Chapter 2

Spectrum - Resolvent

2.1 Spectrum - Resolvent Set - Resolvent

Let E be a Banach space.

2.1.1 Definitions and first properties

Definition 2.1. *Let A be an operator on E.*

(i) *The resolvent set $\rho(A)$ of A is the set of $z \in \mathbb{C}$ such that $(A - z) = (A - z\,\mathrm{Id}_{\mathsf{E}})$ is (boundedly) invertible (see Definition 1.23).*

(ii) *The spectrum $\sigma(A)$ of A is the complementery set of $\rho(A)$ in $\mathbb{C}$.*

Definition 2.2. *Let A be an operator on E.*

(i) *We say that $\lambda \in \mathbb{C}$ is an eigenvalue of A if there exists $\varphi \in \mathsf{Dom}(A)\setminus\{0\}$ such that $A\varphi = \lambda\varphi$ (in other words, $(A - \lambda)$ is not injective).*

(ii) *Let $\lambda \in \mathbb{C}$ be an eigenvalue of A. A vector $\varphi \in \mathsf{Dom}(A)\setminus\{0\}$ such that $A\varphi = \lambda\varphi$ is called an eigenvector of A associated with λ, and $\ker(A - \lambda)$ is the corresponding eigenspace.*

(iii) *The geometric multiplicity of λ is the dimension of $\ker(A - \lambda)$.*

(iv) *We denote by $\sigma_{\mathsf{p}}(A)$ the set of eigenvalues of A.*

Remark 2.3. We know that if E is of finite dimension then $\sigma(A) = \sigma_{\mathsf{p}}(A)$. However, in general we always have $\sigma_{\mathsf{p}}(A) \subset \sigma(A)$, but the inclusion can be strict.

Example 2.4. We consider the multiplication operator M_w defined in Example 1.12. Let $\lambda \in \mathbb{C}$. Notice that $M_w - \lambda = M_{w-\lambda}$. Then λ is an eigenvalue of M_w is and only if

$$\mathsf{Leb}\left(\{x \in \Omega \,:\, w(x) = \lambda\}\right) > 0,$$

and λ belongs to $\sigma(M_w)$ if and only if for all $\varepsilon > 0$ we have

$$\mathsf{Leb}\left(\{x \in \Omega \,:\, |w(x) - \lambda| \leqslant \varepsilon\}\right) > 0.$$

Ex. 2.1-2.6

Remark 2.5. If A is not closed then $(A - z)$ is never closed, and hence never boundedly invertible (see Proposition 1.35). Then $\rho(A) = \varnothing$. This is why we are only interested in closed operators. Notice however that a closed operator can have an empty resolvent set (see Exercise 2.6). On the other hand, by Proposition 1.35 again, if A is closed and $A - z : \mathsf{Dom}(A) \to \mathsf{E}$ is bijective, then $z \in \rho(A)$.

Example 2.6. • Let $\mathsf{E} = L^2(\mathbb{R}^d)$ and $A_0 = -\Delta$ with $\mathsf{Dom}(A_0) = C_0^\infty(\mathbb{R}^d)$. Then for any $z \in \mathbb{C}$ we have $\mathsf{Ran}(A_0 - z) \subset C_0^\infty(\mathbb{R}^d)$ so $A_0 - z$ cannot be invertible. This proves that $\sigma(A_0) = \mathbb{C}$. This is consistent with the fact that A_0 is not closed.
• Now we consider $A = -\Delta$ with $\mathsf{Dom}(A) = H^2(\mathbb{R}^d)$. Then $\sigma(A) = \mathbb{R}_+$ and for $z \in \mathbb{C}\backslash\mathbb{R}_+$ we have

$$\left\|(A-z)^{-1}\right\|_{\mathcal{L}(L^2(\mathbb{R}^d))} = \frac{1}{\mathsf{dist}(z,\mathbb{R}_+)}.$$

Indeed, if we denote by $\mathcal{F}$ the Fourier transform on $L^2(\mathbb{R}^d)$, then $\mathcal{F}$ is a unitary operator. Then $(-\Delta-z)$ is invertible if and only if $\mathcal{F}(-\Delta-z)\mathcal{F}^{-1} = M-z$ is invertible on $L^2(\mathbb{R}^d)$, where $M = \mathcal{F}(-\Delta)\mathcal{F}^{-1}$ is equal to the multiplication operator M_w for $w : \xi \mapsto |\xi|^2$. In particular, we have $\mathsf{Dom}(-\Delta) = \left\{u \in L^2(\mathbb{R}^d) : \mathcal{F}u \in \mathsf{Dom}(M_w)\right\}$. Thus $\sigma(A) = \sigma(M_w) = \mathbb{R}_+$ and for $z \in \mathbb{C}\backslash\mathbb{R}_+$ we have

$$\begin{aligned}\left\|(A-z)^{-1}\right\|_{\mathcal{L}(L^2(\mathbb{R}^d))} &= \left\|\mathcal{F}^{-1}(M-z)^{-1}\mathcal{F}\right\|_{\mathcal{L}(L^2(\mathbb{R}^d))} = \left\|(M-z)^{-1}\right\|_{\mathcal{L}(L^2(\mathbb{R}^d))}\\ &= \sup_{\xi\in\mathbb{R}} \frac{1}{\xi^2 - z} = \frac{1}{\mathsf{dist}(z,\mathbb{R}_+)}.\end{aligned}$$

Ex. 2.7-2.8

Proposition 2.7. *Let A be an operator on E and $z \in \mathbb{C}$. Assume that there exists a sequence (φ_n) in $\mathsf{Dom}(A)$ such that $\|\varphi_n\|_{\mathsf{E}} = 1$ for all $n \in \mathbb{N}$ and*

$$\|(A-z)\varphi_n\|_{\mathsf{E}} \xrightarrow[n\to+\infty]{} 0.$$

Then $z \in \sigma(A)$.

Proof. Assume that $z \in \rho(A)$. Then

$$\|\varphi_n\|_{\mathsf{E}} \leqslant \left\|(A-z)^{-1}\right\|_{\mathcal{L}(\mathsf{E})} \|(A-z)\varphi_n\|_{\mathsf{E}} \xrightarrow[n\to+\infty]{} 0.$$

This gives a contradiction. □

Remark 2.8. The converse is not true in general. Consider for instance the shift operator S_r (see Example 1.2). Then S_r is not surjective, so $0 \in \sigma(S_r)$, but $\|S_r\varphi\| = \|\varphi\|$ for all $\varphi \in \ell^2(\mathbb{N})$.

Ex. 2.9

Proposition 2.9. *Let A be an operator on E. Let $z \in \mathbb{C}$. Assume that there exists $c_0 > 0$ such that*

$$\forall \varphi \in \mathsf{Dom}(A), \quad \|(A-z)\varphi\|_{\mathsf{E}} \geqslant c_0 \|\varphi\|_{\mathsf{E}}. \tag{2.1}$$

We say that z is a regular point of A. Then

(i) *$(A-z)$ is injective ;*

(ii) *If $(A-z)$ is invertible then $\left\|(A-\lambda)^{-1}\right\| \leqslant c_0^{-1}$.*

(iii) *If moreover A is closed, then $(A-z)$ has closed range.*

This means that if z is a regular point of A, then $z \in \rho(A)$ if and only if $\mathsf{Ran}(A-z)$ is dense in E. Moreover, in this case we already have a bound for the inverse.

Proof. We apply Proposition 1.36 to the operator $(A-z)$. □

Proposition 2.10. *Let A be a closed and densely defined operator on $\mathcal{H}$. Then*

$$\sigma(A^*) = \{\overline{z}, z \in \sigma(A)\}.$$

Proof. Let $\lambda \in \mathbb{C}$. By Proposition 1.61 the operator $(A-\lambda)$ is bijective if and only if $(A-\lambda)^* = (A^* - \overline{\lambda})$ is bijective. □

Proposition 2.11. *Let A be a boundedly invertible operator.*

(i) *A^{-1} has a bounded inverse if and only if A is bounded (and in this case we have $(A^{-1})^{-1} = A$).*

(ii) *For $\lambda \in \mathbb{C}^*$ we have $\lambda \in \rho(A)$ if and only if $\lambda^{-1} \in \rho(A^{-1})$.*

Proof. We prove the second statement. Let $\lambda \in \mathbb{C}^*$. Assume that $A^{-1} - \lambda^{-1}$ has a bounded inverse. Since $(A-\lambda) = -\lambda(A^{-1}-\lambda^{-1})A$, the bounded operator $-\lambda^{-1}A^{-1}(A^{-1}-\lambda^{-1})^{-1}$ is a bounded inverse for $(A-\lambda)$. Conversely, if $(A-\lambda)$ has a bounded inverse then $(A^{-1}-\lambda^{-1}) = -\lambda^{-1}(A-\lambda)A^{-1}$ has a bounded inverse given by $-\lambda A(A-\lambda)^{-1} = -\lambda(1+\lambda(A-\lambda)^{-1})$. □

2.1.2 Example: the harmonic oscillator

We consider on $L^2(\mathbb{R})$ the operator H which acts as

$$H = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + x^2 \tag{2.2}$$

on the domain

$$\mathsf{Dom}(H) = \left\{u \in L^2(\mathbb{R}) \,:\, -u'' + x^2 u \in L^2(\mathbb{R})\right\}. \tag{2.3}$$

Proposition 2.12. *The spectrum of H consists of a sequence $(\lambda_k)_{k\in\mathbb{N}}$ of simple eigenvalues. Moreover, for $k \in \mathbb{N}^*$ we have*

$$\lambda_k = (2k+1)$$

and a corresponding eigenfunction is given by

$$\varphi_k(x) = h_k(x) e^{-\frac{x^2}{2}},$$

where $h_k(x) =$ is the k-th Hermite polynomial (in particular it has degree k).

Proof. • We recall that we have introduced the operators a and c is Section 1.2.4. We observe that for $u \in \mathcal{S}(\mathbb{R})$ we have

$$Hu = 2\mathsf{ca}u + u.$$

We also have $[\mathsf{a},\mathsf{c}]u = \mathsf{ac}u - \mathsf{ca}u = u$ so, by induction on k,

$$\mathsf{ac}^k u = k\mathsf{c}^{k-1}u + \mathsf{c}^k a u. \tag{2.4}$$

• We set $\varphi_0(x) = e^{-\frac{x^2}{2}}$. We have $\varphi_0 \in \mathcal{S}(\mathbb{R})$ and $\mathsf{a}\varphi_0 = 0$, so $H\varphi_0 = \varphi_0$. For $k \in \mathbb{N}^*$ we set $\varphi_k = \mathsf{c}^k\varphi_0$. We can check by induction on $k \in \mathbb{N}$ that φ_k is of the form $\varphi_k = P_k\varphi_0$ where P_k is a polynomial of degree k. In particular $\varphi_k \in \mathcal{S}(\mathbb{R})$. We have

$$H\varphi_k = 2\mathsf{cac}^k\varphi_0 + \varphi_k = 2k\mathsf{c}^k\varphi_0 + 2\mathsf{c}^{k+1}\mathsf{a}\varphi_0 + \varphi_k = (2k+1)\varphi_k.$$

This proves that $\lambda_k = 2k+1$ is an eigenvalue of H and φ_k is a corresponding eigenfunction.
• We prove by induction on $j \in \mathbb{N}$ that for all $k > j$ we have $\langle\varphi_j, \varphi_k\rangle = 0$. Since $\mathsf{c}^* = \mathsf{a}$, we have

$$\langle\varphi_j, \varphi_k\rangle = \left\langle \mathsf{c}^j\varphi_0, \mathsf{c}^k\varphi_0\right\rangle = \left\langle \mathsf{a}^k\mathsf{c}^j\varphi_0, \varphi_0\right\rangle.$$

Since $\mathsf{a}\varphi_0 = 0$ the conclusion follows if $j = 0$. For $j \geqslant 1$ we have by

$$\left\langle \mathsf{a}^k\mathsf{c}^j\varphi_0, \varphi_0\right\rangle = j\left\langle \mathsf{a}^{k-1}\mathsf{c}^{j-1}\varphi_0, \varphi_0\right\rangle + \left\langle \mathsf{a}^{k-1}\mathsf{c}^j\mathsf{a}\varphi_0, \varphi_0\right\rangle = 0.$$

This proves that the family of eigenvectors $(\varphi_k)_{k\in\mathbb{N}}$ is orthogonal in $L^2(\mathbb{R})$.
• Let us prove that the family (φ_k) is total in $L^2(\mathbb{R})$. This means that $\overline{\mathsf{span}((\varphi_k)_{k\in\mathbb{N}})} = L^2(\mathbb{R})$. Let $u \in L^2(\mathbb{R})$ be such that $\langle\varphi_k, u\rangle_{L^2(\mathbb{R})} = 0$ for all $k \in \mathbb{N}$. Since P_k is of degree k for all k, we deduce that for any polynomial q we have

$$\int_{\mathbb{R}} \mathsf{q}(x) e^{-\frac{x^2}{2}} u(x)\,\mathrm{d}x = 0.$$

For $\xi \in \mathbb{C}$ we set

$$v(\xi) = \int_{\mathbb{R}} e^{-ix\xi} u(x) e^{-\frac{x^2}{2}}\,\mathrm{d}x.$$

By differentiation under the integral sign we see that v is holomorphic in $\mathbb{C}$ and for $m \in \mathbb{N}$ we have

$$v^{(m)}(0) = \int_{\mathbb{R}} (-ix)^m u(x) e^{-\frac{x^2}{2}}\,\mathrm{d}x = 0.$$

This implies that $v = 0$ on $\mathbb{C}$, and in particular in $\mathbb{R}$. Thus the Fourier transform of $x \mapsto u(x)e^{-\frac{x^2}{2}}$ is 0, so $u = 0$ almost everywhere.

For $k \in \mathbb{N}$ we set

$$\psi_k = \frac{\varphi_k}{\|\varphi_k\|}.$$

Then (ψ_k) is a Hilbert basis of $L^2(\mathbb{R})$, and $H\psi_k = \lambda_k\psi_k$ for all k. Thus the spectrum of H is exactly given by the sequence $(\lambda_k)_{k\in\mathbb{N}}$ of simple eigenvalues (see Exercise 2.2). ☐

2.1.3 Resolvent

Let A be an operator on E with non-empty resolvent set.

Definition 2.13. *Let $z \in \rho(A)$. We say that $(A-z)^{-1}$ is the resolvent of A at z.*

Notice that the operator A is completely characterized by its resolvent. The interest of considering this resolvent is that it is a bounded operator on E, even if A is not. Moreover, the good properties of the resolvent will be useful to study the operator A.

Proposition 2.14. *For $z \in \rho(A)$ we have*

$$(A-z)^{-1}A \subset A(A-z)^{-1} = \operatorname{Id} + z(A-z)^{-1}.$$

Proposition 2.15 (Resolvent Identity)**.** *For $z_1, z_2 \in \rho(A)$ we have*

$$\begin{aligned}(A-z_1)^{-1} - (A-z_2)^{-1} &= (z_1 - z_2)(A-z_1)^{-1}(A-z_2)^{-1}\\ &= (z_1 - z_2)(A-z_2)^{-1}(A-z_1)^{-1}.\end{aligned}$$

Proof. On $\mathsf{Dom}(A)$ we have $(A-z_2) - (A-z_1) = z_1 - z_2$. The first equality follows after composition by $(A-z_1)^{-1}$ on the left and by $(A-z_2)^{-1}$ on the right and the second after composition by $(A-z_1)^{-1}$ on the right and by $(A-z_2)^{-1}$ on the left. □

Remark 2.16. The resolvent identity proves in particular that $(A-z_1)^{-1}$ and $(A-z_2)^{-1}$ commute.

Proposition 2.17. *The resolvent set $\rho(A)$ of A is open (equivalently, its spectrum $\sigma(A)$ is closed) and for all $z_0 \in \rho(A)$ we have*

$$\left\|(A-z_0)^{-1}\right\|_{\mathcal{L}(\mathsf{E})} \geqslant \frac{1}{\mathsf{dist}(z_0, \sigma(A))}.$$

Moreover, the resolvent map $z \mapsto (A-z)^{-1}$ is analytic on $\rho(A)$ and

$$\frac{\mathrm{d}}{\mathrm{d}z}(A-z)^{-1} = (A-z)^{-2}.$$

Proof. Let $z_0 \in \rho(A)$. For $z \in D\big(z_0, \left\|(A-z_0)^{-1}\right\|_{\mathcal{L}(\mathsf{E})}^{-1}\big)$ we have

$$A - z = (A-z_0) - (z-z_0) = \big(1 - (z-z_0)(A-z_0)^{-1}\big)(A-z_0).$$

Since $(z-z_0)(A-z_0)^{-1}$ has norm less that 1 we can apply Proposition 1.8. Then the operator $1-(z-z_0)(A-z_0)^{-1}$ is invertible and

$$\big(1-(z-z_0)(A-z_0)^{-1}\big)^{-1} = \sum_{n\in\mathbb{N}} (z-z_0)^n (A-z_0)^{-n}.$$

Then $A-z$ is invertible and $(A-z)^{-1}$ and

$$(A-z)^{-1} = \sum_{n\in\mathbb{N}} (z-z_0)^n (A-z_0)^{-(n+1)}.$$

In particular

$$\mathsf{dist}\big(z_0, \sigma(A)\big) \geqslant \left\|(A-z_0)^{-1}\right\|_{\mathcal{L}(\mathsf{E})}^{-1}.$$

Moreover, we have written $(A-z)^{-1}$ as a power series around z_0 from which we deduce the last statement. □

Ex. 2.10-2.11

Applying Proposition 1.45 to $(A-z)$, we get the following result for reducing subspaces.

Proposition 2.18. *Let Π be a projection of E such that $\Pi A \subset A\Pi$, $\mathsf{F} = \mathsf{Ran}(\Pi)$ and $\mathsf{G} = \ker(\Pi)$. Then we have $\sigma(A) = \sigma(A_{\mathsf{F}}) \cup \sigma(A_{\mathsf{G}})$ and for $z \in \rho(A) = \rho(A_{\mathsf{F}}) \cap \rho(A_{\mathsf{G}})$ we have*

$$(A-z)^{-1} = (A_{\mathsf{F}} - z)^{-1} \oplus (A_{\mathsf{G}} - z)^{-1}.$$

2.2 Spectrum of bounded operators

2.2.1 General properties

Proposition 2.19. *Let $A \in \mathcal{L}(\mathsf{E})$. Then $\sigma(A)$ is compact and included in $D(0, \|A\|_{\mathcal{L}(\mathsf{E})})$.*

Proof. Let $z \in \mathbb{C}$ such that $|z| > \|A\|$. Then we have

$$A - z = -z\left(\mathrm{Id} - \frac{A}{z}\right).$$

Since

$$\left\|\frac{A}{z}\right\| = \frac{\|A\|}{|z|} < 1,$$

the operator $\mathrm{Id} - \frac{A}{z}$ is invertible with inverse given by the Neumann series $\sum_{k\in\mathbb{N}} (\frac{A}{z})^k$. This proves that $A - z$ is invertible with inverse

$$(A - z)^{-1} = -\sum_{k\in\mathbb{N}} \frac{A^k}{z^{k+1}}. \tag{2.5}$$

In particular, $\sigma(A)$ is included in $D(0, \|A\|_{\mathcal{L}(\mathsf{E})})$ so it is bounded. Since it is closed by Proposition 2.17, it is compact. □

Proposition 2.20. *Assume that $\mathsf{E} \neq \{0\}$. Let $A \in \mathcal{L}(\mathsf{E})$. Then $\sigma(A) \neq \varnothing$.*

Proof. Assume by contradiction that $\rho(A) = \mathbb{C}$. For $z \in \mathbb{C}$ such that $|z| \geqslant 2\|A\|_{\mathcal{L}(\mathsf{E})}$ we have by (2.5)

$$\left\|(A - z)^{-1}\right\|_{\mathcal{L}(\mathsf{E})} \leqslant \frac{1}{|z|} \sum_{k=0}^{\infty} \left(\frac{\|A\|_{\mathcal{L}(\mathsf{E})}}{|z|}\right)^k \leqslant \frac{2}{|z|}. \tag{2.6}$$

Let $\varphi \in \mathsf{E}\setminus\{0\}$ and $\ell \in \mathsf{E}'$. The map $z \mapsto \ell((A - z)^{-1}\varphi)$ is holomorphic on $\mathbb{C}$ and bounded. Thus it is constant by the Liouville Theorem. By the previous estimate, its value must be 0. In particular, $\ell(A^{-1}\varphi) = 0$ for all $\ell \in \mathsf{E}'$. By the Hahn-Banach Theorem, we have $A^{-1}\varphi = 0$. This gives a contradiction and proves that $\rho(A) \neq \mathbb{C}$. □

Remark 2.21. In the real case we know from the finite dimensional case that the spectrum of a bounded operator can be empty.

Remark 2.22. An unbounded operator can have empty resolvent set (see Exercise 2.6) or an empty spectrum (see Exercise 2.7).

Example 2.23. We consider on $\ell^2(\mathbb{N})$ the shift operators of Example 1.2. We have

$$\sigma_{\mathsf{p}}(S_r) = \varnothing \quad \text{and} \quad \sigma_{\mathsf{p}}(S_\ell) = D(0,1).$$

By Proposition 2.19, $\sigma(S_\ell)$ is closed and contained in $\overline{D}(0,1)$, so $\sigma(S_\ell) = \overline{D}(0,1)$. Finally, since $S_r^* = S_\ell$, we also have $\sigma(S_r) = \overline{D}(0,1)$ by Proposition 2.10.

✎ Ex. 2.12

2.2.2 Spectral radius

Definition 2.24. *Let $A \in \mathcal{L}(\mathsf{E})$. We define the spectral radius of A by*

$$r(A) = \sup_{\lambda\in\sigma(A)} |\lambda|.$$

By Proposition 2.19 we already know that $r(A) \leqslant \|A\|_{\mathcal{L}(\mathsf{E})}$. The equality is not true in general. Consider for instance the matrix

$$A_\alpha = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$$

for $\alpha \in \mathbb{C}$. We have $\sigma(A) = \{1\}$ and $\|A\|_{\mathcal{L}(\mathbb{C}^2)} \to +\infty$ as $|\alpha| \to +\infty$. In general we have at least the following result.

Proposition 2.25 (Gelfand's Formula)**.** *Let $A \in \mathcal{L}(\mathsf{E})$. We have*

$$r(A) = \inf_{n \in \mathbb{N}^*} \|A^n\|_{\mathcal{L}(\mathsf{E})}^{\frac{1}{n}} = \lim_{n \to \mathbb{N}^*} \|A^n\|_{\mathcal{L}(\mathsf{E})}^{\frac{1}{n}}.$$

Example 2.26. Check that A_α satisfies the Gelfand Formula.

Proof. • Assume that there exists $N \in \mathbb{N}$ such that $A^N = 0$. Then $A^n = 0$ for all $n \geqslant N$. Let $z \in \mathbb{C} \setminus \{0\}$. Then $(z^{-1}A - 1)$ is invertible with inverse

$$\left(\frac{A}{z} - 1\right)^{-1} = -\sum_{n=0}^{N-1} \left(\frac{A}{z}\right)^n.$$

This proves that $A - z = z(z^{-1}A - 1)$ is invertible. Thus $\sigma(A) \subset \{0\}$. Since $\sigma(A) \neq \emptyset$, we have $\sigma(A) = \{0\}$ and the proposition is proved in this case. Now we assume that $A^n \neq 0$ for all $n \in \mathbb{N}$.

• For $n \in \mathbb{N}$ we set $u_n = \ln(\|A^n\|)$. For $m, p \in \mathbb{N}^*$ we have by (1.1)

$$u_{m+p} \leqslant u_m + u_p.$$

Let $p \in \mathbb{N}^*$. Let $n \in \mathbb{N}^*$ and $(q, r) \in \mathbb{N} \times [\![0, p-1]\!]$ such that $n = qp + r$. Then we have

$$\frac{u_n}{n} \leqslant \frac{qu_p + u_r}{qp + r} \leqslant \frac{u_p}{p} + \frac{u_r}{n},$$

so

$$\limsup_{n \to \infty} \frac{u_n}{n} \leqslant \frac{u_p}{p}.$$

Then for all $p \in \mathbb{N}^*$ we have

$$\limsup_{n \to \infty} \|A^n\|^{\frac{1}{n}} \leqslant \|A^p\|^{\frac{1}{p}}$$

Thus

$$\limsup_{n \in \infty} \|A^n\|^{\frac{1}{n}} \leqslant \inf_{p \in \mathbb{N}^*} \|A^p\|^{\frac{1}{p}}.$$

This implies that

$$\|A^n\|^{\frac{1}{n}} \xrightarrow[n \to \infty]{} \inf_{p \in \mathbb{N}^*} \|A^p\|^{\frac{1}{p}},$$

which gives the second inequality of the proposition.

• We set $\tilde{r}(A) = \lim \|A^n\|^{\frac{1}{n}}$. For $z \in \mathbb{C}$ we have $\ker(A - z) \subset \ker(A^n - z^n)$ and

$$A^n - z^n = (A - z) \sum_{k=0}^{n-1} z^k A^{n-1-k},$$

so $\mathsf{Ran}(A^n - z^n) \subset \mathsf{Ran}(A - z)$. Thus, if $A^n - z^n$ is bijective, then so is $A - z$. Now let $\lambda \in \sigma(A)$. We have $\lambda^n \in \sigma(A^n)$. By Proposition 2.19 we have $|\lambda|^n = |\lambda^n| \leqslant \|A^n\|$, so $|\lambda| \leqslant \|A^n\|^{\frac{1}{n}}$ for all $n \in \mathbb{N}$, and hence $|\lambda| \leqslant \tilde{r}(A)$. This proves that $r(A) \leqslant \tilde{r}(A)$.

• Let $z \in \mathbb{C}$ with $|z| > \tilde{r}(A)$. Then the power series

$$-\sum_{n \in \mathbb{N}} \frac{A^n}{z^{n+1}}$$

is convergent in $\mathcal{L}(\mathsf{E})$ and defines a bounded inverse for $(A - z)$. This proves that $\tilde{r}(A) \leqslant r(A)$ and concludes the proof. □

2.2.3 Normal bounded operators

Definition 2.27. *We say that $A \in \mathcal{L}(\mathcal{H})$ is normal if $AA^* = A^*A$.*

Example 2.28.
- The multiplication operator M_w (see Example 1.4) is normal.
- Since $S_r S_\ell \neq S_\ell S_r$, the shift operators S_ℓ and S_r (see Example 1.2) are not normal.

Remark 2.29. If A is normal and invertible, then A^{-1} is normal.

Proposition 2.30. *Let $A \in \mathcal{L}(\mathcal{H})$ be a normal operator.*

(i) *For $\varphi \in \mathcal{H}$ we have $\|A\varphi\| = \|A^*\varphi\|$. In particular,* $\ker(A^*) = \ker(A)$.

(ii) *If λ and μ are two distinct eigenvalues of A, then* $\ker(A-\lambda)$ *and* $\ker(A-\mu)$ *are orthogonal.*

Proof. • Let $\varphi \in \mathcal{H}$. We have

$$\|A\varphi\|^2 = \langle A^*A\varphi, \varphi\rangle = \langle AA^*\varphi, \varphi\rangle = \|A^*\varphi\|^2,$$

which gives the first statement.
• Let $\varphi \in \ker(A-\lambda)$ and $\psi \in \ker(A-\mu)$. By the first statement we also have $\psi \in \ker((A-\mu)^*) = \ker(A^* - \overline{\mu})$. Then we have

$$(\lambda - \mu)\langle \varphi, \psi\rangle = \langle \lambda\varphi, \psi\rangle - \langle \varphi, \overline{\mu}\psi\rangle = \langle A\varphi, \psi\rangle - \langle \varphi, A^*\psi\rangle = 0.$$

Since $\lambda \neq \mu$, this proves that $\langle \varphi, \psi\rangle = 0$, so $\ker(A-\lambda)$ and $\ker(A-\mu)$ are orthogonal. □

In Section 2.2.2 we have said that the spectral radius of a bounded operator can be smaller that its norm. This is not the case for a normal operator.

Proposition 2.31. *Let $A \in \mathcal{L}(\mathsf{E})$ be normal. We have $r(A) = \|A\|_{\mathcal{L}(\mathcal{H})}$.*

Proof. • Assume that $A = A^*$ (A is selfadjoint). We always have $\|A^2\| \leqslant \|A\|^2$. For $\varphi \in \mathcal{H}$ we have

$$\|A\varphi\|^2 = \langle A^*A\varphi, \varphi\rangle = \langle A^2\varphi, \varphi\rangle \leqslant \|A^2\| \, \|\varphi\|^2.$$

This proves that $\|A\|^2 \leqslant \|A^2\|$, and hence $\|A\|^2 = \|A^2\|$. Since A^{2^k} is selfadjoint for all $k \in \mathbb{N}$, we deduce by induction that $\|A^{2^k}\| = \|A\|^{2^k}$ for all $k \in \mathbb{N}$. Then, by the Gelfand Formula we have

$$r(A) = \lim_{k\to\infty} \|A^{2^k}\|^{\frac{1}{2^k}} = \|A\|.$$

• Now we only assume that A is normal. We have $\|A^*A\| = \|A\|^2$ (exercise). On the other hand, since A^*A is selfadjoint we have $r(A^*A) = \|A^*A\|$, so $r(A^*A) = \|A\|^2$. On the other hand, since A is normal,

$$r(A^*A) = \lim_{n\to\infty} \|(A^*A)^n\|^{\frac{1}{n}} = \lim_{n\to\infty} \|(A^n)^*A^n\|^{\frac{1}{n}} = \lim_{n\to\infty} \|A^n\|^{\frac{2}{n}} = r(A)^2.$$

This proves that $r(A) = \|A\|$. □

Remark 2.32. If $A \in \mathcal{L}(\mathcal{H})$ is a normal operator such that $\sigma(A) = \{0\}$ then $A = 0$. This is not the case in general, since every nilpotent operator has spectrum $\{0\}$.

Theorem 2.33. *Let $A \in \mathcal{L}(\mathcal{H})$ a normal operator. For $z \in \rho(A)$ we have*

$$\|(A-z)^{-1}\|_{\mathcal{L}(\mathcal{H})} = \frac{1}{\mathsf{dist}(z, \sigma(A))}.$$

Proof. Let $z \in \rho(A)$. By Proposition 2.11 we have

$$\sigma((A-z)^{-1}) = \{(\zeta - z)^{-1}, \zeta \in \sigma(A)\}.$$

Since $(A-z)^{-1}$ is normal, we deduce by Proposition 2.31

$$\|(A-z)^{-1}\| = r((A-z)^{-1}) = \sup_{\lambda\in\sigma(A)} |\lambda - z|^{-1} = \frac{1}{\inf_{\lambda\in\sigma(A)} |\lambda - z|} = \frac{1}{\mathsf{dist}(z, \sigma(A))}.$$ □

2.3 Riesz projections

2.3.1 Separation of the spectrum

The interest of the resolvent is that it is a bounded operator which completely characterize the operator. Moreover, since it is analytic, we can use all the tools from complex analysis. In the following section we give a first application of the resolvent for the analysis of an operator.

Let E be a Banach space and let A be a closed operator on E.

Proposition 2.34. *Let $z_0 \in \mathbb{C}$ and $r_0 > 0$. Assume that $\mathcal{C}(z_0, r_0) \subset \rho(A)$. We define*

$$\Pi = -\frac{1}{2i\pi} \int_{\mathcal{C}(z_0,r_0)} (A-\zeta)^{-1} \,\mathrm{d}\zeta = -\frac{1}{2\pi} \int_0^{2\pi} \big(A - (z_0 + r_0 e^{i\theta})\big)^{-1} r_0 e^{i\theta} \,\mathrm{d}\theta.$$

We also set $\mathsf{F} = \mathsf{Ran}(\Pi)$ and $\mathsf{G} = \ker(\Pi)$.

(i) Π *is a (not necessarily orthogonal) projection of* E.

(ii) $\mathsf{F} \subset \mathsf{Dom}(A)$.

(iii) $\Pi A \subset A \Pi$.

(iv) $\sigma(A_{\mathsf{F}}) = \sigma(A) \cap D(z_0, r_0)$ *and* $\sigma(A_{\mathsf{G}}) = \sigma(A) \backslash \overline{D}(z_0, r_0)$.

Remark 2.35. In Proposition 2.34 we consider for simplicity the case where Π is defined by an integral on a circle. But we can similarly consider the integral on any rectifiable simple closed curve in $\rho(A)$ (see [Kat80, § III.6.4]).

Remark 2.36. Π is defined by the integral on a line segment of a continuous function with values in the Banach space $\mathcal{L}(\mathsf{E})$. This can be understood in the sense of Riemann integrals and this defines a bounded operator on E. In particular we have in $\mathcal{L}(\mathsf{E})$

$$\Pi = \lim_{n \to +\infty} \Pi_n, \quad \text{where} \quad \Pi_n = -\frac{1}{n} \sum_{k=1}^{n} \big(A - (z_0 + r_0 e^{i\theta_{n,k}})\big)^{-1} r_0 e^{i\theta_{n,k}}, \quad \theta_{n,k} = \frac{2k\pi}{n}.$$

Then if T is a closed operator with $\mathsf{Dom}(A) \subset \mathsf{Dom}(T)$, we have

$$T\Pi = -\frac{1}{2i\pi} \int_{\mathcal{C}(z_0,r_0)} T(A-\zeta)^{-1} \,\mathrm{d}\zeta.$$

Indeed, for $\varphi \in \mathcal{H}$ and $n \in \mathbb{N}^*$ we have $\Pi_n \varphi \in \mathsf{Dom}(A) \subset \mathsf{Dom}(T)$, $\Pi_n \varphi \to \Pi \varphi$ and

$$T\Pi_n \varphi = -\frac{1}{n} \sum_{k=1}^{n} T\big(A - (z_0 + r_0 e^{i\theta_{n,k}})\big)^{-1} r_0 e^{i\theta_{n,k}} \varphi$$

Proof. • For $\varphi \in \mathsf{E}$ and $\ell \in \mathsf{E}'$ we have

$$\ell(\Pi\varphi) = -\frac{1}{2i\pi} \int_{\mathcal{C}(z_0,r_0)} \ell\big((A-z)^{-1}\varphi\big) \,\mathrm{d}z.$$

Since $\rho(A)$ is open in $\mathbb{C}$, there exists $R_1 \in]0, r_0[$ and $R_2 > r_0$ such that $D(0, R_2) \backslash \overline{D}(0, R_1) \subset \rho(A)$. Let $\varphi \in \mathsf{E}$ and $\ell \in \mathsf{E}^*$. Since the map $\zeta \mapsto \ell\big((A-\zeta)^{-1}\varphi\big)$ is holomorphic on $\rho(A)$, we can replace r_0 by any $r \in]R_1, R_2[$ in the expression of Π.

• Let $r_1, r_2 \in]R_1, R_2[$ with $r_1 < r_2$. We can write

$$\Pi^2 = \frac{1}{(2i\pi)^2} \int_{\zeta_1 \in \mathcal{C}(z_0,r_1)} \int_{\zeta_2 \in \mathcal{C}(z_0,r_2)} (A-\zeta_1)^{-1} (A-\zeta_2)^{-1} \,\mathrm{d}\zeta_2 \,\mathrm{d}\zeta_1.$$

By the resolvent identity we have

$$\Pi^2 = \frac{1}{(2i\pi)^2} \int_{\zeta_1 \in \mathcal{C}(z_0,r_1)} \int_{\zeta_2 \in \mathcal{C}(z_0,r_2)} \frac{(A-\zeta_1)^{-1} - (A-\zeta_2)^{-1}}{\zeta_1 - \zeta_2} \,\mathrm{d}\zeta_2 \,\mathrm{d}\zeta_1.$$

Then, by the Fubini Theorem,

$$\Pi^2 = -\frac{1}{(2i\pi)^2}\int_{\zeta_1\in\mathcal{C}(z_0,r_1)} (A-\zeta_1)^{-1}\left(\int_{\zeta_2\in\mathcal{C}(z_0,r_2)} \frac{1}{\zeta_2-\zeta_1}\,\mathrm{d}\zeta_2\right)\mathrm{d}\zeta_1 - \frac{1}{(2i\pi)^2}\int_{\zeta_2\in\mathcal{C}(z_0,r_2)} (A-\zeta_2)^{-1}\left(\int_{\zeta_1\in\mathcal{C}(z_0,r_1)} \frac{1}{\zeta_1-\zeta_2}\,\mathrm{d}\zeta_1\right)\mathrm{d}\zeta_2.$$

We look at the integral in brackets for each term. For the second term, for any $\zeta_2 \in \mathcal{C}(z_0,r_2)$ the map $\zeta_1 \mapsto 1/(\zeta_1-\zeta_2)$ is holomorphic on $D(z_0,r_2)$, so the integral vanishes. For the first term, we get by the Cauchy Theorem that the integral is equal to $2i\pi$ for all $\zeta_1 \in \mathcal{C}(z_0,r_1)$. Then

$$\Pi^2 = -\frac{1}{2i\pi}\int_{\zeta_2\in\mathcal{C}(z_0,r_2)} (A-\zeta_2)^{-1}\,\mathrm{d}\zeta_2 = \Pi.$$

This proves that Π is a projection of E.

- Let $\varphi \in \mathsf{F}$ and $\psi \in \mathsf{E}$ such that $\varphi = \Pi\psi$. For $n \in \mathbb{N}^*$ we set $\varphi_n = \Pi_n\psi \in \mathsf{Dom}(A)$. Then $\varphi_n \to \varphi$ in E. Moreover,

$$\begin{aligned} A\varphi_n &= -\frac{1}{n}\sum_{k=1}^{n} A\big(A-(z_0+r_0e^{i\theta_{n,k}})\big)^{-1} r_0 e^{i\theta_{n,k}}\psi \\ &= -\frac{1}{n}\sum_{k=1}^{n}\big(\mathrm{Id} + (z_0+r_0e^{i\theta_{n,k}})\big(A-(z_0+r_0e^{i\theta_{n,k}})\big)^{-1}\big) r_0 e^{i\theta_{n,k}}\psi \\ &\xrightarrow[n\to\infty]{} -\frac{1}{2i\pi}\int_{\mathcal{C}(z_0,r_0)} \big(\mathrm{Id} + \zeta(A-\zeta)^{-1}\big)\psi\,\mathrm{d}\zeta = -\frac{1}{2i\pi}\int_{\mathcal{C}(z_0,r_0)} \zeta(A-\zeta)^{-1}\psi\,\mathrm{d}\zeta. \end{aligned}$$

Since A is closed this proves that $\varphi \in \mathsf{Dom}(A)$ (and $A\varphi = -\frac{1}{2i\pi}\int_{\mathcal{C}(z_0,r_0)} \zeta(A-\zeta)^{-1}\psi\,\mathrm{d}\zeta$).

- Let $\varphi \in \mathsf{Dom}(A)$. Since A commutes with its resolvent, we have $A\Pi_n\varphi = \Pi_n A\varphi$ for all $n \in \mathbb{N}^*$. Since $\Pi_n\varphi \to \Pi\varphi$ and $A\Pi_N\varphi = \Pi_N A\varphi \to \Pi A\varphi$, we get by closedness of A that $\Pi\varphi \in \mathsf{Dom}(A)$ and $A\Pi\varphi = \Pi A\varphi$.
- Let $z \in \rho(A_\mathsf{F})\backslash D(z_0,r_0)$. Let $r \in]R_1,r_0[$. We have on F

$$\begin{aligned} (A_\mathsf{F}-z)^{-1} &= (A_\mathsf{F}-z)^{-1}\Pi \\ &= -\frac{1}{2i\pi}\int_{\zeta\in\mathcal{C}(z_0,r)} (A_\mathsf{F}-z)^{-1}(A_\mathsf{F}-\zeta)^{-1}\,\mathrm{d}\zeta \\ &= -\frac{1}{2i\pi}\int_{\zeta\in\mathcal{C}(z_0,r)} \frac{(A_\mathsf{F}-z)^{-1}-(A_\mathsf{F}-\zeta)^{-1}}{z-\zeta}\,\mathrm{d}\zeta \\ &= \frac{1}{2i\pi}\int_{\zeta\in\mathcal{C}(z_0,r)} \frac{(A_\mathsf{F}-\zeta)^{-1}}{z-\zeta}\,\mathrm{d}\zeta. \end{aligned}$$

The right-hand side is bounded uniformly in $z \in \rho(A_\mathsf{F})\backslash D(z_0,r_0)$. By Proposition 2.17 this implies that

$$\sigma(A_\mathsf{F}) \subset D(z_0,r_0). \tag{2.7}$$

Now let $z \in \rho(A_\mathsf{G}) \cap D(z_0,r_0)$ and $r \in]r_0,R_2[$. We have on G

$$\begin{aligned} (A_\mathsf{G}-z)^{-1} &= (A_\mathsf{G}-z)^{-1}(1-\Pi) \\ &= (A_\mathsf{G}-z)^{-1} - \frac{1}{2i\pi}\int_{\zeta\in\mathcal{C}(z_0,r)} \frac{(A_\mathsf{G}-z)^{-1}-(A_\mathsf{G}-\zeta)^{-1}}{\zeta-z}\,\mathrm{d}\zeta \\ &= \frac{1}{2i\pi}\int_{\zeta\in\mathcal{C}(z_0,r)} \frac{(A_\mathsf{G}-\zeta)^{-1}}{z-\zeta}\,\mathrm{d}\zeta. \end{aligned}$$

This is bounded uniformly in $z \in \rho(A_\mathsf{G}) \cap D(z_0,r_0)$, so

$$\sigma(A_\mathsf{G}) \subset \mathbb{C}\backslash\overline{D}(0,r_0). \tag{2.8}$$

Finally, with Proposition 2.18 and (2.7)-(2.8) we deduce that $\sigma(A_\mathsf{F}) = \sigma(A) \cap D(0,r_0)$ and $\sigma(A_\mathsf{G}) = \sigma(A)\backslash\overline{D}(0,r_0)$. □ Ex. 2.13-2.14

2.3.2 Isolated eigenvalues

Definition 2.37. *We consider an operator A on E. Assume that $\lambda \in \mathbb{C}$ is an isolated point in the spectrum of A. Let $r_0 > 0$ such that $\sigma(A) \cap D(\lambda, r_0) = \{\lambda\}$ and $r \in]0, r_0[$. Then the Riesz projection of A at λ is*

$$\Pi_\lambda = -\frac{1}{2i\pi} \int_{\mathcal{C}(\lambda,r)} (A-z)^{-1} \, \mathrm{d}z \tag{2.9}$$

Remark 2.38. The definition of Π_λ does not depend on the choice of $r \in]0, r_0[$. More generally, we can replace $\mathcal{C}(\lambda, r)$ any closed curve in $D(\lambda_r 0) \setminus \{\lambda\}$ enclosing λ exactly once in the direct sense.

Definition 2.39. *Let λ be an isolated element of $\sigma(A)$. The algebraic multiplicity of λ is $\mathsf{dim}(\mathsf{Ran}(\Pi_\lambda))$, where Π_λ is the Riesz projection at λ.*

Example 2.40. Let $\alpha, \beta \in \mathbb{C}$ distinct and

$$M = \begin{pmatrix} \alpha & 1 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & \beta & 1 \\ 0 & 0 & 0 & 0 & \beta \end{pmatrix}$$

Then $\sigma(M) = \{\alpha, \beta\}$ and α is an eigenvalue of geometric multiplicity 2. For $z \in \mathbb{C} \setminus \{\alpha, \beta\}$ we have

$$(M-z)^{-1} = \begin{pmatrix} (\alpha-z)^{-1} & -(\alpha-z)^{-2} & 0 & 0 & 0 \\ 0 & (\alpha-z)^{-1} & 0 & 0 & 0 \\ 0 & 0 & (\alpha-z)^{-1} & 0 & 0 \\ 0 & 0 & 0 & (\beta-z)^{-1} & -(\beta-z)^{-2} \\ 0 & 0 & 0 & 0 & (\beta-z)^{-1} \end{pmatrix}.$$

Then for $r \in]0, |\alpha - \beta|\,[$ we have

$$\Pi_\alpha = -\frac{1}{2i\pi} \int_{\mathcal{C}(\alpha,r)} (M-z)^{-1} \, \mathrm{d}z = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

so α has algebraic multplicity 3 and Π_α is the projection of $\mathbb{C}^5$ on $\ker((M-\alpha)^2)$ parallel to $\ker((M-\beta)^2)$.

Proposition 2.41. *We use the notation of Proposition 2.34.*

(i) *Let $\lambda \in D(z_0, r_0)$ and $m \in \mathbb{N}^*$. Then $\ker((A-\lambda)^m) \subset \mathsf{F}$.*

(ii) *Let $\lambda \in \mathbb{C} \setminus \overline{D}(z_0, r_0)$ and $m \in \mathbb{N}^*$. Then $\ker((A-\lambda)^m) \subset \mathsf{G}$.*

Proof. • Let $\varphi \in \mathsf{Dom}(A)$ such that $(A-\lambda)\varphi \in \mathsf{F}$. For $\zeta \in \mathcal{C}(z_0, r_0)$ we have

$$(A-\zeta)^{-1}\varphi = (\lambda-\zeta)^{-1}\varphi - (\lambda-\zeta)^{-1}(A-\zeta)^{-1}(A-\lambda)\varphi,$$

Then

$$\begin{aligned} \Pi\varphi &= -\frac{1}{2i\pi} \int_{\mathcal{C}(z_0,r)} \left((\lambda-\zeta)^{-1}\varphi - (\lambda-\zeta)^{-1}(A-\zeta)^{-1}(A-\lambda)\varphi \right) \mathrm{d}\zeta \\ &= \varphi + \frac{1}{2i\pi} \int_{\mathcal{C}(z_0,r)} (\lambda-\zeta)^{-1}(A-\zeta)^{-1}(A-\lambda)\varphi \, \mathrm{d}\zeta. \end{aligned}$$

Since

$$\forall \zeta \in \mathcal{C}(z_0, r), \quad (A-\zeta)^{-1}(A-\lambda)(1-\Pi)\varphi = (A-\zeta)^{-1}(1-\Pi)(A-\lambda)\varphi = 0,$$

we deduce

$$(1-\Pi)\varphi = (1-\Pi)^2\varphi = -\frac{1}{2i\pi}\int_{\mathcal{C}(z_0,r)} (\lambda-\zeta)^{-1}(A-\zeta)^{-1}(1-\Pi)(A-\lambda)\varphi \,\mathrm{d}\zeta = 0.$$

This proves that $\varphi \in \mathsf{F}$. Then we can prove by induction on $m \in \mathbb{N}^*$ that $\ker((A-\lambda)^m) \subset \mathsf{F}$. The second statement is similar. □

Remark 2.42. Let λ be an isolated element of $\sigma(A)$. Since $\ker(A-\lambda) \subset \mathsf{Ran}(\Pi_A(\lambda))$ the geometric multiplicity of λ (which can be 0 if λ is not an eigenvalue) is not greater than its algebraic multiplicity.

Proposition 2.43. *Assume that λ is an isolated point of $\sigma(A)$ such that $\mathsf{Ran}(\Pi_\lambda)$ is of finite dimension $m \in \mathbb{N}^*$. Then λ is an eigenvalue and*

$$\mathsf{Ran}(\Pi_\lambda) = \ker((A-\lambda)^m).$$

Proof. The restriction A_F of A to $\mathsf{F} = \mathsf{Ran}(\Pi_\lambda)$ is an operator on the finite dimensional space F, with $\sigma(A_\mathsf{F}) = \{\lambda\}$. Then the result follows from the finite dimensional case. □

Remark 2.44. Notice that (see Exercise 2.14)

- an isolated point λ of $\sigma(A)$ is not necessarily an eigenvalue (in this case we have $\mathsf{dim}(\mathsf{Ran}(\Pi_\lambda)) = +\infty$ by Proposition 2.43);
- as isolated eigenvalue of finite geometric multiplicity can have infinite algebraic multiplicity.

Definition 2.45. *Let A be a closed operator on E. Let $\lambda \in \mathbb{C}$. We say that λ belongs to the discrete spectrum $\sigma_{\mathsf{disc}}(A)$ of A and λ is an isolated eigenvalue of A with finite algebraic multiplicity.*

Example 2.46.
- Assume that E has infinite dimension. Then $\sigma_{\mathsf{disc}}(\mathrm{Id}_\mathsf{E}) = \varnothing$ (the spectrum is given by the eigenvalue 1, but it has infinite dimension.
- The harmonic oscillator (see Section 2.1.2) has purely discrete spectrum: $\sigma_{\mathsf{disc}}(H) = \sigma(H)$.
- The usual Laplacian on $\mathbb{R}^d$ (see Example 2.6) has empty discrete spectrum: $\sigma(-\Delta) = \varnothing$.

✎ Ex. 2.15

2.3.3 Additional topic: regularity of the spectrum with respect to a parameter

Lemma 2.47. *Let Π_1 and Π_2 be two projections on E. Assume that $\|\Pi_2 - \Pi_1\|_{\mathcal{L}(\mathsf{E})} < 1$. Then*

$$\mathsf{dim}(\mathsf{Ran}(\Pi_1)) = \mathsf{dim}(\mathsf{Ran}(\Pi_2)).$$

Proof. Let $\pi : \mathsf{Ran}(\Pi_2) \to \mathsf{Ran}(\Pi_1)$ be the restriction of Π_1 to $\mathsf{Ran}(\Pi_2)$. This is a continuous linear map. For $\varphi \in \ker(\pi)$ we have $\Pi_2(\varphi) = \varphi$ and $\Pi_1(\varphi) = 0$ so

$$\|\varphi\| = \|\Pi_2(\varphi) - \Pi_1(\varphi)\| \leqslant \|\Pi_2 - \Pi_2\| \|\varphi\|,$$

so $\varphi = 0$. This implies that $\mathsf{dim}(\mathsf{Ran}(\Pi_1)) \geqslant \mathsf{dim}(\mathsf{Ran}(\Pi_1))$. Interverting the roles of Π_1 and Π_2 gives the reverse inequality and concludes the proof. □

Proposition 2.48. *Let ω be a connected subset of $\mathbb{C}$. Let $(A_\alpha)_{\alpha\in\mathbb{C}}$ be a family of linear operators on E. Assume that there exists $\lambda_0 \in \mathbb{C}$ and $r_0 > 0$ such that $\mathcal{C}(\lambda_0, r_0) \subset \rho(A_\alpha)$ for all $\alpha \in \omega$. Assume that the map*

$$\left\{ \begin{array}{ccc} \omega \times \mathcal{C}(\lambda_0, r_0) & \to & \mathcal{L}(\mathsf{E}) \\ (\alpha, z) & \mapsto & (A_\alpha - z)^{-1} \end{array} \right.$$

is continuous. We denote by Π_α the Riesz projection of A_α on $\mathcal{C}(\lambda_0, r)$.

(i) $\mathsf{dim}(\mathsf{Ran}(\Pi_\alpha))$ *does not depend on* $\alpha \in \omega$.

(ii) *Assume that* $\mathsf{dim}(\mathsf{Ran}(\Pi_\alpha)) = 1$. *Then for all* $\alpha \in \omega$ *the operator* A_α *has a unique simple eigenvalue* λ_α *in* $D(\lambda_0, r)$. *Moreover the maps* $\alpha \mapsto \lambda_\alpha$ *and* $\alpha \mapsto \Pi_\alpha$ *are continuous on* ω. *If moreover* $\alpha \mapsto (A_\alpha - z)^{-1}$ *is holomorphic on* ω *for all* $z \in \mathcal{C}(\lambda_0, r_0)$, *then* $\alpha \mapsto \Pi_\alpha$ *and* $\alpha \mapsto \lambda_\alpha$ *are holomorphic.*

Proof. • Let $\alpha_0 \in \omega$. Since $\mathcal{C}(\lambda_0, r)$ is compact, there exists a neighborhood $\mathcal{V}$ of α_0 in ω such that for all $\alpha \in \mathcal{V}$ and $\zeta \in \mathcal{C}(\lambda_0, r)$ we have

$$\left\| (A_\alpha - \zeta)^{-1} - (A_{\alpha_0} - \zeta)^{-1} \right\| \leqslant \frac{1}{2r_0}.$$

Then we have

$$\|\Pi_\alpha - \Pi_{\alpha_0}\| \leqslant \frac{1}{2},$$

and, by Lemma 2.47, $\mathsf{Ran}(\Pi_\alpha) = \mathsf{Ran}(\Pi_{\alpha_0})$ for all $\alpha \in \mathcal{V}$. Then $\mathsf{Ran}(\Pi_\alpha)$ is locally constant, so it is constant on the connected set ω.

• By continuity under the integral sign, we see that Π_α is continuous with respect to α. If $(A_\alpha - \zeta)^{-1}$ is holomorphic with respect to α for all $\zeta \in \mathcal{C}(l_0, r)$, then Π_α is holomorphic by complex differentiation under the integral sign.

• Now assume that $\mathsf{Ran}(\Pi_\alpha) = 1$ for all $\alpha \in \omega$. Let $\alpha_0 \in \omega$ and $\psi \in \mathsf{Ran}(\Pi_{\alpha_0})$ with $\|\psi\| = 1$. Then ψ is an eigenvector corresponding to an eigenvalue $\lambda_{\alpha_0} \in D(\lambda_0, r)$. For $\alpha \in \omega$ we set $\psi_\alpha = \Pi_\alpha \psi$. For α close to α_0 we have $\psi_\alpha \neq 0$. Then ψ_α is an eigenvector of A_α corresponding to an eigenvalue λ_α, and it is continuous (holomorphic if the resolvent is holomorphic) with respect to α. Finally we have $(A_\alpha - z)^{-1}\psi_\alpha = (\lambda_\alpha - z)^{-1}\psi_\alpha$. Taking the inner product with ψ gives

$$\left\langle \psi, (A_\alpha - z)^{-1}\psi_\alpha \right\rangle = (\lambda_\alpha - z)^{-1} \langle \psi, \psi_\alpha \rangle .$$

We have $\langle \psi, \psi_\alpha \rangle = 1$ when $\alpha = \alpha_0$, so this does not vanish on a neighborhood of α_0. This gives

$$(\lambda_\alpha - z)^{-1} = \frac{\left\langle \psi, (A_\alpha - z)^{-1}\psi_\alpha \right\rangle}{\langle \psi, \psi_\alpha \rangle}.$$

Thus $(\lambda_\alpha - z)^{-1}$ is continuous (holomorphic if the resolvent is holomorphic) for α an a neighborhood of α_0, and so is λ_α. ☐

Proposition 2.49 (Analytic family of type A)**.** *Let* ω *be an open subset of* $\mathbb{C}$. *Let* $(A_\alpha)_{\alpha \in \omega}$ *be a family of closed operators on* E. *We assume that*

(i) *the operators* A_α, $\alpha \in \omega$, *have the same domain* $\mathcal{D}$ *;*

(ii) *for all* $\psi \in \mathcal{D}$ *the map* $\alpha \mapsto A_\alpha \psi \in \mathcal{H}$ *is holomorphic on* ω.

Let $\alpha_0 \in \omega$ *and* $z_0 \in \rho(A_{\alpha_0})$. *Then there exists* $r > 0$ *such that* $z \in \rho(A_\alpha)$ *for all* $\alpha \in D(\alpha_0, r)$ *and* $z \in D(z_0, r)$ *and the map*

$$(\alpha, z) \mapsto (A_\alpha - z)^{-1}$$

is continuous on $D(\alpha_0, r) \times D(z_0, r)$ *and analytic in* $D(\alpha_0, r)$ *for all* $z \in D(z_0, r)$.

Proof. For $\alpha \in \omega$ and $z \in \mathbb{C}$ we have

$$(A_\alpha - z) = \Big(1 + \big((A_\alpha - A_{\alpha_0}) - (z - z_0)\big)(A_{\alpha_0} - z_0)^{-1}\Big)(A_{\alpha_0} - z_0).$$

Since $(A_{\alpha_0} - z_0)^{-1}$ maps $\mathcal{H}$ to $\mathcal{D}$, the operators $A_\alpha(A_{\alpha_0} - z_0)^{-1}$ and $A_{\alpha_0}(A_{\alpha_0} - z_0)^{-1}$ are well defined on $\mathcal{H}$. Since they are closed, they are bounded by the closed graph theorem. Then the function $\alpha \mapsto A_\alpha(A_{\alpha_0} - z)^{-1}$ is weakly holomorphic, and hence holomorphic by Proposition A.7. In particular it is continuous, so there exists $r > 0$ so small that $\left\|(A_{\alpha_0} - z_0)^{-1}\right\| < 1/(4r)$, $D(\alpha_0, r) \subset \omega$ and for all $\alpha \in D(\alpha_0, r)$ we have

$$\left\|(A_\alpha - A_{\alpha_0})(A_{\alpha_0} - z_0)^{-1}\right\|_{\mathcal{L}(\mathcal{H})} \leqslant \frac{1}{4}.$$

Then the map $(\alpha, z) \mapsto \big(1 + \big(A_\alpha - A_{\alpha_0}\big) - (z - z_0)\big)(A_{\alpha_0} - z_0)^{-1}\big)^{-1}$ is well defined and continuous on $D(\alpha_0, r) \times D(z_0, r)$, and analytic with respect to α for all $z \in D(z_0, r)$. We deduce that the same holds for $\alpha \mapsto (A_\alpha - z)^{-1}$. ☐

Proposition 2.50 (Analytic family of type B)**.** *Let $\mathcal{V}$ be a Hilbert space continuously and densely embedded in $\mathcal{H}$. Let ω be an open subset of $\mathbb{C}$. Let $(q_\alpha)_{\alpha\in\omega}$ be a family of continuous forms on $\mathcal{V}$ such that $\varphi \mapsto q_\alpha(\varphi) \in \mathbb{C}$ is analytic for all $\varphi \in \mathcal{V}$. Assume that there exist $\alpha_0 \in \omega$ and $z_0 \in \mathbb{C}$ such that $q_{\alpha_0} - z_0$ is coercive. Then there exists $r > 0$ such that $q_\alpha - z$ is coercive for all $\alpha \in D(\alpha_0, r)$ and $z \in D(z_0, r)$. For $\alpha \in D(\alpha_0, r)$ we denote by A_α the operator on $\mathcal{H}$ given by the representation theorem (see Theorem 1.71 and Remark 1.72). Then the map*

$$(\alpha, z) \mapsto (A_\alpha - z)^{-1}$$

is continuous on $D(\alpha_0, r) \times D(z_0, r)$ and holomorphic with respect to $\alpha \in D(\alpha_0, r)$ for all $z \in D(z_0, r)$.

Proof. We denote by Q_α the operator in $\mathcal{L}(\mathcal{V}, \mathcal{V}')$ associated with q_α (see (1.12)). For $\alpha \in \omega$ we have in $\mathcal{L}(\mathcal{V}, \mathcal{V}')$

$$(Q_\alpha - z) = \Big(1 + \big((Q_\alpha - Q_{\alpha_0}) - (z - z_0)\big)(Q_{\alpha_0} - z)^{-1}\Big)(Q_{\alpha_0} - z)$$

Since $(Q_{\alpha_0} - z)^{-1}$ maps $\mathcal{V}'$ to $\mathcal{V}$, the operators $Q_\alpha(Q_{\alpha_0} - z)^{-1}$ and $Q_{\alpha_0}(Q_{\alpha_0} - z)^{-1}$ are bounded on $\mathcal{V}'$. Then the function $\alpha \mapsto Q_\alpha(Q_{\alpha_0} - z)^{-1}$ is weakly holomorphic, and hence holomorphic by Proposition A.7. In particular it is continuous, so there exists $r > 0$ such that $\big\|(Q_{\alpha_0} - z_0)^{-1}\big\|_{\mathcal{L}(\mathcal{V}', \mathcal{V})} \leqslant 1/(4r)$, $D(\alpha_0, r) \subset \omega$ and for all $\alpha \in D(\alpha_0, r)$ we have

$$\big\|(Q_\alpha - Q_{\alpha_0})(Q_{\alpha_0} - z)^{-1}\big\|_{\mathcal{L}(\mathcal{V}')} \leqslant \frac{1}{4}.$$

Then the map $(\alpha, z) \mapsto \big(1 + \big((Q_\alpha - Q_{\alpha_0}) - (z - z_0)\big)(Q_{\alpha_0} - z)^{-1}\big)^{-1} \in \mathcal{L}(\mathcal{V}')$ is well defined and continuous on $D(\alpha_0, r) \times D(z_0, r)$, and analytic on $D(\alpha_0, r)$ for all $z \in D(z_0, r)$. We deduce that the same holds for $\alpha \mapsto (Q_\alpha - z)^{-1}$ in $\mathcal{L}(\mathcal{V}', \mathcal{V})$. Since $(Q_\alpha - z)^{-1}$ and $(A_\alpha - z)^{-1}$ coincide on $\mathcal{H}$, the conclusion follows. □

For the perturbation of a double eigenvalue, we refer to Exemple II.1.1 (page 64) in [Kat80]

2.4 Exercises

***Exercise* 2.1.** Let $a = (a_n)_{n\in\mathbb{N}} \in \ell^\infty(\mathbb{N})$. We consider the operator M_a given in Example 1.3. Prove that

$$\sigma_{\mathsf{p}}(M_a) = \{a_n, n \in \mathbb{N}\} \quad \text{and} \quad \sigma(M_a) = \overline{\sigma_{\mathsf{p}}(M_a)}.$$

***Exercise* 2.2.** Let $\mathcal{H}$ be a Hilbert space. Let A be a closed operator on $\mathcal{H}$. Assume that there exist a Hilbert basis $(\beta_n)_{n\in\mathbb{N}}$ of $\mathcal{H}$ and a complex sequence $(\lambda_n)_{n\in\mathbb{N}}$ such that

$$\mathsf{Dom}(A) = \left\{\varphi = \sum_{n=0}^{\infty} \varphi_n \beta_n \, : \, \sum_{n=0}^{\infty} |\lambda_n \varphi_n|^2 < \infty\right\},$$

and $A\beta_n = \lambda_n \beta_n$ for all $n \in \mathbb{N}$. Prove that

$$\sigma(A) = \overline{\{\lambda_n, n \in \mathbb{N}\}}.$$

***Exercise* 2.3.** We define on $\mathbb{R}$ the function w defined by

$$w(x) = \begin{cases} \frac{1}{x+1} & \text{if } x > 0, \\ 0 & \text{if } x \leqslant 0 \end{cases}$$

Then we consider on $L^2(\mathbb{R})$ the operator M_w of multiplication by w.
1. What is $\sigma(M_w)$?
2. What is $\sigma_{\mathsf{p}}(M_w)$? For each eigenvalue λ of M_w, give a corresponding eigenvector.

***Exercise* 2.4.** Let $A \in \mathcal{L}(\mathcal{H})$. Let $U \in \mathcal{L}(\mathcal{H})$ be unitary. Prove that

$$\sigma(U^*AU) = \sigma(A) \quad \text{and} \quad \sigma_{\mathsf{p}}(U^*AU) = \sigma(A).$$

***Exercise* 2.5.** We consider on $\ell^2(\mathbb{Z})$ the operator H_0 which maps the sequence $u = (u_n)_{n\in\mathbb{Z}}$ to the sequence $H_0 u$ defined by

$$\forall n \in \mathbb{Z}, \quad (H_0 u)_n = u_{n+1} + u_{n-1} - 2u_n.$$

1. Prove that $H_0 \in \mathcal{L}(\ell^2(\mathbb{Z}))$.
2. We denote by $L^2(\mathbb{S}^1)$ the set of L^2-functions on the torus $\mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$. Functions on $\mathbb{S}^1$ can also be seen as 2π-periodic functions on $\mathbb{R}$. For $v \in L^2(\mathbb{S}^1)$ we have

$$\|v\|^2_{L^2(S^1)} = \frac{1}{2\pi}\int_{-\pi}^{\pi} |v(s)|^2 \, \mathrm{d}s.$$

Given a sequence $u = (u_n)_{n\in\mathbb{Z}}$ we define $\Theta u \in L^2(\mathbb{S}^1)$ by

$$(\Theta u)(s) = \sum_{n\in\mathbb{Z}} u_n e^{ins}.$$

Prove that Θ is a unitary operator from $\ell^2(\mathbb{Z})$ to $L^2(\mathbb{S}^1)$.
3. Prove that $\Theta H_0 \Theta^{-1}$ is a multiplication operator on $\mathbb{S}^1$.
4. Compute the spectrum of $\Theta H_0 \Theta^{-1}$ and deduce the spectrum of H_0 (use Exercise 2.4).

***Exercise* 2.6.** We consider on $L^2(\mathbb{C})$ ($\mathbb{C}$ is endowed with its usual Lebesgue measure) the operator A defined by $(Au)(y) = yu(y)$ on the domain

$$\mathsf{Dom}(A) = \left\{u \in L^2(\mathbb{C}) \, : \, yu \in L^2(\mathbb{C})\right\}.$$

1. Prove that A is closed.
2. Prove that $\sigma(A) = \mathbb{C}$.

***Exercise* 2.7.** We consider on $L^2(0,1)$ the operator

$$A = \partial_x$$

defined on the domain

$$\mathsf{Dom}(A) = \left\{u \in H^1(0,1) \, : \, u(0) = 0\right\}.$$

1. Prove that A is closed.
2. Prove that $\sigma(A) = \varnothing$.

***Exercise* 2.8.** We set

$$\mathcal{H} = \left\{u \in L^2(\mathbb{R}) \, : \, u \text{ is even}\right\}.$$

1. Prove that $\mathcal{H}$ is a Hilbert space.
2. We want to consider on $\mathcal{H}$ the operator defined by $Au = -u''$. What is the natural domain for A (in particular, we want A to be closed) ?
3. Then what is the spectrum of A ?

***Exercise* 2.9.** For $u = (u_n)_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$ we set

$$U(\dots, u_{-2}, u_{-1}, u_0, u_1, u_2, \dots) = (\dots, u_{-1}, u_0, u_1, u_2, u_3, \dots).$$

1. Prove that $\|U\|_{\mathcal{L}(\ell^2(\mathbb{Z}))} = 1$.
2. Prove that U is invertible and $U^{-1} = U^*$ (U is a unitary operator).
3. Prove that $\sigma(U) \subset \mathbb{U} = \{z \in \mathbb{C} \, : \, |z| \neq 1\}$.
4. Let $\lambda \in \mathbb{U}$. For $k \in \mathbb{N}$ we consider

$$u^{(k)} = (\dots, 0, 0, 1, \lambda, \lambda^2, \dots, \lambda^k, 0, 0, \dots).$$

Compute $\left\|u^{(k)}\right\|_{\ell^2(\mathbb{Z})}$ and $\left\|(U - \lambda)u^{(k)}\right\|_{\ell^2(\mathbb{Z})}$. Prove that $\lambda \in \sigma(S)$.

***Exercise* 2.10.** Compute, for all $n \in \mathbb{N}$ and $z \in \rho(A)$,

$$\frac{\mathrm{d}^n}{\mathrm{d}z^n}(A - z)^{-1}.$$

***Exercise* 2.11.** Using the resolvent identity, give another proof of the facts that the resolvent map $R_A : z \mapsto (A-z)^{-1}$ is continuous and then holomorphic on $\rho(A)$ with $R_A' = R_A^2$.

***Exercise* 2.12.** We consider on $\ell^2(\mathbb{Z})$ the operator A defined by

$$A(\ldots, u_{-2}, u_{-1}, u_0, u_1, u_2, \ldots) = (\ldots, u_{-1}, 0, u_1, u_2, u_3, \ldots)$$

(replace u_0 by 0 and then shift to the left). What is the spectrum of A ?

***Exercise* 2.13.** Let A be a closed and densely defined operator on E. Let $\lambda_0 \in \mathbb{C}$ and $r_0 > 0$ such that $D(\lambda_0, r_0) \cap \sigma(A) \neq 0$. Let

$$\Pi = -\frac{1}{2i\pi} \int_{\mathcal{C}(\lambda_0, r_0)} (A-\zeta)^{-1} \, \mathrm{d}\zeta.$$

Prove that $1 \in \sigma(\Pi)$.

***Exercise* 2.14.** We consider on $\ell^2(\mathbb{N}^*)$ the operator A defined by

$$A(u_1, u_2, u_3, \ldots, u_k, \ldots) = \left(0, \frac{u_1}{2}, \frac{u_2}{4}, \frac{u_3}{8}, \ldots, \frac{u_k}{2^k}, \ldots\right).$$

1. Prove that $A \in \mathcal{L}(\ell^2(\mathbb{N}^*))$ and compute $\|A\|_{\mathcal{L}(\ell^2(\mathbb{N}^*))}$.
2. Compute $\sigma(A)$.
3. Compute $\sigma_{\mathsf{p}}(A)$.
4. Let $z \in \mathbb{C}\backslash\{0\}$ and $f = (f_k)_{k\in\mathbb{N}^*} \in \ell^2(\mathbb{N}^*)$. Compute $(A-z)^{-1} f$.
5. Compute the Riesz projection of A at point 0.

***Exercise* 2.15.** Let E_1 and E_2 be two Banach spaces and $\mathsf{E} = \mathsf{E}_1 \oplus \mathsf{E}_2$. Let A_1 and A_2 be two closed operators, on E_1 and E_2 respectively. For $\varphi = \varphi_1 + \varphi_2 \in \mathsf{E}$ we set $A = A_1\varphi_1 + A_2\varphi_2$.
1. Prove that this defines a closed operator A on E.
2. Prove that $\sigma(A) = \sigma(A_1) \cup \sigma(A_2)$.
3. Prove that $\sigma_{\mathsf{p}}(A) = \sigma_{\mathsf{p}}(A_1) \cup \sigma_{\mathsf{p}}(A_2)$.
4. Assume that λ is an isolated eigenvalue of A. Prove that the geometric (algebraic) multiplicity of λ as an eigenvalue of A is the sum of the geometric (algebraic) multiplicities of λ as an eigenvalue of A_1 and A_2.

***Exercise* 2.16.** Let $A \in \mathcal{L}(\mathsf{E})$. Let $P \in \mathbb{C}[X]$. Prove that

$$\sigma(P(A)) = \{P(\lambda), \lambda \in \sigma(A)\}.$$

***Exercise* 2.17** (Regular points)**.** Let A be an operator on the Hilbert space $\mathcal{H}$. Let z be a regular point of A (see Proposition 2.9). We denote by $d_A(z) = \mathsf{dim}(\mathsf{Ran}(A-z)^{\perp})$ the defect number of A. We also denote by $\pi(A)$ the set of regular points of A.
1. Prove that $\pi(A)$ is open (more precisely, if $z_0 \in \pi(A)$ and $c_0 > 0$ is the constant given by (2.1), show that $D(z_0, c_{z_0}) \subset \pi(A)$).
2. Assume that A is closable.
a. Let $z_0 \in \pi(A)$. Assume that $z \in \pi(A)$ is such that $d_A(z) \neq d_A(z_0)$. Prove that there exists $\varphi \in \mathsf{Dom}(A)\backslash\{0\}$ such that

$$\langle (A-z)\varphi, (A-z_0)\varphi \rangle = 0.$$

b. Let $c_0 > 0$ is the constant given by (2.1) for z_0 and assume that $|z - z_0| < c_0$. Prove that $d_A(z) = d_A(z_0)$.
c. Prove that the defect number is constant on each connected component of $\pi(A)$.

***Exercise* 2.18.** Let A be a closed operator on E. Let $\lambda \in \sigma_{\mathsf{disc}}(A)$. Let $r_0 > 0$ be such that $D(\lambda, r_0) \cap \sigma(A) = \{\lambda\}$. For $r \in]0, r_0[$ and $n \in \mathbb{Z}$ we set

$$R_n = \frac{1}{2i\pi} \int_{\mathcal{C}(\lambda, r)} \frac{(A-\zeta)^{-1}}{(\zeta-\lambda)^{n+1}} \, \mathrm{d}\zeta.$$

1. Prove that for $n_1, n_2 \in \mathbb{Z}\backslash\{0\}$ we have $R_{n_1} R_{n_2} = -R_{n_1+n_2+1}$.

2. We set $N = -R_{-2}$. Prove that for all $n \geqslant 2$ we have $R_{-n} = -N^{n-1}$.
3. We denote by Π the Riesz projection at λ. Prove that $N\Pi = \Pi N = N$. Deduce that N has finite rank.
4. Prove that for $z \in D(\lambda, r_0)\backslash\{\lambda\}$ we can write $(A-z)^{-1}$ as the Laurent series

$$(A-z)^{-1} = \sum_{n\in\mathbb{Z}} (z-\lambda)^n R_n,$$

and in particular that the power series $\sum_{m\geqslant 0} \rho^n R_{-m}$ is convergent for any $\rho \in \mathbb{C}$.
5. Prove that N is nilpotent and that $R_{-n} = 0$ for n large enough.