^2 \right)^{\frac{1}{2}} \\ &\leqslant \eta + \varepsilon. \end{aligned}$$

This proves that

$$\mu_n \leqslant \sup_{\substack{\psi \in F \\ \|\psi\|_{\mathcal{H}} = 1}} \langle A\varphi, \varphi \rangle_{\mathcal{H}} \leqslant \eta + \varepsilon.$$

Finally $\mu_n = \eta$. □

Remark 3.58.
- Let F be a finite dimensional subspace of $\mathsf{Dom}(A)$. Since the unit sphere $\mathbb{S}_{\mathsf{F}}$ of F is compact and the map $\varphi \mapsto \langle A\varphi, \varphi \rangle$ is continuous on $\mathbb{S}_{\mathsf{F}}$, we have

$$\sup_{\varphi \in \mathsf{F} \setminus \{0\}} \frac{\langle A\varphi, \varphi \rangle_{\mathcal{H}}}{\|\varphi\|_{\mathcal{H}}^2} = \sup_{\varphi \in \mathbb{S}_{\mathsf{F}}} \langle A\varphi, \varphi \rangle_{\mathcal{H}} = \max_{\varphi \in \mathbb{S}_{\mathsf{F}}} \langle A\varphi, \varphi \rangle_{\mathcal{H}} = \max_{\varphi \in \mathsf{F} \setminus \{0\}} \frac{\langle A\varphi, \varphi \rangle_{\mathcal{H}}}{\|\varphi\|_{\mathcal{H}}^2}.$$

- Let $n \in \mathcal{N}$. We have seen that

$$\inf_{\substack{\mathsf{F} \subset \mathsf{Dom}(A) \\ \dim(\mathsf{F}) = n}} \sup_{\varphi \in \mathsf{F} \setminus \{0\}} \frac{\langle A\varphi, \varphi \rangle_{\mathcal{H}}}{\|\varphi\|_{\mathcal{H}}^2} = \lambda_n = \sup_{\varphi \in \mathsf{F}_n \setminus \{0\}} \frac{\langle A\varphi, \varphi \rangle_{\mathcal{H}}}{\|\varphi\|_{\mathcal{H}}^2},$$

so the infinimum is a minimum.
- When $n > N$, the infimum is not necessarily reached. Consider for instance the usual Laplacian H_0 on $\mathbb{R}^d$. We have $\min \sigma(H_0) = \sigma_{\mathsf{ess}}(H_0) = 0$ and there is no $\varphi \in H^2(\mathbb{R}^d)$ such that $\langle H_0 \varphi, \varphi \rangle = 0$.

Ex. 3.9 This Min-max Theorem has an equivalent Max-min version. See Exercise 3.9.

Corollary 3.59. *Let $a < \inf \sigma_{\mathsf{ess}}(A)$. Assume that there exists a subspace V of $\mathsf{Dom}(A)$ of dimension $n \in \mathbb{N}^*$ such that*

$$\forall \varphi \in V, \quad \langle A\varphi, \varphi \rangle_{\mathcal{H}} \leqslant a \left\| \varphi \right\|_{\mathcal{H}}^2 .$$

Then A has at least n eigenvalues (counted with multiplicities) not greater that a.

Proposition 3.60. *Let A be a lower bounded selfadjoint operator A on $\mathcal{H}$. Let q_A be the corresponding quadratic form and let $\mathcal{V}_A$ be the form domain of A (see Definition 3.29).*

(i) *We have*

$$\min \sigma(A) = \inf_{\varphi \in \mathcal{V}_A \setminus \{0\}} \frac{\mathsf{q}_A(\varphi)}{\left\| \varphi \right\|_{\mathcal{H}}^2}. \tag{3.9}$$

(ii) *The right-hand side of (3.9) is a minimum if and only if $\min \sigma(A)$ is an eigenvalue, and in this case the minimizers are the eigenvectors corresponding to the eigenvalue $\min \sigma(A)$.*

Proof. • We set

$$\mu_1 = \min \sigma(A) = \inf_{\varphi \in \mathsf{Dom}(A) \setminus \{0\}} \frac{\langle A\varphi, \varphi \rangle}{\left\| \varphi \right\|^2} \quad \text{and} \quad \tilde{\mu}_1 = \inf_{\varphi \in \mathcal{V}_A \setminus \{0\}} \frac{\mathsf{q}_A(\varphi)}{\left\| u \right\|_{\mathcal{H}}^2}.$$

Since $\mathsf{Dom}(A) \subset \mathcal{V}_A$ and $\mathsf{q}_A(\varphi) = \langle A\varphi, \varphi \rangle$ for $\varphi \in \mathsf{Dom}(A)$, we have $\tilde{\mu}_1 \leqslant \mu_1$. After translation we can assume that $\mu_1 > 0$. Then by definition of the form domain, $\mathsf{Dom}(A)$ is dense in $\mathcal{V}_A$ for the norm defined by q_A, so we also have $\mu_1 \leqslant \tilde{\mu}_1$. This gives the first statement.

• Now assume that μ_1 is an eigenvalue of A. Then for a corresponding eigenvector φ we have

$$\frac{\mathsf{q}_A(\varphi)}{\left\| \varphi \right\|^2} = \frac{\langle A\varphi, \varphi \rangle}{\left\| \varphi \right\|^2} = \mu_1,$$

so $\tilde{\mu}_1$ is a minimum and φ is a minimizer. Conversely, assume that φ is a minimizer for $\tilde{\mu}_1$ with $\left\| \varphi \right\| = 1$. Let $\psi \in \mathsf{Dom}(A)$. The map

$$\Phi : t \mapsto \frac{\mathsf{q}_A(\varphi + t\psi)}{\left\| \varphi + t\psi \right\|_{\mathcal{H}}^2}$$

is well defined for $|t|$ small enough, it is smooth and it reaches its minimum at $t = 0$. Thus $\Phi'(0) = 0$, which implies that

$$\mathrm{Re}\, \mathsf{q}_A(\varphi, \psi) = \tilde{\mu}_1 \,\mathrm{Re} \langle \varphi, \psi \rangle .$$

Since we can replace ψ by $i\psi$, this gives

$$\forall \psi \in \mathsf{Dom}(A), \quad \mathsf{q}_A(\varphi, \psi) = \langle \tilde{\mu}_1 \varphi, \psi \rangle .$$

This proves that $\varphi \in \mathsf{Dom}(A)$ and $A\varphi = \tilde{\mu}_1 \varphi$. Then $\tilde{\mu}_1$ is an eigenvalue of A and φ is a corresponding eigenvector. □

Example 3.61. Let Ω be a bounded open set of $\mathbb{R}^d$. We denote by H_0 the Dirichlet Laplacian on Ω ($H_0 = -\Delta$, $\mathsf{Dom}(H_0) = H^2(\Omega) \cap H_0^1(\Omega)$). The form domain of H_0 is $H_0^1(\Omega)$ and the corresponding quadratic form is $\mathsf{q}_{H_0} : u \mapsto \left\| \nabla u \right\|_{L^2(\Omega)}^2$. We will see in Chapter 4 that H_0 has no essential spectrum. Then by Proposition 3.60 the first eigenvalue of H_0 is given by

$$\lambda_1(H_0) = \inf_{u \in H_0^1(\Omega) \setminus \{0\}} \frac{\left\| \nabla u \right\|_{L^2(\Omega)}^2}{\left\| u \right\|_{L^2(\Omega)}^2}.$$

By the Poincaré inequality we have $\lambda_1(H_0) > 0$.

3.3 Additional topic: polar decomposition

Proposition 3.62 (Square root of a bounded non-negative operator)**.** *Let $A \in \mathcal{L}(\mathcal{H})$ be non-negative. There exists a unique non-negative bounded operator S such that $S^2 = A$. Moreover, S commutes with A, and any operator which commutes with A also commutes with S. We can write $S = \sqrt{A}$.*

Proof. • Assume that the existence is proved when $\|A\| \leqslant 1$. Then in general we can multiply A by $\varepsilon = \|A\|^{-1}$, so that $\|\varepsilon A\| \leqslant 1$. Then we set $S = \varepsilon^{-\frac{1}{2}} S_\varepsilon$, where S_ε is the square root of εA. Then $S^2 = \varepsilon^{-1}\varepsilon A = A$ and, since S_ε commutes with εA, S commutes with A.
• Now assume that $\|A\| \leqslant 1$. We set $B = \mathrm{Id} - A$. For $\varphi \in \mathcal{H}$ we have

$$\langle B\varphi, \varphi\rangle = \|\varphi\|^2 - \langle A\varphi, \varphi\rangle \leqslant \|\varphi\|^2 .$$

We also have

$$\langle B\varphi, \varphi\rangle = \|\varphi\|^2 - \langle A\varphi, \varphi\rangle \geqslant \|\varphi\|^2 - \|A\| \|u\|^2 \geqslant 0.$$

Then by the Cauchy-Schwarz inequality we have for $\varphi, \psi \in \mathcal{H}$,

$$|\langle B\varphi, \psi\rangle| \leqslant \langle B\varphi, \varphi\rangle^{\frac{1}{2}} \langle B\psi, \psi\rangle^{\frac{1}{2}} \leqslant \|\varphi\| \|\psi\| .$$

This proves that $\|B\| \leqslant 1$. Now we use the power series for the function $z \mapsto \sqrt{1-z}$, absolutely convergent [1] on $\overline{D}(0,1)$:

$$\forall z \in \overline{D}(0,1), \quad \sqrt{1-z} = 1 - \sum_{n=1}^{\infty} a_n z^n, \quad a_n = \frac{(2n)!}{(2n-1)(n!)^2 4^n}.$$

Then we set

$$S = 1 - \sum_{n=1}^{\infty} a_n B^n.$$

Then by Cauchy product for a power series we have $S^2 = \mathrm{Id} - B = A$. Moreover S commute with B and hence with A. Similarly, any operator which commutes with A commutes with B and hence with S.
• Now we prove uniqueness. Assume that S' is another solution. In particular S and S' commute. If we set

$$T = (S - S')S(S - S') \quad \text{and} \quad T' = (S - S')S'(S - S')$$

We observe that

$$T + T' = (S - S')(S + S')(S - S-) = (S - S')(S^2 - S'^2) = 0.$$

Since T and T' are non-negative, they are both 0 by Proposition **??**. Then

$$(S - S')^4 = (S - S')(T - T') = 0.$$

This implies that $(S - S')^2 = 0$ and finally $S - S' = 0$. □

Definition 3.63. *For $A \in \mathcal{L}(\mathcal{H})$ we set $|A| = \sqrt{A^*A}$.*

This definition makes sense since A^*A is always a non-negative operator.

Definition 3.64. *We say that $U \in \mathcal{L}(\mathcal{H})$ is a partial isometry if for all $\varphi \in \ker(U)^\perp$ we have $\|U\varphi\| = \|\varphi\|$.*

[1] For $x \in [0,1[$ we have

$$\sqrt{1-x} = 1 - \sum_{n=1}^{\infty} a_n x^n.$$

Since all the coefficients are positive we have

$$\sum_{n=1}^{\infty} a_n = \lim_{x\to 1} \sum_{n=1}^{\infty} a_n x^n = 1 - \sqrt{1-1} = 1 < +\infty.$$

This proves that $\sum_{n=1}^{\infty} a_n < +\infty$.

Proposition 3.65. *Let $A \in \mathcal{L}(\mathcal{H})$. There exists a unique partial isometry U such that* $\ker(U) = \ker(A)$ *and*

$$A = U\,|A|\,.$$

Proof. • Assume that U_1 and U_2 are solutions. We have $U_1\,|A| = U_2\,|A|$ so $U_1 = U_2$ on $\mathsf{Ran}(|A|)$, and then on $\overline{\mathsf{Ran}(|A|)}$ by continuity. On the other hand, on $\overline{\mathsf{Ran}(|A|)}^{\perp} = \ker(|A|) = \ker(A)$ (see Proposition 1.58) we have $U_1 = U_2 = 0$ so, finally, $U_1 = U_2$.
• For $\varphi \in \mathcal{H}$ we have $\||A|\,\varphi\| = \|A\varphi\|$. Then if $\varphi_1, \varphi_2 \in \mathcal{H}$ are such that $|A|\,\varphi_1 = |A|\,\varphi_2$, we also have $A\varphi_1 = A\varphi_2$. Thus we can define U on $\mathsf{Ran}(|A|)$ by

$$U\,|A|\,\varphi = A\varphi.$$

This is a linear isometry from $\mathsf{Ran}(|A|)$ to $\mathsf{Ran}(A)$. It can be extended to a linear isometry from $\overline{\mathsf{Ran}(|A|)}$ to $\overline{\mathsf{Ran}(A)}$. Then we extend U by 0 on $\overline{\mathsf{Ran}(|A|)}^{\perp} = \ker(A)$. In particular, $\ker(A) \subset \ker(U)$. On the other hand, since U is an isometry on $\ker(A)^{\perp}$, we can check that $\ker(U) = \ker(A)$. Then U is an isometry on $\ker(U)^{\perp}$, so this is a partial isometry. □

3.4 Exercises

***Exercise* 3.1.** Let Ω be an open subset of $\mathbb{R}^d$. We consider on $L^2(\Omega)$ the operators H_0 and H which act as $-\Delta$ on the domains $\mathsf{Dom}(H_0) = C_0^\infty(\Omega)$ and $\mathsf{Dom}(H) = H^2(\Omega)$. Are H_0 and H symmetric operators ?

***Exercise* 3.2.** Let $\Pi \in \mathcal{L}(\mathcal{H})$ be a projection of $\mathcal{H}$ ($\Pi^2 = \Pi$). Prove that Π is an orthogonal projection if and only if it is selfadjoint.

***Exercise* 3.3.** Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces. Let $U : \mathcal{H}_1 \to \mathcal{H}_2$ be a unitary operator. Let A_1 be an operator on $\mathcal{H}_1$ and A_2 an operator on $\mathcal{H}_2$. Assume that $\mathsf{Dom}(A_2) = U\mathsf{Dom}(A_1)$ and $A_2 = UA_1U^*$. Prove that A_1 is selfadjoint on $\mathcal{H}_1$ if and only if A_2 is selfadjoint on $\mathcal{H}_2$.

***Exercise* 3.4.** Let A be a symmetric operator on the Hilbert space $\mathcal{H}$. Assume that A is not selfadjoint but $\mathsf{Ran}(A - i) = \mathcal{H}$ *or* $\mathsf{Ran}(A + i) = \mathcal{H}$. Prove that A has no selfadjoint extension.

***Exercise* 3.5.** Let $m > 0$. We consider the Hilbert space $\mathscr{H} = H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$ the operator

$$\mathcal{W} = \begin{pmatrix} 0 & 1 \\ \Delta - m & 0 \end{pmatrix}$$

defined on the domain $\mathsf{Dom}(\mathcal{W}) = H^2(\mathbb{R}^d) \times H^1(\mathbb{R}^d)$. Prove that $\mathcal{W}$ is skew-adjoint if $\mathscr{H}$ is endowed with the Hilbert structure corresponding to the norm defined by

$$\|(u,v)\|_{\mathscr{H}}^2 = \|\nabla u\|_{L^2(\mathbb{R}^d)}^2 + m\,\|u\|_{L^2(\mathbb{R}^d)}^2 + \|v\|_{L^2(\mathbb{R}^d)}^2\,.$$

***Exercise* 3.6.** Let A_0 be the operator of Example 3.38.
1. What is the adjoint of A_0 ?
2. Compute $\ker(A_0^* - z)$ for $z \in \mathbb{C}\backslash\mathbb{R}$.
3. For $u \in H^2(0,1)$ we set

$$Bu = \begin{pmatrix} u(0) \\ u'(0) \\ u(1) \\ u'(1) \end{pmatrix}.$$

Prove that there exists a matrix $M \in M_4(\mathbb{C})$ (to be explicited) such that an operator A is a selfadjoint extension of A_0 if and only if there exists a subspace F of $\mathbb{C}^4$ such that $MF = F^{\perp}$ and

$$A = -\frac{d^2}{dx^2}, \quad \mathsf{Dom}(A) = \left\{u \in H^2(0,1) \,:\, Bu \in F\right\}.$$

4. Give some examples of selfadjoint extensions of A_0.
5. What is the Friedrichs extension A_F of A_0 ?

***Exercise* 3.7.** Give an example of an operator A and $\lambda \in \mathbb{C}$ such that $\lambda \in \sigma(A)$ but there is no corresponding Weyl sequence.

***Exercise* 3.8.** We consider the Laplacian $H = -\Delta$ on $L^2(\mathbb{R})$, with domain $H^2(\mathbb{R})$. Let $\lambda > 0$. Construct a sequence (φ_n) in $H^2(\mathbb{R})$ such that $\|\varphi_n\| = 1$, $\|(H-\lambda)\varphi_n\| \to 0$ and φ_n goes weakly to 0 in $L^2(\mathbb{R})$.

***Exercise* 3.9.** Prove the following version of the Min-Max Theorem. Let A be a self-adjoint operator on $\mathcal{H}$. Assume that A is semi-bounded from below. For $n \in \mathbb{N}^*$ (with $n \leqslant \mathsf{dim}(\mathcal{H})$ if $\mathcal{H}$ is of finite dimension) we set

$$\mu_n(A) = \sup_{\varphi_1,\dots,\varphi_{n-1} \in \mathcal{H}} \inf_{\substack{\varphi \in \mathsf{span}(\varphi_1,\dots,\varphi_{n-1})^\perp \\ \varphi \in \mathsf{Dom}(A)\setminus\{0\}}} \frac{\langle A\varphi, \varphi\rangle_{\mathcal{H}}}{\|\varphi\|_{\mathcal{H}}^2}.$$

The sequence $(\mu_n)_{n\in\mathbb{N}^*}$ is non-decreasing and for $n \in \mathbb{N}^*$ one of the following statements hold.

(i) $\mu_n(A) < \inf \sigma_{\mathsf{ess}}(A)$ and μ_n is the n-th eigenvalue of A counted with multiplicities,

(ii) $\mu_n(A) = \inf \sigma_{\mathsf{ess}}(A)$.