Final Exam

Monday, November 28 (3h)

Five pages of notes are allowed. French or English can be used for the answers. Unless otherwise specified, all the answers have to be justified and the clarity of the writing will be taken into account.

Exercise 1. We consider on $\ell^2(\mathbb{N})$ the operator A defined on the domain

$$\mathsf{Dom}(A) = \left\{ u = (u_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N}) \ : \ \sum_{n=0}^{\infty} n^2 |u_n|^2 < +\infty \right\}$$

by

$$\forall u = (u_n)_{n\in\mathbb{N}} \in \mathsf{Dom}(A), \quad Au = (ne^{in}u_n)_{n\in\mathbb{N}}.$$

1. Prove that A is densely defined.
2. Prove that A is closed.
3. What is the adjoint of A ?

Exercise 2. Let E_1, E_2 and E_3 be three Banach spaces such that $\mathsf{E}_1 \subset \mathsf{E}_2 \subset \mathsf{E}_3$. We assume that the embedding $i : \mathsf{E}_1 \to \mathsf{E}_2$ is compact and that the embedding $j : \mathsf{E}_2 \to \mathsf{E}_3$ is continuous. Let $\varepsilon > 0$. Prove that there exists $C_\varepsilon > 0$ such that for all $\varphi \in \mathsf{E}_1$ we have

$$\|\varphi\|_{\mathsf{E}_2} \leqslant \varepsilon \|\varphi\|_{\mathsf{E}_1} + C_\varepsilon \|\varphi\|_{\mathsf{E}_3}.$$

Exercise 3. For $u = (u_n)_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$ we define $H_0 u \in \ell^2(\mathbb{Z})$ by

$$\forall n \in \mathbb{Z}, \quad (H_0 u)_n = 2u_n - u_{n+1} - u_{n-1}.$$

1. Prove that this defines a bounded operator H_0 on $\ell^2(\mathbb{Z})$.
2. We denote by L^2_{per} the space of 2π-periodic functions in $L^2_{\mathrm{loc}}(\mathbb{R})$ (this is equivalent to considering $L^2(S^1)$, where S^1 is the circle, or one dimensional torus). It is endowed with the norm defined by

$$\|v\|^2_{L^2_{\mathsf{per}}} = \frac{1}{2\pi} \int_{-\pi}^{\pi} |v(x)|^2 \, \mathrm{d}x.$$

For $u = (u_n)_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$ we define $\mathcal{F}u \in L^2_{\mathsf{per}}$ by

$$\forall x \in \mathbb{R}, \quad (\mathcal{F}u)(x) = \sum_{n\in\mathbb{Z}} u_n e^{-inx}.$$

We recall that $\mathcal{F} : \ell^2(\mathbb{Z}) \to L^2_{\mathsf{per}}$ is a unitary operator. Prove that $\mathcal{F} H_0 \mathcal{F}^{-1}$ is the operator M of multiplication by $2(1 - \cos(x))$ on L^2_{per}.
3. Give without proof the spectrum of M.
4. Prove that $\sigma(H_0) = \sigma(M)$.
5. Prove that H_0 has no eigenvalue.
6. Let $(\beta_n)_{n\in\mathbb{Z}}$ be a real-valued sequence such that $\beta_n > 0$ for all $n \in \mathbb{Z}$ and $\beta_n \to 0$ as $n \to \pm\infty$. We denote by B the operator on $\ell^2(\mathbb{Z})$ which maps $u = (u_n) \in \ell^2(\mathbb{Z})$ to $Bu = (\beta_n u_n)_{n\in\mathbb{N}}$. For $\alpha \in \mathbb{R}$ we set $H_\alpha = H_0 + \alpha B$. Prove that H_α is selfadjoint for all $\alpha \in \mathbb{R}$.
7. Let $\alpha \in \mathbb{R}$. What is the essential spectrum of H_α ?
8. Prove that there exists $\alpha \in \mathbb{R}$ such that H_α has at least one eigenvalue.
9. Let $N \in \mathbb{N}^*$. Prove that there exists $\alpha \in \mathbb{R}$ such that H_α has at least N eigenvalues (counted with multiplicities).

Exercise 4. We consider on $\mathcal{H} = L^2(0,1)$ the operator A defined by

$$\mathsf{Dom}(A) = \left\{u \in H^2(0,1) \,:\, u(0) = 0 \text{ and } u'(1) = 0\right\}$$

and $Au = -u''$ for all $u \in \mathsf{Dom}(A)$. We recall that if $u \in L^2(0,1)$ is such that $u'' \in L^2(0,1)$ then $u' \in L^2(0,1)$, and moreover the graph norm on $\mathsf{Dom}(A)$ is equivalent to the norm $\|\cdot\|_{H^2(0,1)}$.
1. Prove that A is selfadjoint.
2. Prove that $A \geqslant 0$.
3. Prove that $(-A)$ generates a contractions semigroup on $L^2(0,1)$.
4. Prove that $\ker(A) = \{0\}$ (we recall that if $u \in H^2(0,1)$ satisfies $-u'' = 0$ in the sense of distributions, then it is of class C^2).
5. Prove that $\min \sigma(A) > 0$.
6. Prove that there exists $\gamma > 0$ such that for all $t \geqslant 0$ we have $\left\|e^{-tA}\right\|_{\mathcal{L}(L^2(0,1))} \leqslant e^{-t\gamma}$.

Exercise 5. Let $\mathcal{H}$ be a Hilbert space. Let $(S_t)_{t\geqslant 0}$ be a strongly continuous semigroup on $\mathcal{H}$ and let A be its generator. Prove that the generator of the semigroup $(S_t^*)_{t\geqslant 0}$ is A^* (the proof that $(S_t^*)_{t\geqslant 0}$ is a strongly continuous semigroup is not required).

Exercise 6. Let $\mathcal{H} = L^2(\mathbb{R})$.
1. We set $\mathsf{Dom}(T) = \{u \in C_0^\infty(\mathbb{R}) \,:\, u(0) = 0\}$, and for $u \in \mathsf{Dom}(T)$ we set $Tu = -u'' + u$. Prove that this defines a symmetric and non-negative operator T on $\mathcal{H}$.
2. Prove that T is not selfadjoint.
3. We set $\mathcal{V}_N = H^1(\mathbb{R})$. For $v \in \mathcal{V}_N$ we set $q_N(v) = \|v\|_{H^1(\mathbb{R})}^2 = \|v'\|_{L^2(\mathbb{R})}^2 + \|v\|_{L^2(\mathbb{R})}^2$. What is the operator A_N (domain and action) associated with the quadratic form q_N by the representation theorem on $\mathcal{H}$? Prove that A_N is a selfadjoint extension of T.
4. We set $\mathcal{V}_D = \left\{v \in H^1(\mathbb{R}) \,:\, v(0) = 0\right\}$. For $v \in \mathcal{V}_D$ we set $q_D(v) = \|v\|_{H^1(\mathbb{R})}^2$. What is the operator A_D (domain and action) associated with the quadratic form q_D by the representation theorem on $\mathcal{H}$? Prove that A_D is a selfadjoint extension of T.
5. Give all the selfadjoint extensions of T on $\mathcal{H}$.