Appendix A

Appendices

A.1 Reminder of functional analysis

Proposition A.1 (Cor. 1.4 in [Bre11])**.** *Let E be a normed vector space and let E' be the space of semilinear forms on E. For $\varphi \in \mathsf{E}$ we have*

$$\|\varphi\| = \sup_{\substack{\ell \in E' \\ \|\ell\|_{E'} \leqslant 1}} \left| \langle \ell, \varphi \rangle_{\mathsf{E}',\mathsf{E}} \right|.$$

In particular, if $\varphi \neq 0$ there exists $\ell \in \mathsf{E}'$ such that $\langle \ell, \varphi \rangle_{\mathsf{E}',\mathsf{E}} \neq 0$.

Theorem A.2 (Open mapping theorem, th.2.6 in [Bre11])**.**

Proposition A.3 (Cor. 2.7 in [Bre11])**.** *Let E and F be two Banach spaces. Let A be a continuous linear operator from E to F. If A is bijective, then A^{-1} is a continuous linear operator from F to E.*

Theorem A.4 (Closed Graph Theorem, th. 2.9 in [Bre11])**.** *Let E and F be two Banach spaces. Let A be a linear map from E to F. Assume that the graph of A is closed in $\mathsf{E} \times \mathsf{F}$. Then A is continuous.*

Proposition A.5. *Let $\mathcal{H}$ be a Hilbert space and F be a subset of $\mathcal{H}$. Then we have*

$$(F^{\perp})^{\perp} = \overline{F}.$$

(see Proposition 1.9 in [Bre11] for the version in normed vector spaces)

Proof. • We have $F \subset F^{\perp\perp}$ and $F^{\perp\perp}$ is closed, so $\overline{F} \subset F^{\perp\perp}$
• We have $F^{\perp} = \overline{F}^{\perp}$ and $\mathcal{H} = \overline{F} \oplus \overline{F}^{\perp}$. Let $\varphi \in F^{\perp\perp}$. There exist $\overline{\varphi} \in \overline{F}$ and $\overline{\varphi}^{\perp} \in \overline{F}^{\perp} = F^{\perp}$ such that $\varphi = \overline{\varphi} + \overline{\varphi}^{\perp}$. Then $0 = \langle \varphi, \overline{\varphi}^{\perp} \rangle = \|\overline{\varphi}^{\perp}\|^2$, so $\varphi = \overline{\varphi} \in \overline{F}$. □

A.2 Holomorphic functions in a Banach space

Let E be a Banach space.

Definition A.6. *Let Ω be an open subset of $\mathbb{C}$ and f be a function from Ω to E. We say that f is holomorphic on Ω if for all $z_0 \in \mathbb{C}$ the limit*

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. In this case we denote by $f'(z_0)$ this limit and we say that f' is the derivative of f.

Proposition A.7. *Let ω be an open subset of $\mathbb{C}$. Let $\varphi : \omega \to \mathsf{E}$ and $B : \omega \to \mathcal{L}(\mathsf{E}, \mathsf{F})$.*

(i) *Assume that for all $\ell \in \mathsf{E}'$ the map $z \mapsto \langle \ell, \varphi(z) \rangle_{\mathsf{E}',\mathsf{E}}$ is holomorphic in ω. Then φ is holomorphic .*

(ii) *Assume that for all $\psi \in \mathsf{E}$ the map $z \mapsto B(z)\psi \in \mathsf{F}$ is holomorphic in ω. Then B is holomorphic.*

(iii) *Assume that for all $\psi \in \mathsf{E}$ and $\ell \in \mathsf{F}'$ the map $z \mapsto \langle \ell, B(z)\psi \rangle_{\mathsf{F}',\mathsf{F}} \in \mathbb{C}$ is holomorphic in ω. Then B is holomorphic.*

Proof. • Let $z_0 \in \omega$ and $r > 0$ such that $D(z_0, 2r) \subset \omega$. Let

$$\Phi = \left\{ \frac{\varphi(z) - \varphi(z_0)}{z - z_0}, z \in D(z_0, 2r) \setminus \{z_0\} \right\}.$$

For all $\ell \in \mathsf{E}'$ the set $\ell(\Phi)$ is bounded in $\mathbb{C}$. By the uniform boundedness principle (see [Bre11, Cor.2.4]), Φ is bounded. In particular φ is continuous at z_0. This proves that φ is continous on ω.

• Let z_0 and $r > 0$ as above. For $\ell \in \mathsf{E}'$ and $z \in D(z_0, r)$, we write the integrals over $\mathcal{C}(z_0, r)$ as the limit of the Riemann sums to see that

$$\langle \ell, \varphi(z) \rangle = \frac{1}{2i\pi} \int_{\mathcal{C}(z_0,r)} \frac{\langle \ell, \varphi(\zeta) \rangle}{\zeta - z} \, \mathrm{d}\zeta = \left\langle \ell, \frac{1}{2i\pi} \int_{\mathcal{C}(z_0,r)} \frac{\varphi(\zeta)}{\zeta - z} \, \mathrm{d}\zeta \right\rangle.$$

By the Hahn-Banach theorem, this implies that

$$\varphi(z) = \frac{1}{2i\pi} \int_{\mathcal{C}(z_0,r)} \frac{\varphi(\zeta)}{\zeta - z} \, \mathrm{d}\zeta,$$

which in turn implies that φ is holomorphic.

• The second statement is proved similarly. Given z_0 and r as above we set

$$\mathcal{B} = \left\{ \frac{B(z) - B(z_0)}{z - z_0}, z \in D(z_0, 2r) \setminus \{z_0\} \right\}.$$

Then $\mathcal{B}$ is bounded by the uniform boundedness principle, which implies that B is continuous.

• Then for $\psi \in \mathsf{E}$ we write

$$B(z)\psi = \frac{1}{2i\pi} \int_{\mathcal{C}(z_0,r)} \frac{B(\zeta)\psi}{\zeta - z} \, \mathrm{d}\zeta = \left(\frac{1}{2i\pi} \int_{\mathcal{C}(z_0,r)} \frac{B(z)}{\zeta - z} \, \mathrm{d}\zeta \right) \psi.$$

This proves that

$$B(z) = \frac{1}{2i\pi} \int_{\mathcal{C}(z_0,r)} \frac{B(z)}{\zeta - z} \, \mathrm{d}\zeta,$$

and gives the second statement.

• Finally, (iii) is a direct consequence of (i) and (ii). □