Chapter 2

Spectrum of general (unbounded) operators

[Draft version, November 21, 2022]

2.1 Unbounded operators - Spectrum

Let E and F be two Banach spaces.

2.1.1 Definitions and examples

Definition 2.1. *A linear operator (or unbounded operator) from* E *to* F *is a linear map* A *from a linear subspace* $\mathsf{Dom}(A)$ *of* E *(the domain of* A*) to* F*. An operator on* E *is an operator from* E *to itself.*

Definition 2.2. *We say that the operator* A *is densely defined if* $\mathsf{Dom}(A)$ *is dense in* E*.*

Example 2.3. A bounded operator $A \in \mathcal{L}(\mathsf{E}, \mathsf{F})$ is a particular case of unbounded operator with $\mathsf{Dom}(A) = \mathsf{E}$.

Example 2.4. Let (Ω, μ) be a measure space. Let f be a measurable function on Ω. We consider on $L^2(\Omega, \mu)$ the multiplication operator

$$M_f : \varphi \mapsto f\varphi,$$

defined on the domain

$$\mathsf{Dom}(M_f) = \left\{\varphi \in L^2(\Omega) \; : \; f\varphi \in L^2(\Omega)\right\}. \tag{2.1}$$

Remark 2.5. One has to be careful when dealing with unbounded operators. For instance, if A_1 and A_2 are two operators on E, then the sum $A_1 + A_2$ is only defined on the domain $\mathsf{Dom}(A_1) \cap \mathsf{Dom}(A_2)$ (which can be $\{0\}$) and the composition $A_2 \circ A_1$ is defined on $\{\varphi \in \mathsf{Dom}(A_1) \; : \; A_1\varphi \in \mathsf{Dom}(A_2)\}$.

Definition 2.6. *Let* A *and* B *be two linear operators from* E *to* F*. We say that* A *is an extension of* B *and we write* $B \subset A$ *if* $\mathsf{Dom}(B) \subset \mathsf{Dom}(A)$ *and* $A\varphi = B\varphi$ *for all* $\varphi \in \mathsf{Dom}(B)$*.*

Ex. 2.1

Example 2.7. Let Ω be an open subset of $\mathbb{R}^d$. Let f be a continuous function on Ω. We can define M_f on $L^2(\Omega)$ as above (with domain (2.1)). We can also define M_f^0 by $M_f^0 u = fu$ for $u \in \mathsf{Dom}(M_f^0) = C_0^\infty(\Omega)$. Then we have $M_f^0 \subset M_f$.

Example 2.8. Let Ω be an open subset of class C^2 in $\mathbb{R}^d$. We denote by H_0, $\tilde{H}$, H_D and H_N the operators on $L^2(\Omega)$ which are all equal to $-\Delta$, but defined on different domains:

- $\mathsf{Dom}(H_0) = C_0^\infty(\Omega)$,

- $\mathsf{Dom}(\tilde{H}) = H^2(\Omega)$,
- $\mathsf{Dom}(H_D) = H^2(\Omega) \cap H_0^1(\Omega)$,
- $\mathsf{Dom}(H_N) = \{u \in H^2(\Omega) : \partial_\nu u = 0 \text{ on } \partial\Omega\}$.

These four operators are densely defined. Moreover we have $H_0 \subset H_D \subset \tilde{H}$ and $H_0 \subset H_N \subset \tilde{H}$.

Definition 2.9. *Let A be an operator from E to F. The graph of A is*

$$\mathsf{Gr}(A) = \{(\varphi, A\varphi), \varphi \in \mathsf{Dom}(A)\} \subset \mathsf{E} \times \mathsf{F}.$$

Remark 2.10. If A and S are two linear operators from E to F then $S \subset A$ if and only if $\mathsf{Gr}(S) \subset \mathsf{Gr}(A)$.

Definition 2.11. *Let A be an operator on E. We define on $\mathsf{Dom}(A)$ the graph norm by*

$$\|\varphi\|_A^2 := \|(\varphi, A\varphi)\|_{\mathsf{E}\times\mathsf{F}}^2 = \|A\varphi\|_{\mathsf{F}}^2 + \|\varphi\|_{\mathsf{E}}^2 .$$

Remark 2.12. If $A \in \mathcal{L}(\mathsf{E})$ then the graph norm is equivalent to the original norm on E.

Example 2.13. We consider on $L^2(\mathbb{R}^d)$ the Laplace operator $H = -\Delta$, with domain $\mathsf{Dom}(H) = H^2(\mathbb{R}^d)$. Then the graph norm of H is equivalent to the usual Sobolev norm:

$$\|-\Delta u\|_{L^2(\mathbb{R}^d)}^2 + \|u\|_{L^2(\mathbb{R}^d)}^2 \simeq \|u\|_{H^2(\mathbb{R}^d)}^2 .$$

This is not the case on any open subset Ω of $\mathbb{R}^d$.

2.1.2 Spectrum of unbounded operators

Definition 2.14. *Let A be a linear operator from E to F. We say that A is invertible (or that it is boundedly invertible, or that it has a bounded inverse) if there exists $B \in \mathcal{L}(\mathsf{F}, \mathsf{E})$ such that $\mathsf{Ran}(S) \subset \mathsf{Dom}(A)$, $BA = \mathrm{Id}_{\mathsf{Dom}(A)}$ and $AB = \mathrm{Id}_{\mathsf{F}}$. In this case we write $B = A^{-1}$.*

Remark 2.15. Notice that if A is invertible then it is a bijective map from $\mathsf{Dom}(A)$ to F. But if $\mathsf{Dom}(A) \neq \mathsf{E}$ then A^{-1} is only a right inverse of A.

Remark 2.16. If A is injective we can always define an (unbounded) inverse A^{-1}, even if A is not surjective. We define A^{-1} as an operator from F to E with domain $\mathsf{Dom}(A^{-1}) = \mathsf{Ran}(A)$ and we have $A^{-1}A = \mathrm{Id}_{\mathsf{Dom}(A)}$, $AA^{-1} = \mathrm{Id}_{\mathsf{Ran}(A)}$. We will never consider unbounded inverses in this course.

Definition 2.17. *Let A be an operator on E. Then $\lambda \in \mathbb{C}$ belongs to the resolvent set $\rho(A)$ of A if $A - \lambda$ is invertible (according to Definition 2.14, this means that $(A - \lambda)$ is bijective as a map from $\mathsf{Dom}(A)$ to E and its inverse $(A - \lambda)^{-1} : \mathsf{E} \to \mathsf{Dom}(A) \subset \mathsf{E}$ defines a bounded operator on E). The spectrum $\sigma(A)$ is the complementary set of $\rho(A)$ in $\mathbb{C}$.*

Definition 2.18. *Let A be an operator on E. We say that $\lambda \in \mathbb{C}$ is an eigenvalue of A if there exists $\varphi \in \mathsf{Dom}(A) \backslash \{0\}$ such that $A\varphi = \lambda\varphi$. Such a φ is an eigenvector associated to the eigenvalue λ. The geometric multiplicity of λ is the dimension of* $\ker(A - \lambda)$. *We denote by $\sigma_{\mathsf{p}}(A)$ the set of eigenvalues of A.*

As for bounded operators, we have $\sigma_{\mathsf{p}}(A) \subset \sigma(A)$ but the inclusion can be strict.

Example 2.19. Let M_w be the multiplication operator defined in Example 2.4. Let $z \in \mathbb{C}$. Then, as in the bounded case, $z \in \sigma(M_f)$ if and only if

$$\forall \varepsilon > 0, \quad \mu\big(\{x \in \Omega : |w(x) - z| \leqslant \varepsilon\}\big) > 0,$$

and $z \in \sigma_{\mathsf{p}}(M_f)$ if and only if

$$\mu\big(\{x \in \Omega : |w(x) - z| = 0\}\big) > 0,$$

Example 2.20. • Let $\mathsf{E} = L^2(\mathbb{R}^d)$ and $A_0 = -\Delta$ with $\mathsf{Dom}(A_0) = C_0^\infty(\mathbb{R}^d)$. Then for any $z \in \mathbb{C}$ we have $\mathsf{Ran}(A_0 - z) \subset C_0^\infty(\mathbb{R}^d)$ so $A_0 - z$ cannot be invertible. This proves that $\sigma(A_0) = \mathbb{C}$.
• Now we consider $A = -\Delta$ with $\mathsf{Dom}(A) = H^2(\mathbb{R}^d)$. Then $\sigma(A) = \mathbb{R}_+$ and for $z \in \mathbb{C}\backslash\mathbb{R}_+$ we have

$$\left\|(A-z)^{-1}\right\|_{\mathcal{L}(L^2(\mathbb{R}^d))} = \frac{1}{\mathsf{dist}(z, \mathbb{R}_+)}.$$

Indeed, if we denote by $\mathcal{F}$ the Fourier transform on $L^2(\mathbb{R}^d)$, then $\mathcal{F}$ is a unitary operator. Then $(-\Delta - z)$ is invertible if and only if $\mathcal{F}(-\Delta - z)\mathcal{F}^{-1} = M - z$ is invertible on $L^2(\mathbb{R}^d)$, where $M = \mathcal{F}(-\Delta)\mathcal{F}^{-1}$ is equal to the multiplication operator M_w for $w : \xi \mapsto |\xi|^2$. Thus $\sigma(A) = \sigma(M_w) = \mathbb{R}_+$ and for $z \in \mathbb{C}\backslash\mathbb{R}_+$ we have

$$\begin{aligned}\left\|(A-z)^{-1}\right\|_{\mathcal{L}(L^2(\mathbb{R}^d))} &= \left\|\mathcal{F}^{-1}(M-z)^{-1}\mathcal{F}\right\|_{\mathcal{L}(L^2(\mathbb{R}^d))} = \left\|(M-z)^{-1}\right\|_{\mathcal{L}(L^2(\mathbb{R}^d))}\\ &= \frac{1}{\mathsf{dist}(z, \mathbb{R}_+)}.\end{aligned}$$

2.1.3 Basic properties of the spectrum and the resolvent

Proposition 2.21. *Let A be an operator on E and $z \in \mathbb{C}$. Assume that there exists a sequence (φ_n) in $\mathsf{Dom}(A)$ such that $\|\varphi_n\|_{\mathsf{E}} = 1$ for all $n \in \mathbb{N}$ and*

$$\|(A-z)\varphi_n\|_{\mathsf{E}} \xrightarrow[n\to+\infty]{} 0.$$

Then $z \in \sigma(A)$.

Proof. Assume that $z \in \rho(A)$. Then

$$\|\varphi_n\|_{\mathsf{E}} \leqslant \left\|(A-z)^{-1}\right\|_{\mathcal{L}(\mathsf{E})} \|(A-z)\varphi_n\|_{\mathsf{E}} \xrightarrow[n\to+\infty]{} 0.$$

This gives a contradiction. □

Example 2.22. An unbounded operator can have an empty spectrum (compare with Proposition 1.21). We consider on $L^2(0,1)$ the operator

$$A = \partial_x$$

defined on the domain

$$\mathsf{Dom}(A) = \left\{u \in H^1(0,1) \,:\, u(0) = 0\right\}.$$

Then $\sigma(A) = \varnothing$.

Indeed, for $z \in \mathbb{C}$ we define $R_z : L^2(0,1) \to L^2(0,1)$ as follows. For $f \in L^2(0,1)$ and $x \in [0,1]$ we set

$$(R_z f)(x) = \int_0^x e^{z(x-y)} f(y)\,\mathrm{d}y.$$

We can check that R_z defines a bounded inverse for $(A - z)$, which proves that z belongs to $\rho(A)$. Notice that we can replace $H^1(0,1)$ and $L^2(0,1)$ by $C^1([0,1])$ and $C^0([0,1])$.

Proposition 2.23. *Let A be a closed operator on E.*

(i) *For $\varphi \in \mathsf{Dom}(A)$ and $z \in \rho(A)$ we have $(A-z)^{-1}A\varphi = A(A-z)^{-1}\varphi$.*

(ii) *The resolvent set $\rho(A)$ of A is open (and, equivalently, its spectrum $\sigma(A)$ is closed). Moreover, for $z_0 \in \rho(A)$ the disk $D(z_0, \left\|(A-z_0)^{-1}\right\|_{\mathcal{L}(\mathsf{E})}^{-1})$ is included in $\rho(A)$, which implies*

$$\left\|(A-z)^{-1}\right\|_{\mathcal{L}(\mathsf{E})} \geqslant \frac{1}{\mathsf{dist}(z, \sigma(A))}.$$

(iii) *The resolvent $R_A : z \mapsto (A-z)^{-1}$ is analytic on $\rho(A)$ and $R_A' = R_A^2$.*

(iv) *For $z_1, z_2 \in \rho(A)$ we have the resolvent identity*

$$(A-z_1)^{-1} - (A-z_2)^{-1} = (z_1-z_2)(A-z_1)^{-1}(A-z_2)^{-1} = (z_1-z_2)(A-z_2)^{-1}(A-z_1)^{-1}.$$

In particular, $(A-z_1)^{-1}$ and $(A-z_2)^{-1}$ commute.

The proofs are the same as for the bounded case. ✎ Ex. 2.2

2.2 Closed operators

2.2.1 Closed operators

Proposition-Definition 2.24. *Let A be an operator E. We say that A is closed if the following equivalent assertions are satisfied.*

(i) *If a sequence $(\varphi_n)_{n\in\mathbb{N}} \in \mathsf{Dom}(A)^{\mathbb{N}}$ is such that φ_n goes to some φ in E and $A\varphi_n$ goes to some ψ in F, then φ belongs to $\mathsf{Dom}(A)$ and $A\varphi = \psi$;*

(ii) *$\mathsf{Gr}(A)$ is closed in $\mathsf{E} \times \mathsf{F}$;*

(iii) *$\mathsf{Dom}(A)$, endowed with the norm $\|\cdot\|_A$, is complete (hence a Banach space).*

Remark 2.25. Let A be a closed operator on E. Then A defines a bounded operator from the Banach space $\mathsf{Dom}(A)$ to E.

Example 2.26. A bounded operator is closed.

Example 2.27.
- We consider on $L^2(\mathbb{R})$ the operator A defined on the domain $\mathsf{Dom}(A) = C_0^\infty(\mathbb{R})$ by $(Au)(x) = x^2u(x)$, $x \in \mathbb{R}$. We define $v : \mathbb{R} \to \mathbb{R}$ by $v(x) = (1+x^2)^{-2}$. Let $\chi \in C_0^\infty(\mathbb{R}, [0,1])$ be equal to 1 on $[-1,1]$. For $n \in \mathbb{N}^*$ and $x \in \mathbb{R}$ we set $\chi_n(x) = \chi(x/n)$. Then $\chi_n v$ goes to v in $L^2(\mathbb{R})$, $\chi_n v \in \mathsf{Dom}(A)$ for all $n \in \mathbb{N}^*$ and $A(\chi_n v)$ has a limit in $L^2(\mathbb{R})$. However v does not belong to $\mathsf{Dom}(A)$. This proves that A is not closed.
- We now consider the operator $A : u \mapsto x^2u$ on the domain
$$\mathsf{Dom}(A) = \left\{u \in L^2(\mathbb{R}) \,:\, x^2u \in L^2(\mathbb{R})\right\}.$$
Assume that $(u_n)_{n\in\mathbb{N}}$ is a sequence in $\mathsf{Dom}(A)$ which goes to some u in $L^2(\mathbb{R})$ and such that Au_n has a limit $v \in L^2(\mathbb{R})$. The function x^2u belongs to $L^2_{\mathrm{loc}}(\mathbb{R})$ and for all $\phi \in C_0^\infty(\mathbb{R})$ we have
$$\int_{\mathbb{R}} x^2u(x)\phi(x)\,\mathrm{d}x = \lim_{n\to+\infty} \int_{\mathbb{R}} x^2u_n(x)\phi(x)\,\mathrm{d}x = \int_{\mathbb{R}} v(x)\phi(x)\,\mathrm{d}x.$$
This proves that $x^2u(x) = v(x)$ for almost all $x \in \mathbb{R}$. In particular, $u \in \mathsf{Dom}(A)$ and $Au = v$. This proves that A is closed.

Example 2.28. The Laplace operator $A = -\Delta$ with $\mathsf{Dom}(A) = C_0^\infty(\mathbb{R}^d)$ is not closed in $L^2(\mathbb{R}^d)$. Let $u \in H^2(\mathbb{R}^d)\backslash C_0^\infty(\mathbb{R}^d)$ and let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $C_0^\infty(\mathbb{R}^d)$ which goes to u in $H^2(\mathbb{R}^d)$. Then u_n goes to u in $L^2(\mathbb{R}^d)$, the sequence $(Au_n)_{n\in\mathbb{N}}$ has a limit in $L^2(\mathbb{R}^d)$ but $u \notin \mathsf{Dom}(A)$. This proves that the Laplace operator is not closed if the domain is $C_0^\infty(\mathbb{R}^d)$. However it is closed with domain $H^2(\mathbb{R}^d)$.

Example 2.29. This example generalizes Examples 2.27 and 2.28. Let Ω be an open subset of $\mathbb{R}^d$. Let $m \in \mathbb{N}$ and consider smooth functions b_α on Ω for all $\alpha \in \mathbb{N}^d$ such that $|\alpha| \leqslant m$. Then we consider the differential operator
$$P = \sum_{|\alpha|\leqslant m} b_\alpha(x)\partial_x^\alpha. \tag{2.2}$$
We denote by P^* the formal adjoint of P, defined for $\phi \in C_0^\infty(\Omega)$ by
$$P^*\phi = \sum_{|\alpha|\leqslant m} (-1)^{|\alpha|}\partial_x^\alpha(\overline{b_\alpha}\phi) = \sum_{|\alpha|\leqslant m} (-1)^{|\alpha|} \sum_{\beta\leqslant\alpha} \binom{\alpha}{\beta} (\partial_x^{\alpha-\beta}\overline{b_\alpha})\partial^\beta. \tag{2.3}$$
Given $u \in L^2(\Omega)$, we have $Pu \in L^2(\Omega)$ (in the sense of distributions) if and only if there exists $v \in L^2(\Omega)$ such that
$$\forall \phi \in C_0^\infty(\Omega), \quad \int_\Omega u\overline{P^*\phi}\,\mathrm{d}x = \int_\Omega v\overline{\phi}\,\mathrm{d}x,$$
and in this case we write $Pu = v$.

We define an unbounded operator A on $L^2(\Omega)$ by setting $Au = Pu$ for any u in the domain
$$\mathsf{Dom}(A) = \left\{u \in L^2(\Omega) \, : \, Pu \in L^2(\Omega)\right\},$$
where Pu is understood in the sense of distributions. This operator A is closed. Indeed, let (u_n) be a sequence in $\mathsf{Dom}(A)$ such that u_n goes to some u and Au_n goes to some v in $L^2(\Omega)$. For $\phi \in C_0^\infty(\Omega)$ we have
$$\begin{aligned}\int_\Omega u(x)\overline{(P^*\phi)(x)}\,\mathrm{d}x &= \lim_{n\to\infty}\int_\Omega u_n(x)\overline{(P^*\phi)(x)}\,\mathrm{d}x = \lim_{n\to\infty}\int_\Omega (Pu_n)(x)\overline{\phi(x)}\,\mathrm{d}x\\ &= \lim_{n\to\infty}\int_\Omega (Au_n)(x)\overline{\phi(x)}\,\mathrm{d}x = \int_\Omega v(x)\overline{\phi(x)}\,\mathrm{d}x.\end{aligned}$$
This proves that in the sense of distributions we have $Pu = v \in L^2(\Omega)$. Therefore $u \in \mathsf{Dom}(A)$ and $Au = v$. This proves that A is closed.

2.2.2 Spectrum of closed operators

Remark 2.30. Let A be an operator from E to F, with domain $\mathsf{Dom}(A)$. Assume that A has a bounded inverse $A^{-1} \in \mathcal{L}(\mathsf{F}, \mathsf{E})$. Then A^{-1} is closed, which implies that A is closed ($\mathsf{Gr}(A)$ is closed in $\mathsf{E} \times \mathsf{F}$ if and only if $\mathsf{Gr}(A^{-1})$ is closed in $\mathsf{F} \times \mathsf{E}$). We can also give a direct proof. Assume that (φ_n) is a sequence in E such that φ_n has a limit φ in E and $A\varphi_n$ has a limit ψ in F. Then $A\varphi_n \to \psi$ and $A^{-1}(A\varphi_n) \to \varphi$. Since A^{-1} is closed, this implies that $\psi \in \mathsf{Dom}(A^{-1}) = \mathsf{F}$ (nothing new here) and $\varphi = A^{-1}\psi$, so $\varphi \in \mathsf{Ran}(A^{-1}) = \mathsf{Dom}(A)$ and $A\varphi = \psi$. This proves that A is closed.

In particular we have the following result.

Proposition 2.31. *Let A be an operator on E. If A is not closed then $\rho(A) = \emptyset$.*

This is why we will only consider the spectral theory of closed operators.

Proposition 2.32. *Let $A : \mathsf{Dom}(A) \subset E \to \mathsf{E}$ be a closed operator. Then $\lambda \in \mathbb{C}$ belongs to the resolvent set $\rho(A)$ of A if and only if $A - \lambda : \mathsf{Dom}(A) \to \mathsf{E}$ is bijective.*

Proof. We already know that if $\lambda \in \rho(A)$ then $A - \lambda : \mathsf{Dom}(A) \to \mathsf{E}$ is bijective. Conversely, assume that $(A - \lambda)$ is bijective. Since it is closed, $\mathsf{Dom}(A)$ is a Banach space and $(A - \lambda)^{-1}$ belongs to $\mathcal{L}(\mathsf{E}, \mathsf{Dom}(A))$, hence to $\mathcal{L}(\mathsf{F}, \mathsf{E})$, by the open mapping theorem (see Theorem A.2). □

Ex. 2.3

Remark 2.33. A closed operator can have empty resolvent set (see Exercise 2.7).

Proposition 2.34. *Let A be an operator on E. Let $z \in \mathbb{C}$. Assume that there exists $c_0 > 0$ such that*
$$\forall \varphi \in \mathsf{Dom}(A), \quad \left\|(A - z)\varphi\right\|_{\mathsf{E}} \geqslant c_0 \left\|\varphi\right\|_{\mathsf{E}}. \tag{2.4}$$
We say that z is a regular point of A. Then

(i) *$(A - z)$ is injective ;*

(ii) *If $(A - z)$ is invertible then $\left\|(A - \lambda)^{-1}\right\| \leqslant c_0^{-1}$.*

(iii) *If moreover A is closed, then $(A - z)$ has closed range.*

This means that if z is a regular point of A, then $z \in \rho(A)$ if and only if $\mathsf{Ran}(A - z)$ is dense in E. Moreover, in this case we already have a bound for the inverse.

Proof. We prove the last statement. Let (ψ_n) be a sequence in $\mathsf{Ran}(A - z)$ which converges to some ψ in E. For $n \in \mathbb{N}$ we consider $\varphi_n \in \mathsf{Dom}(A)$ such that $(A - z)\varphi_n = \psi_n$. Since $((A - z)\varphi_n)$ is a Cauchy sequence, so is (φ_n) by (2.4). Since E is complete, φ_n converges to some φ in E. Finally, since A is closed, $\varphi \in \mathsf{Dom}(A)$ and $\psi = (A - z)\varphi \in \mathsf{Ran}(A - z)$. This proves that $\mathsf{Ran}(A - z)$ is closed in E. □

Ex. 2.3 to 2.8

2.2.3 Closable operators

We have seen in Examples 2.27 and 2.28 that an operator which is not closed can be closed if it is defined on a bigger domain.

Definition 2.35. *We say that on operator A is closable if it has a closed extension.*

Of course, a closed operator is closable.

Proposition 2.36. *Let A be an operator from E to F. The following assertions are equivalent.*

(i) *A is closable;*

(ii) *If $(\varphi_n)_{n\in\mathbb{N}}$ is a sequence in $\mathsf{Dom}(A)$ such that $\varphi_n \to 0$ in E and $A\varphi_n$ has a limit ψ in F, then $\psi = 0$;*

(iii) *$\overline{\mathsf{Gr}(A)}$ is the graph of a closed operator $\overline{A}$ from E to F.*

Definition 2.37. *If the assertions of Proposition 2.36 are satisfied, then the closure of A is the operator $\overline{A}$ such that $\mathsf{Gr}(\overline{A}) = \overline{\mathsf{Gr}(A)}$.*

Proof. • Assume that A is closable and let $\tilde{A}$ be a closed extension of A. Let (φ_n) be a sequence in $\mathsf{Dom}(A)$ such that $\varphi_n \to 0$ in E and $A\varphi_n \to \psi$ in F. Then (φ_n) is also a sequence in $\mathsf{Dom}(\tilde{A})$ and $\tilde{A}\varphi_n \to \psi$. Since $\tilde{A}$ is closed we have $\psi = \tilde{A}0 = 0$.
• Now assume that if a sequence $(\varphi_n)_{n\in\mathbb{N}}$ in $\mathsf{Dom}(A)$ is such that $\varphi_n \to 0$ in E and $A\varphi_n$ has a limit ψ in F, then we necessarily have $\psi = 0$. We denote by $\mathsf{Dom}(\overline{A})$ the closure of $\mathsf{Dom}(A)$ for the graph norm. Let $\varphi \in \mathsf{Dom}(\overline{A})$ and let (φ_n) be a sequence in $\mathsf{Dom}(A)$ which goes to φ for the graph norm. Then $(A\varphi_n)$ is a Cauchy sequence in F, and we denote by $\overline{A}\varphi$ its limit. This definition does not depend on the choice of the sequence (φ_n) since if (ζ_n) is another sequence which goes to φ for the graph norm, we have $\varphi_n - \zeta_n \to 0$ and $A\varphi_n - A\zeta_n$ has a limit, so this limit is 0. This defines a linear map $\overline{A}$ from $\mathsf{Dom}(\overline{A})$ to F, so $\overline{A}$ is an extension of A.

By definition we have $\mathsf{Gr}(\overline{A}) \subset \overline{\mathsf{Gr}(A)}$. Now let $(\varphi, \psi) \in \overline{\mathsf{Gr}(A)}$. There exists a sequence (φ_n, ψ_n) in $\mathsf{Gr}(A)$ such that $\varphi_n \to \varphi$ in E and $\psi_n = A\varphi_n \to \psi$ in F. By definition of $\overline{A}$ we have $\varphi \in \mathsf{Dom}(\overline{A})$ and $\psi = \overline{A}\varphi$, so $(\varphi, \psi) \in \mathsf{Gr}(\overline{A})$. This proves that $\mathsf{Gr}(\overline{A}) = \overline{\mathsf{Gr}(A)}$. Since $\overline{A}$ has a closed graph, this is a closed operator and (iii) is proved.
• Finally, assume (iii). Since $\mathsf{Gr}(A) \subset \mathsf{Gr}(\overline{A})$, $\overline{A}$ is an extension of A, so $\overline{A}$ is a closed extension of A and (i) holds. □

We have already seen examples of operators which are not closed but closable. Here is an example of operator which is not closable.

Example 2.38. We consider on $L^2(\mathbb{R}^d)$ the operators H_0 and H which acts as $-\Delta$ on the domains

$$\mathsf{Dom}(H_0) = C_0^\infty(\mathbb{R}^d) \quad \mathsf{Dom}(H) = H^2(\mathbb{R}^d).$$

Then $H = \overline{H_0}$.

Example 2.39. We consider the operator A from $L^2(\mathbb{R})$ to $\mathbb{C}$ defined on $\mathsf{Dom}(A) = C_0^\infty(\mathbb{R})$ by $Au = u(0)$. Then there exists a sequence $(u_n)_{n\in\mathbb{N}}$ in $C_0^\infty(\mathbb{R})$ such that $u_n \to 0$ in $L^2(\mathbb{R})$ but $u_n(0) \to 1$ in $\mathbb{R}$, so A is not closable.

Proposition 2.40. *If A is a closable operator, then $\overline{A}$ is the smallest closed extension of A (if B is a closed extension of A we have $\overline{A} \subset B$ or, equivalently, $\mathsf{Gr}(\overline{A}) \subset \mathsf{Gr}(B)$).*

Proof. Let B be a closed extension. Then $\mathsf{Gr}(B)$ is closed and contains $\mathsf{Gr}(A)$, so it contains $\mathsf{Gr}(\overline{A}) = \overline{\mathsf{Gr}(A)}$. □

Definition 2.41. *Let A be a closed operator from E to F. Let $\mathcal{D}$ be a linear subspace of $\mathsf{Dom}(A)$. We say that $\mathcal{D}$ is a core of A if $A_{|\mathcal{D}}$ is closable and $\overline{A_{|D}} = A$. Equivalently, $\mathcal{D}$ is dense in $\mathsf{Dom}(A)$ for the graph norm, or for any $\varphi \in \mathsf{Dom}(A)$ there exists a sequence (φ_n) in $\mathcal{D}$ such that $\varphi_n \to \varphi$ in E and $A\varphi_n \to A\varphi$ in F.*

Example 2.42. We consider on $L^2(\mathbb{R}^d)$ the Laplacian $A = -\Delta$, $\mathsf{Dom}(A) = H^2(\mathbb{R}^d)$. Any subspace $\mathcal{D}$ of $H^2(\mathbb{R}^d)$ which contains $C_0^\infty(\mathbb{R}^d)$ is a core of A.

2.3 Adjoint of an unbounded operator

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces.

2.3.1 Definition and properties

Definition 2.43. *Let A be a densely defined operator from $\mathcal{H}_1$ to $\mathcal{H}_2$. Let $\psi \in \mathcal{H}_2$. We say that ψ belongs to $\mathsf{Dom}(A^*)$ if there exists $\psi^* \in \mathcal{H}_1$ such that*

$$\forall \varphi \in \mathsf{Dom}(A), \quad \langle A\varphi, \psi\rangle_{\mathcal{H}_2} = \langle \varphi, \psi^*\rangle_{\mathcal{H}_1}.$$

In this case ψ^* is unique and we set $A^*\psi = \psi^*$. This defines an operator A^* from $\mathcal{H}_2$ to $\mathcal{H}_1$ with domain $\mathsf{Dom}(A^*)$. We say that A^* is the adjoint of A. Ex. 2.9

By definition, we have

$$\forall \varphi \in \mathsf{Dom}(A), \forall \psi \in \mathsf{Dom}(A^*), \quad \langle A\varphi, \psi\rangle_{\mathcal{H}_2} = \langle \varphi, A^*\psi\rangle_{\mathcal{H}_1}.$$

Notice that if A is not densely defined, then $A^*\psi$ is not uniquely defined. We will never consider this situation.

Remark 2.44. Let A be a densely defined operator from $\mathcal{H}_1$ to $\mathcal{H}_2$ and $\psi \in \mathcal{H}_2$. By the Riesz representation theorem, we see that ψ belongs to $\mathsf{Dom}(A^*)$ if and only if there exists $C > 0$ such that

$$\forall \varphi \in \mathsf{Dom}(A), \quad \left|\langle A\varphi, \psi\rangle_{\mathcal{H}_2}\right| \leqslant C \left\|\varphi\right\|_{\mathcal{H}_1}.$$

Moreover, in this case we have $\|A^*\psi\|_{\mathcal{H}_1} \leqslant C$.

Proposition 2.45. *Let A and B be two densely defined operators from $\mathcal{H}_1$ to $\mathcal{H}_2$ such that $B \subset A$. Then $A^* \subset B^*$.*

Proposition 2.46. *Let A be a densely defined operator from $\mathcal{H}_1$ to $\mathcal{H}_2$. Then we have*

$$\ker(A^*) = \mathsf{Ran}(A)^\perp, \quad \ker(A^*)^\perp = \overline{\mathsf{Ran}(A)}.$$

Ex. 2.10

Proposition 2.47. *Let A be a densely defined operator from $\mathcal{H}_1$ to $\mathcal{H}_2$. Then A^* is closed.*

Proof. Let (ψ_n) be a sequence in $\mathsf{Dom}(A^*)$ such that ψ_n goes to some ψ in $\mathcal{H}_2$ and $A^*\psi_n$ goes to some ζ in $\mathcal{H}_1$. For $\varphi \in \mathsf{Dom}(A)$ we have

$$\langle A\varphi, \psi\rangle_{\mathcal{H}_2} - \langle \varphi, \zeta\rangle_{\mathcal{H}_1} = \lim_{n\to+\infty} \langle A\varphi, \psi_n\rangle_{\mathcal{H}_2} - \langle \varphi, A^*\psi_n\rangle_{\mathcal{H}_1} = 0.$$

This proves that $\psi \in \mathsf{Dom}(A^*)$ and $A^*\psi = \zeta$. Thus A^* is closed. □

Proposition 2.48. *Let A be a densely defined operator from $\mathcal{H}_1$ to $\mathcal{H}_2$. Then A is closable if and only if $\mathsf{Dom}(A^*)$ is dense in $\mathcal{H}_2$. Moreover, in this case we have $(\overline{A})^* = A^*$ and $\overline{A} = (A^*)^*$. In particular, A is closed if and only if $A = (A^*)^*$.*

We can write A^{**} instead of $(A^*)^*$.

Proof. • We define

$$\Theta : \begin{cases} \mathcal{H}_1 \times \mathcal{H}_2 & \to & \mathcal{H}_2 \times \mathcal{H}_1, \\ (x_1, x_2) & \mapsto & (-x_2, x_1). \end{cases}$$

Then $\Theta^* = \Theta^{-1} : (y_2, y_1) \mapsto (y_1, -y_2)$.

- Let $(\psi, \tilde\psi) \in \mathcal{H}_2 \times \mathcal{H}_1$. We have

$$\begin{aligned}(\psi, \tilde\psi) \in \mathsf{Gr}(A^*) &\iff \forall \varphi \in \mathsf{Dom}(A), \quad -\langle T\varphi, \psi\rangle_{\mathcal{H}_2} + \left\langle \varphi, \tilde\psi\right\rangle_{\mathcal{H}_1} = 0 \\ &\iff \forall \varphi \in \mathsf{Dom}(A), \quad \langle \Theta(\varphi, A\varphi), (\psi, \tilde\psi)\rangle_{\mathcal{H}_2\times\mathcal{H}_1} = 0 \\ &\iff (\psi, \tilde\psi) \in (\Theta\mathsf{Gr}(A))^\perp,\end{aligned}$$

so

$$\mathsf{Gr}(A^*) = \big(\Theta\mathsf{Gr}(A)\big)^\perp = \Theta\big(\mathsf{Gr}(A)^\perp\big). \tag{2.5}$$

Then

$$\mathsf{Gr}(A^*)^\perp = \overline{\Theta \mathsf{Gr}(A)} = \Theta \overline{\mathsf{Gr}(A)}.$$

After composition by Θ^* we get

$$\overline{\mathsf{Gr}(A)} = \Theta^*\big(\mathsf{Gr}(A^*)^\perp\big). \tag{2.6}$$

- Assume that $\mathsf{Dom}(A^*)$ is dense in $\mathcal{H}_2$. Then we can define $A^{**} = (A^*)^*$. By Proposition 2.47, this defines a closed operator from $\mathcal{H}_1$ to $\mathcal{H}_2$. Let $\varphi \in \mathsf{Dom}(A)$. For all $\psi \in \mathsf{Dom}(A^*)$ we have

$$\langle A^*\psi, \varphi\rangle = \langle \psi, A\varphi\rangle,$$

so $\varphi \in \mathsf{Dom}(A^{**})$ and $A^{**}\varphi = A\varphi$. This proves that A^{**} is an extension of A, and in particular A is closable.
- Now assume that A is closable and let $\psi \in \mathsf{Dom}(A^*)^\perp$. Then, by (2.6),

$$(0,\psi) = \Theta^*(-\psi, 0) \in \Theta^*(\mathsf{Gr}(A^*)^\perp) = \overline{\mathsf{Gr}(A)} = \mathsf{Gr}(\overline{A}).$$

so $\psi = 0$. Thus $\mathsf{Dom}(A^*)$ is dense in $\mathcal{H}_2$. Moreover, by (2.5) applied with $\overline{A}$ we have

$$\mathsf{Gr}((\overline{A})^*) = \Theta\big(\mathsf{Gr}(\overline{A})^\perp\big) = \Theta\big(\overline{\mathsf{Gr}(A)}^\perp\big) = \Theta\big(\mathsf{Gr}(A)^\perp\big) = \mathsf{Gr}(A^*).$$

This proves that $(\overline{A})^* = A^*$. Since A^* is densely defined, we can consider its adjoint A^{**}. By (2.5) applied first to A^* (with Θ replaced by $-\Theta^*$) and then to A, we have

$$\mathsf{Gr}(A^{**}) = \Theta^*\big(\mathsf{Gr}(A^*)^\perp\big) = \Theta^*\Big(\big(\Theta \overline{\mathsf{Gr}(A)}^\perp\big)^\perp\Big) = \big(\overline{\mathsf{Gr}(A)}^\perp\big)^\perp = \overline{\mathsf{Gr}(A)} = \mathsf{Gr}(\overline{A}).$$

This proves that $A^{**} = \overline{A}$. □

Proposition 2.49. *Let A be a closed and densely defined operator from $\mathcal{H}_1$ to $\mathcal{H}_2$. Then $A^* : \mathsf{Dom}(A^*) \to \mathcal{H}_1$ is bijective if and only if $A : \mathsf{Dom}(A) \to \mathcal{H}_2$ is bijective, and in this case we have $(A^*)^{-1} = (A^{-1})^*$.* ✎ *Ex. 2.11*

Proposition 2.50. *Let A be a closed and densely defined operator on $\mathcal{H}$. We have*

✎ *Ex. 2.12*

$$\sigma(A^*) = \overline{\sigma(A)}.$$

2.3.2 Examples: adjoints of some differential operators

General differential operators with smooth and bounded coefficients

Let Ω be an open subset of $\mathbb{R}^d$. We define on $\mathcal{H} = L^2(\Omega)$ the operator A_0 which acts as the differential operator P (see (2.2)) on the domain $\mathsf{Dom}(A_0) = C_0^\infty(\Omega)$. Then $v \in L^2(\Omega)$ belongs to $\mathsf{Dom}(A_0^*)$ if and only if there exists $w \in L^2(\Omega)$ such that

$$\forall \phi \in C_0^\infty(\Omega), \quad \int_\Omega P\phi(x)\overline{v(x)}\,\mathrm{d}x = \int_\Omega \phi(x)\overline{w(x)}\,\mathrm{d}x.$$

By definition, this means that $P^*v = w$ (see (2.3)) in the sense of distributions. Then A_0^* acts as P^* on the domain

$$\mathsf{Dom}(A_0^*) = \left\{v \in L^2(\Omega) \,:\, P^*v \in L^2(\Omega)\right\}.$$

Then A_0 is closed by Proposition 2.47 or by Example 2.29. The domain of A_0^* contains $C_0^\infty(\Omega)$, so it is dense. By Proposition 2.48 this implies that A_0 is closable. This is consistent with the fact that we already know by Example 2.29 that A_0 has a closed extension. Notice that A_0 may have several closed extensions (see for instance the discussion of Section 3.1.5).

The Laplace operator

As a particular case, we consider the Laplace operator. We define the operators which acts as $-\Delta$ on the domains

$$\mathsf{Dom}(H_0) = C_0^\infty(\Omega) \quad \text{and} \quad \mathsf{Dom}(H) = \left\{u \in L^2(\Omega) \,:\, \Delta u \in L^2(\Omega)\right\}.$$

When $\Omega = \mathbb{R}^d$, the domain of H is just $H^2(\mathbb{R}^d)$. We recall that this is not true for a general Ω (it can happen that $u \in L^2(\Omega)$, $\Delta u \in L^2(\Omega)$ but u is not in $H^2(\Omega)$).

Since the formal adjoint of the Laplacian is the Laplacian itself we have in general $H_0^* = H$. Since $H_0 \subset H$ we have $H^* \subset H_0^* = H$ by Proposition 2.45.

When $\Omega = \mathbb{R}^d$ we actually have $H^* = H_0^*$. This follows from the fact that $H = \overline{H_0}$ and Proposition 2.48. We can also give a direct proof. Let $\psi \in \mathsf{Dom}(H_0^*) = H^2(\mathbb{R}^d)$. For $\varphi \in \mathsf{Dom}(H) = H^2(\mathbb{R}^d)$ we have by the Green formula

$$\langle H\varphi, \psi\rangle = \langle -\Delta\varphi, \psi\rangle = \langle \varphi, -\Delta\psi\rangle = \langle \varphi, H_0^*\psi\rangle,$$

so $\psi \in \mathsf{Dom}(H)$. In general, since functions in $\mathsf{Dom}(H)$ or $\mathsf{Dom}(H_0^*)$ are not necessarily in $H^2(\Omega)$, we cannot apply the usual Green formula.

In dimension 1, it is still true that $\mathsf{Dom}(H_0^*) = H^2(\Omega)$. And we can see that in general we do not necessarily have $H^* = H_0^*$. We consider the case $\Omega =]0,1[$. Let $v \in \mathsf{Dom}(H^*)$ and $w = H^* v$. For all $u \in \mathsf{Dom}(H) = H^2(0,1)$ we have

$$-\int_0^1 u''(x)\overline{v(x)}\,\mathrm{d}x = \langle Hu, v\rangle_{L^2(0,1)} = \langle u, w\rangle_{L^2(0,1)} = \int_0^1 u(x)\overline{w(x)}\,\mathrm{d}x.$$

On the other hand, since $v \in \mathsf{Dom}(A_0^*) = H^2(0,1)$ we also have by the Green formula

$$\begin{aligned}-\int_0^1 u(x)''\overline{v(x)}\,\mathrm{d}x &= -u'(1)\overline{v(1)} + u'(0)\overline{v(1)} + \int_0^1 u'(x)\overline{v'(x)}\,\mathrm{d}x\\ &= -u'(1)\overline{v(1)} + u'(0)\overline{v(1)} + u(1)\overline{v'(1)} - u(0)\overline{v'(0)} - \int_0^1 u(x)\overline{v''(x)}\,\mathrm{d}x.\end{aligned}$$

This implies that $w = -v''$ and $v(0) = v(1) = v'(0) = v'(1) = 0$. Then $\mathsf{Dom}(H_0^*)$ is not included in $\mathsf{Dom}(H^*)$.

Creation and annihilation operators

We consider on $\mathcal{H} = L^2(\mathbb{R})$ the creation and annihilation operators defined on the domain $C_0^\infty(\mathbb{R})$ by

$$\forall u \in C_0^\infty(\mathbb{R}), \quad \mathsf{a}_0 u = \frac{u' + xu}{\sqrt{2}} \quad \text{and} \quad \mathsf{c}_0 u = \frac{-u' + xu}{\sqrt{2}}.$$

Then we set

$$\mathsf{a} = \overline{\mathsf{a}_0} \quad \text{and} \quad \mathsf{c} = \overline{\mathsf{c}_0}.$$

We have

$$\mathsf{Dom}(\mathsf{a}) = \left\{u \in L^2(\mathbb{R}) \,:\, u' + xu \in L^2(\mathbb{R})\right\}, \quad \mathsf{Dom}(\mathsf{c}) = \left\{u \in L^2(\mathbb{R}) \,:\, -u' + xu \in L^2(\mathbb{R})\right\}.$$

Finally we have

$$\mathsf{a}^* = \mathsf{c} \quad \text{and} \quad \mathsf{c}^* = \mathsf{a}.$$

2.4 Example: the harmonic oscillator

We consider on $L^2(\mathbb{R})$ the operator H which acts as

$$H = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + x^2 \tag{2.7}$$

on the domain

$$\mathsf{Dom}(H) = \left\{u \in L^2(\mathbb{R}) \,:\, -u'' + x^2 u \in L^2(\mathbb{R})\right\}. \tag{2.8}$$

Proposition 2.51. *The spectrum of H consists of a sequence $(\lambda_k)_{k\in\mathbb{N}}$ of simple eigenvalues. Moreover, for $k \in \mathbb{N}^*$ we have*

$$\lambda_k = (2k+1)$$

and a corresponding eigenfunction is given by

$$\varphi_k(x) = h_k(x)e^{-\frac{x^2}{2}},$$

where $h_k(x) =$ is the k-th Hermite polynomial (in particular it has degree k).

Proof. • We recall that we have introduced the operators a and c is Section 2.3.2. We observe that for $u \in \mathcal{S}(\mathbb{R})$ we have

$$Hu = 2\mathsf{ca}u + u.$$

We also have $[\mathsf{a},\mathsf{c}]u = \mathsf{ac}u - \mathsf{ca}u = u$ so, by induction on k,

$$\mathsf{ac}^k u = k\mathsf{c}^{k-1}u + \mathsf{c}^k au. \tag{2.9}$$

• We set $\varphi_0(x) = e^{-\frac{x^2}{2}}$. We have $\varphi_0 \in \mathcal{S}(\mathbb{R})$ and $\mathsf{a}\varphi_0 = 0$, so $H\varphi_0 = \varphi_0$. For $k \in \mathbb{N}^*$ we set $\varphi_k = \mathsf{c}^k\varphi_0$. We can check by induction on $k \in \mathbb{N}$ that φ_k is of the form $\varphi_k = P_k\varphi_0$ where P_k is a polynomial of degree k. In particular $\varphi_k \in \mathcal{S}(\mathbb{R})$. We have

$$H\varphi_k = 2\mathsf{cac}^k\varphi_0 + \varphi_k = 2k\mathsf{c}^k\varphi_0 + 2\mathsf{c}^{k+1}\mathsf{a}\varphi_0 + \varphi_k = (2k+1)\varphi_k.$$

This prove that $\lambda_k = 2k+1$ is an eigenvalue of H and φ_k is a corresponding eigenfunction.
• We prove by induction on $j \in \mathbb{N}$ that for all $k > 0$ we have $\langle \varphi_j, \varphi_k \rangle = 0$. Since $\mathsf{c}^* = \mathsf{a}$, we have

$$\langle \varphi_j, \varphi_k \rangle = \left\langle \mathsf{c}^j\varphi_0, \mathsf{c}^k\varphi_0 \right\rangle = \left\langle \mathsf{a}^k\mathsf{c}^j\varphi_0, \varphi_0 \right\rangle.$$

Since $\mathsf{a}\varphi_0 = 0$ the conclusion follows if $j = 0$. For $j \geqslant 1$ we have by

$$\left\langle \mathsf{a}^k\mathsf{c}^j\varphi_0, \varphi_0 \right\rangle = j\left\langle \mathsf{a}^{k-1}\mathsf{c}^{j-1}\varphi_0, \varphi_0 \right\rangle + \left\langle \mathsf{a}^{k-1}\mathsf{c}^j\mathsf{a}\varphi_0, \varphi_0 \right\rangle = 0.$$

This proves that the family of eigenvectors $(\varphi_k)_{k\in\mathbb{N}}$ is orthogonal in $L^2(\mathbb{R})$.
• Let us prove that the family (φ_k) is total in $L^2(\mathbb{R})$. This means that $\overline{\mathsf{span}((\varphi_k)_{k\in\mathbb{N}})} = L^2(\mathbb{R})$. Let $u \in L^2(\mathbb{R})$ be such that $\langle \varphi_k, u \rangle_{L^2(\mathbb{R})} = 0$ for all $k \in \mathbb{N}$. Since P_k is of degree k for all k, we deduce that for any polynomial q we have

$$\int_{\mathbb{R}} \mathsf{q}(x)e^{-\frac{x^2}{2}}u(x)\,\mathrm{d}x = 0.$$

For $\zeta \in \mathbb{C}$ we set

$$v(\zeta) = \int_{\mathbb{R}} e^{-ix\xi}u(x)e^{-\frac{x^2}{2}}\,\mathrm{d}x.$$

By differentiation under the integral sign we see that v is holomorphic in $\mathbb{C}$ and for $m \in \mathbb{N}$ we have

$$v^{(m)}(0) = \int_{\mathbb{R}} (-ix)^m u(x)e^{-\frac{x^2}{2}}\,\mathrm{d}x = 0.$$

This implies that $v = 0$ on $\mathbb{C}$, and in particular in $\mathbb{R}$. Thus the Fourier transform of $x \mapsto u(x)e^{-\frac{x^2}{2}}$ is 0, so $u = 0$ almost everywhere.

For $k \in \mathbb{N}$ we set

$$\psi_k = \frac{\varphi_k}{\|\varphi_k\|}.$$

Then (ψ_k) is a Hilbert basis of $L^2(\mathbb{R})$, and $H\psi_k = \lambda_k\psi_k$ for all k. Thus the spectrum of H is exactly given by the sequence $(\lambda_k)_{k\in\mathbb{N}}$ of simple eigenvalues. □

2.5 A representation theorem

2.5.1 The abstract result

Let $\mathcal{H}$ be a Hilbert space. We identify $\mathcal{H}$ with its dual $\mathcal{H}'$. Then if $\mathcal{V}$ is another Hilbert space continuously embedded in $\mathcal{H}$ we have

$$\mathcal{V} \subset \mathcal{H} \simeq \mathcal{H}' \subset \mathcal{V}'.$$

Notice that if we have already identified $\mathcal{H}$ with $\mathcal{H}'$ we cannot identify $\mathcal{V}$ with $\mathcal{V}'$.

Theorem 2.52 (Representation theorem)**.** *Let $\mathcal{H}$ and $\mathcal{V}$ be two Hilbert spaces such that $\mathcal{V}$ is densely and continuously embedded in $\mathcal{H}$. Let q be a continuous and coercive sesquilinear form on $\mathcal{V}$. We set*

$$\mathsf{Dom}(A) = \{\varphi \in \mathcal{V} : \exists C_\varphi > 0, \forall \psi \in \mathcal{V}, |\mathsf{q}(\varphi,\psi)| \leqslant C_\varphi \|\psi\|_{\mathcal{H}}\},$$

and for $\varphi \in \mathsf{Dom}(A)$ we define $A\varphi \in \mathcal{H}$ by

$$\forall \psi \in \mathcal{V}, \quad \mathsf{q}(\varphi,\psi) = \langle A\varphi, \psi\rangle_{\mathcal{H}}.$$

This defines on $\mathcal{H}$ an operator A with domain $\mathsf{Dom}(A)$ such that

(i) $\mathsf{Dom}(A)$ *is dense in $\mathcal{V}$ and in $\mathcal{H}$;*

(ii) *A is closed ;*

(iii) *A is invertible.*

Moreover, the operator on $\mathcal{H}$ associated to the form q^* is A^*.

Proof. • Let $\varphi \in \mathsf{Dom}(A)$. The map $\psi \mapsto \mathsf{q}(\varphi,\psi)$ extends to a bounded semilinear form on $\mathcal{H}$. Then, by the Riesz theorem, there exists a vector $A\varphi \in \mathcal{H}$ such that $\mathsf{q}(\varphi,\psi) = \langle A\varphi,\psi\rangle_{\mathcal{H}}$ for all $\psi \in \mathcal{V}$. This defines on $\mathcal{H}$ an operator A with domain $\mathsf{Dom}(A)$ (the linearity of A is left as an exercise).
• Let $\zeta \in \mathcal{H}$. The map $\psi \in \mathcal{V} \mapsto \langle \zeta,\psi\rangle_{\mathcal{H}}$ is a continuous semilinear map on $\mathcal{V}$ so, by the Lax-Milgram theorem, there exists $\varphi \in \mathcal{V}$ such that

$$\forall \psi \in \mathcal{V}, \quad \langle \zeta,\psi\rangle_{\mathcal{H}} = \mathsf{q}(\varphi,\psi).$$

Then we have $\varphi \in \mathsf{Dom}(A)$ and $A\varphi = \zeta$. This proves that A is surjective.
• For $\varphi \in \mathsf{Dom}(A)$ we have

$$\|A\varphi\|_{\mathcal{H}} \|\varphi\|_{\mathcal{H}} \geqslant |\langle A\varphi,\varphi\rangle_{\mathcal{H}}| = |\mathsf{q}(\varphi,\varphi)| \geqslant \alpha \|\varphi\|_{\mathcal{V}}^2 \geqslant \alpha \tilde{C}^{-1} \|\varphi\|_{\mathcal{H}}^2,$$

where $\tilde{C} > 0$ is such that $\|\psi\|_{\mathcal{H}}^2 \leqslant \tilde{C} \|\psi\|_{\mathcal{V}}^2$ for all $\psi \in \mathcal{V}$. Thus,

$$\|A\varphi\|_{\mathcal{H}} \geqslant \alpha \tilde{C}^{-1} \|\varphi\|_{\mathcal{H}}. \tag{2.10}$$

This proves in particular that A is injective. Since A is surjective, it is invertible and $\|A^{-1}\|_{\mathcal{L}(\mathcal{H})} \leqslant \alpha^{-1}\tilde{C}$. This implies that A is closed (see Remark 2.30).
• Let $\psi \in \mathcal{V}$ be in the orthogonal of $\mathsf{Dom}(A)$ in $\mathcal{V}$. Let $T \in \mathcal{L}(\mathcal{V})$ be given by the Lax-Milgram Theorem (Theorem 1.59). Since T^* is bijective, there exists $\zeta \in \mathcal{V}$ such that $T^*\zeta = \psi$. Then for all $\varphi \in \mathsf{Dom}(A)$ we have

$$0 = \langle \varphi,\psi\rangle_{\mathcal{V}} = \langle \varphi, T^*\zeta\rangle_{\mathcal{V}} = \langle T\varphi,\zeta\rangle_{\mathcal{V}} = \mathsf{q}(\varphi,\psi) = \langle A\varphi,\zeta\rangle_{\mathcal{H}}.$$

Since A is surjective, this implies that $\zeta = 0$, and hence $\psi = 0$. Then $\mathsf{Dom}(A)$ is dense in $\mathcal{V}$ for the topology of $\mathcal{V}$, and hence for the topology of $\mathcal{H}$. Since $\mathcal{V}$ is dense in $\mathcal{H}$, $\mathsf{Dom}(A)$ is also dense in $\mathcal{H}$.
• We denote by $\tilde{A}$ the operator associated to q^*. Since q^* is continuous and coercive, $\tilde{A}$ is also a densely defined, closed and invertible operator on $\mathcal{H}$. Let $\psi \in \mathsf{Dom}(\tilde{A})$. For all $\varphi \in \mathsf{Dom}(A)$ we have

$$\langle A\varphi,\psi\rangle = \mathsf{q}(\varphi,\psi) = \overline{\mathsf{q}^*(\psi,\varphi)} = \overline{\langle \tilde{A}\psi,\varphi\rangle} = \langle \varphi, \tilde{A}\psi\rangle.$$

This proves that $\tilde{A} \subset A^*$. Conversely, if $\psi \in \mathsf{Dom}(A^*)$ then for all $\varphi \in \mathsf{Dom}(A)$ we have

$$|\mathsf{q}^*(\psi, \varphi)| = |\mathsf{q}(\varphi, \psi)| = |\langle A\varphi, \psi\rangle| = |\langle \varphi, A^*\psi\rangle| \leqslant \|A^*\psi\|_{\mathcal{H}} \|\varphi\|_{\mathcal{H}}.$$

Since $\mathsf{Dom}(A)$ is dense in $\mathcal{V}$ and $\mathcal{H}$, we deduce that for all $\varphi \in \mathcal{V}$ we have

$$|\mathsf{q}^*(\psi, \varphi)|^{\leqslant} \|A^*\psi\|_{\mathcal{H}} \|\varphi\|_{\mathcal{H}},$$

so $\psi \in \mathsf{Dom}(\tilde{A})$. This proves that $\mathsf{Dom}(A^*) \subset \mathsf{Dom}(\tilde{A})$, so $\tilde{A} = A^*$. □

Remark 2.53. Let q be a continuous quadratic form on $\mathcal{V}$. Assume that there exists $\beta \in \mathbb{C}$ such that the form $\mathsf{q}_\beta : \varphi \mapsto \mathsf{q}(\varphi) + \beta \|\varphi\|_{\mathcal{H}}$ is coercive on $\mathcal{V}$. Let A_β be the operator on $\mathcal{H}$ given by Theorem 2.52 and $A = A_\beta - \beta$ with $\mathsf{Dom}(A) = \mathsf{Dom}(A_\beta)$. Then A is closed and densely defined, and $(A + \beta)$ is invertible. Notice that this definition of A does not depend on the choice of β.

Remark 2.54. Let q be a continuous coercive quadratic form on $\mathcal{V}$ and $Q \in \mathcal{L}(\mathcal{V}, \mathcal{V}')$ defined by (1.8) (invertible by Theorem 1.59). Let A the operator on $\mathcal{H}$ be given by Theorem 2.52. Then for all $\varphi \in \mathcal{H} \subset \mathcal{V}'$ we have $Q^{-1}\varphi = A^{-1}\varphi$.

2.5.2 Examples: Laplacian, Dirichlet and Neumann boundary conditions

Example 2.55. We consider on $H^1(\mathbb{R})$ the quadratic form

$$\mathsf{q} : u \mapsto \|u\|^2_{H^1(\mathbb{R})}.$$

We apply Theorem 2.52 with $\mathcal{V} = H^1(\mathbb{R})$ and $\mathcal{H} = L^2(\mathbb{R})$. We have

$$\mathsf{Dom}(A) = \left\{u \in H^1(\mathbb{R}) : u'' \in L^2(\mathbb{R})\right\} = H^2(\mathbb{R}).$$

Indeed, if $u \in H^2(\mathbb{R})$ then for all $v \in H^1(\mathbb{R})$ we have

$$|q(u, v)| = \left|-\int_{\mathbb{R}} u''\overline{v}\,\mathrm{d}x + \int_{\mathbb{R}} u\overline{v}\,\mathrm{d}x\right| \leqslant (\|u''\| + \|u\|)\|v\|,$$

so $u \in \mathsf{Dom}(A)$. Conversely, assume that $u \in \mathsf{Dom}(A)$. Then for all $v \in H^1(\mathbb{R})$ we have

$$\left|\int_{\mathbb{R}} u'\overline{v}'\,\mathrm{d}x\right| \leqslant |q(u, v)| + \|u\|\|v\| \leqslant (C_u + \|u\|)\|v\|.$$

This proves that $u'' \in L^2$, and hence $u \in H^2(\mathbb{R})$. Finally, for $u \in \mathsf{Dom}(A)$ we have

$$\forall v \in H^1(\mathbb{R}), \quad \langle Au, v\rangle = q(u, v) = \left\langle -u'' + u, v\right\rangle,$$

so

$$Au = -u'' + u.$$

Example 2.56. We consider on $H^1(0, 1)$ the quadratic form

$$\mathsf{q}_N : u \mapsto \|u\|^2_{H^1(0,1)}.$$

We apply Theorem 2.52 with $\mathcal{V} = H^1(0, 1)$ and $\mathcal{H} = L^2(0, 1)$. We denote by A_N the corresponding operator. Let $u \in \mathsf{Dom}(A_N)$. For all $\phi \in C_0^\infty(]0, 1[) \subset H^1(0, 1)$ we have as above

$$\left|\int_0^1 u'\overline{\phi}'\,\mathrm{d}x\right| \leqslant (C_u + \|u\|)\|\phi\|.$$

This implies that $u'' \in L^2(0, 1)$. Then for all $\phi \in C_0^\infty(]0, 1[)$ we have

$$\langle A_N u, \phi\rangle = q_N(u, \phi) = \int_0^1 u'\overline{\phi}'\,\mathrm{d}x + \int_0^1 u\overline{\phi}\,\mathrm{d}x = \left\langle -u'' + u, \phi\right\rangle.$$

This proves that $A_N u = -u'' + u$. Then for all $v \in H^1(0,1)$ we have

$$\langle A_N u, v\rangle = q_N(u,v) = \int_0^1 u'\overline{v}'\,\mathrm{d}x + \int_0^1 u\overline{v}\,\mathrm{d}x = u'(1)\overline{v}(1) - u'(0)\overline{v}(0) + \left\langle -u'' + u, v\right\rangle$$

This proves that for all $v \in H^1(0,1)$

$$u'(1)\overline{v}(1) - u'(0)\overline{v}(0) = 0.$$

This implies that

$$u'(0) = u'(1) = 0. \tag{2.11}$$

Conversely, assume that $u \in H^2(0,1)$ satisfies (2.11). Then we can compute as above that

$$\forall v \in H^1(0,1), \quad q(u,v) = \left\langle -u'' + u, v\right\rangle.$$

Then $u \in \mathsf{Dom}(A_N)$. Finally we have

$$\mathsf{Dom}(A_N) = \left\{u \in H^2(0,1)\ :\ u'(0) = u'(1) = 0\right\}$$

and, for all $u \in \mathsf{Dom}(A_N)$,

$$A_N u = -u'' + u.$$

Example 2.57. We consider on $H_0^1(0,1)$ the quadratic form

$$\mathsf{q}_D : u \mapsto \|u\|^2_{H^1(0,1)}\,.$$

We apply Theorem 2.52 with $\mathcal{V} = H_0^1(0,1)$ and $\mathcal{H} = L^2(0,1)$. We denote by A_D the corresponding operator. Let $u \in \mathsf{Dom}(A_D)$. As above we see that $u \in H^2(0,1)$ and $A_D u = -u'' + u$. On the other hand, if $u \in H^2(0,1) \cap H_0^1(0,1)$ we have $q(u,v) = \langle -u'' + u, v\rangle$ for all $v \in H_0^1(0,1)$ (there are no boundary terms since u and v vanish at the boundary). Finally we have

$$\mathsf{Dom}(A_D) = H^2(0,1) \cap H_0^1(0,1),$$

and for all $u \in \mathsf{Dom}(A_D)$

$$A_D u = -u'' + u.$$

Example 2.58. By Remark 2.53 we can define the operators associated to the form

$$u \mapsto \int_0^1 |u(x)|^2\,\mathrm{d}x$$

defined on $H^1(\mathbb{R})$ and $H^1(0,1)$ (note that this form is already coercive on $H_0^1(0,1)$). ✐ Ex. 2.13

2.6 Riesz projections

2.6.1 Separation of the spectrum

The interest of the resolvent is that it is a bounded operator which completely characterize the operator. Moreover, since it is analytic, we can use all the tools from complex analysis. In the following section we give a first application of the resolvent for the analysis of an operator.

Let E be a Banach space.

Proposition 2.59. *Let A be an operator on E. Let Π be a projection of E such that*

$$\Pi A \subset A\Pi$$

(for all $\varphi \in \mathsf{Dom}(A)$ we have $\Pi\varphi \in \mathsf{Dom}(A)$ and $A\Pi\varphi = \Pi A\varphi$). Let $\mathsf{F} = \mathsf{Ran}(\Pi)$ and $\mathsf{G} = \ker(\Pi)$.

(i) *F and G are closed subspace of E and $\mathsf{E} = \mathsf{F} \oplus \mathsf{G}$.*

(ii) *A maps $\mathsf{Dom}(A)\cap\mathsf{F}$ to F and $\mathsf{Dom}(A)\cap\mathsf{G}$ to G. We denote by A_F and A_G the restrictions of A to F and G, with $\mathsf{Dom}(A_\mathsf{F}) = \mathsf{Dom}(A)\cap\mathsf{F}$ and $\mathsf{Dom}(A_\mathsf{G}) = \mathsf{Dom}(A)\cap\mathsf{G}$.*

(iii) *If $\mathsf{Dom}(A)$ is dense in E then $\mathsf{Dom}(A_\mathsf{F})$ is dense in F and $\mathsf{Dom}(A_\mathsf{G})$ is dense in G.*

(iv) *If A is closed then A_F and A_G are closed.*

(v) *We have $\sigma(A) = \sigma(A_\mathsf{F}) \cup \sigma(A_\mathsf{G})$ and for $z \in \rho(A) = \rho(A_\mathsf{F}) \cap \rho(A_\mathsf{G})$ we have*

$$(A-z)^{-1} = (A_\mathsf{F} - z)^{-1} \oplus (A_\mathsf{G} - z)^{-1}.$$

Proof. • G is closed since it is the kernel of the bounded operator Π, and F is closed since it is the kernel of $(1-\Pi)$. Let $\varphi \in \mathsf{F}\cap\mathsf{G}$. We have $\varphi = \Pi\varphi = 0$, so $\mathsf{F}\cap\mathsf{G} = \{0\}$. On the other hand, for $\varphi \in \mathsf{E}$ we have $\varphi = A\varphi + (\varphi - A\varphi)$ with $A\varphi \in \mathsf{F}$ and $\varphi - A\varphi \in \mathsf{G}$, so $\mathsf{E} = \mathsf{F} + \mathsf{G}$.
• For $\varphi \in \mathsf{Dom}(A)\cap\mathsf{F}$ we have $\Pi A\varphi = A\Pi\varphi = A\varphi$, so $A\varphi \in \ker(1-\Pi) = \mathsf{F}$. We proceed similarly for $\mathsf{G} = \ker(\Pi)$.
• Assume that $\mathsf{Dom}(A)$ is dense in E. Let $\varphi \in \mathsf{F}$. There exists a sequence (φ_n) in $\mathsf{Dom}(A)$ which converges to φ in E. For $n \in \mathbb{N}$ we have $\Pi\varphi_n \in \mathsf{Dom}(A)$ by assumption. Then $\Pi\varphi_n \in \mathsf{Dom}(A)\cap\mathsf{F}$ converges to $\Pi\varphi = \varphi$. We proceed similarly for G.
• Assume that A is closed. Let (φ_n) be a sequence in $\mathsf{Dom}(A_\mathsf{F})$ such that $\varphi_n \to \varphi$ and $A_\mathsf{F}\varphi_n \to \psi$ in F. Then $\varphi_n \to \varphi$ and $A\varphi \to \psi$ in E. Since A is closed, this proves that $\varphi \in \mathsf{Dom}(A)$ and $A\varphi = \psi$. Since $\varphi \in \mathsf{F}$ we also have $\varphi \in \mathsf{Dom}(A_\mathsf{F})$ and $A_F\varphi = \psi$. This proves that A_F is closed.
• Let $z \in \rho(A)$. The restriction of $(A-z)^{-1}$ to F is an inverse for $(A_\mathsf{F} - z)$, so $\rho(A) \subset \rho(A_\mathsf{F})$. Similarly, $\rho(A) \subset \rho(A_\mathsf{G})$. Conversely, if $z \in \rho(A_\mathsf{F}) \cap \rho(A_\mathsf{G})$ then $(A_\mathsf{F} - z)^{-1} \oplus (A_\mathsf{G} - z)^{-1}$ is an inverse for $A - z = (A_\mathsf{F} - z) \oplus (A_\mathsf{G} - z)$, so $\rho(A) = \rho(A_\mathsf{F}) \cap \rho(A_\mathsf{G})$. □

Ex. 2.14

Proposition 2.60. *Let $z_0 \in \mathbb{C}$ and $r_0 > 0$ such that $\mathcal{C}(z_0, r_0) \subset \rho(A)$. We set*

$$\Pi = -\frac{1}{2i\pi}\int_{\mathcal{C}(z_0,r_0)} (A-\zeta)^{-1}\,\mathrm{d}\zeta = -\frac{1}{2\pi}\int_0^{2\pi} \left(A - (z_0 + r_0 e^{i\theta})\right)^{-1} r_0 e^{i\theta}\,\mathrm{d}\theta.$$

We set $\mathsf{F} = \mathsf{Ran}(\Pi)$ and $\mathsf{G} = \ker(\Pi)$.

(i) Π is a (not necessarily orthogonal) projection of E.

(ii) $\mathsf{F} \subset \mathsf{Dom}(A)$.

(iii) $\Pi A \subset A\Pi$.

(iv) $\sigma(A_\mathsf{F}) = \sigma(A) \cap D(z_0, r_0)$ and $\sigma(A_\mathsf{G}) = \sigma(A)\setminus\overline{D}(z_0, r_0)$.

Remark 2.61. In Proposition 2.60 we consider for simplicity the case where Π is defined by an integral on a circle. But we can similarly consider the integral on any rectifiable simple closed curve in $\rho(A)$ (see [Kat80, § III.6.4]).

Proof. • Π is defined by the integral on a line segment of a continuous function with values in the Banach space $\mathcal{L}(\mathsf{E})$. This can be understood in the sense of Riemann integrals and this defines a bounded operator on E. In particular we have in $\mathcal{L}(\mathsf{E})$

$$\Pi = \lim_{n\to+\infty} \Pi_n, \quad \text{where} \quad \Pi_n = -\frac{1}{n}\sum_{k=1}^{n} \left(A - (z_0 + r_0 e^{\frac{ik}{2\pi}})\right)^{-1} r_0 e^{\frac{ik}{2\pi}}.$$

Then for $\varphi \in \mathsf{E}$ and $\ell \in \mathsf{E}^*$ we have

$$\ell(\Pi\varphi) = -\frac{1}{2i\pi}\int_{\mathcal{C}(z_0,r_0)} \ell\big((A-z)^{-1}\varphi\big)\,\mathrm{d}z.$$

Since $\rho(A)$ is open in $\mathbb{C}$, there exists $R_1 \in [0, r_0[$ and $R_2 > r_0$ such that $D(0,R_2)\setminus\overline{D}(0,R_1) \subset \rho(A)$. Let $\varphi \in \mathsf{E}$ and $\ell \in \mathsf{E}^*$. Since the map $\zeta \mapsto \ell\big((A-\zeta)^{-1}\varphi\big)$ is holomorphic on $\rho(A)$, we can replace r_0 by any $r \in]R_1, R_2[$ in the expression of Π.

- Let $r_1, r_2 \in]R_1, R_2[$ with $r_1 < r_2$. We can write
$$\Pi^2 = \frac{1}{(2i\pi)^2} \int_{\zeta_1 \in \mathcal{C}(z_0,r_1)} \int_{\zeta_2 \in \mathcal{C}(z_0,r_2)} (A-\zeta_1)^{-1}(A-\zeta_2)^{-1} \, \mathrm{d}\zeta_2 \, \mathrm{d}\zeta_1.$$
By the resolvent identity we have
$$\Pi^2 = \frac{1}{(2i\pi)^2} \int_{\zeta_1 \in \mathcal{C}(z_0,r_1)} \int_{\zeta_2 \in \mathcal{C}(z_0,r_2)} \frac{(A-\zeta_1)^{-1} - (A-\zeta_2)^{-1}}{\zeta_1 - \zeta_2} \, \mathrm{d}\zeta_2 \, \mathrm{d}\zeta_1.$$
Then, by the Fubini Theorem,
$$\begin{aligned} \Pi^2 = &-\frac{1}{(2i\pi)^2} \int_{\zeta_1 \in \mathcal{C}(z_0,r_1)} (A-\zeta_1)^{-1} \left(\int_{\zeta_2 \in \mathcal{C}(z_0,r_2)} \frac{1}{\zeta_2 - \zeta_1} \, \mathrm{d}\zeta_2 \right) \mathrm{d}\zeta_1 \\ &- \frac{1}{(2i\pi)^2} \int_{\zeta_2 \in \mathcal{C}(z_0,r_2)} (A-\zeta_2)^{-1} \left(\int_{\zeta_1 \in \mathcal{C}(z_0,r_1)} \frac{1}{\zeta_1 - \zeta_2} \, \mathrm{d}\zeta_1 \right) \mathrm{d}\zeta_2. \end{aligned}$$
We look at the integral in brackets for each term. For the second term, for any $\zeta_2 \in \mathcal{C}(z_0, r_2)$ the map $\zeta_1 \mapsto 1/(\zeta_1 - \zeta_2)$ is holomorphic on $D(z_0, r_2)$, so the integral vanishes. For the first term, we get by the Cauchy Theorem that the integral is equal to $2i\pi$ for all $\zeta_1 \in \mathcal{C}(z_0, r_1)$. Then
$$\Pi^2 = -\frac{1}{2i\pi} \int_{\zeta_2 \in \mathcal{C}(z_0,r_2)} (A-\zeta_2)^{-1} \, \mathrm{d}\zeta_2 = \Pi.$$
This proves that Π is a projection of E.
- Let $\varphi \in \mathsf{F}$ and $\psi \in \mathsf{E}$ such that $\varphi = \Pi\psi$. For $n \in \mathbb{N}^*$ we set $\varphi_n = \Pi_n \psi \in \mathsf{Dom}(A)$. Then $\varphi_n \to \varphi$ in E. Moreover,
$$\begin{aligned} A\varphi_n &= -\frac{1}{n} \sum_{k=1}^n A\big(A - (z_0 + r_0 e^{i\theta_k})\big)^{-1} r_0 e^{i\theta_k} \psi \\ &= -\frac{1}{n} \sum_{k=1}^n \big(\mathrm{Id} + (z_0 + r_0 e^{i\theta_k})\big(A - (z_0 + r_0 e^{i\theta_k})\big)^{-1}\big) r_0 e^{i\theta_k} \psi \\ &\xrightarrow[n\to\infty]{} -\frac{1}{2i\pi} \int_{\mathcal{C}(z_0,r_0)} \big(\mathrm{Id} + \zeta(A-\zeta)^{-1}\big)\psi \, \mathrm{d}\zeta = -\frac{1}{2i\pi} \int_{\mathcal{C}(z_0,r_0)} \zeta(A-\zeta)^{-1}\psi \, \mathrm{d}\zeta. \end{aligned}$$
Since A is closed this proves that $\varphi \in \mathsf{Dom}(A)$ (and $A\varphi = -\frac{1}{2i\pi} \int_{\mathcal{C}(z_0,r_0)} \zeta(A-\zeta)^{-1}\psi \, \mathrm{d}\zeta$).
- Let $\varphi \in \mathsf{Dom}(A)$. Since A commutes with its resolvent, we have $A\Pi_n\varphi = \Pi_n A\varphi$ for all $n \in \mathbb{N}^*$. Since $\Pi_n\varphi \to \Pi\varphi$ and $A\Pi_N\varphi = \Pi_N A\varphi \to \Pi A\varphi$, we get by closedness of A that $\Pi\varphi \in \mathsf{Dom}(A)$ and $A\Pi\varphi = \Pi A\varphi$.
- Let $z \in \rho(A_\mathsf{F}) \backslash D(z_0, r_0)$. Let $r \in]R_1, r_0[$. We have on F
$$\begin{aligned} (A_\mathsf{F} - z)^{-1} &= (A_\mathsf{F} - z)^{-1}\Pi \\ &= -\frac{1}{2i\pi} \int_{\zeta \in \mathcal{C}(z_0,r)} (A_\mathsf{F} - z)^{-1}(A_\mathsf{F} - \zeta)^{-1} \, \mathrm{d}\zeta \\ &= -\frac{1}{2i\pi} \int_{\zeta \in \mathcal{C}(z_0,r)} \frac{(A_\mathsf{F} - z)^{-1} - (A_\mathsf{F} - \zeta)^{-1}}{z - \zeta} \, \mathrm{d}\zeta \\ &= \frac{1}{2i\pi} \int_{\zeta \in \mathcal{C}(z_0,r)} \frac{(A_\mathsf{F} - \zeta)^{-1}}{z - \zeta} \, \mathrm{d}\zeta. \end{aligned}$$
The right-hand side is bounded uniformly in $z \in \rho(A_\mathsf{F}) \backslash D(z_0, r_0)$. By Proposition 2.23 this implies that
$$\sigma(A_\mathsf{F}) \subset D(z_0, r_0). \tag{2.12}$$
Now let $z \in \rho(A_\mathsf{G}) \cap D(z_0, r_0)$ and $r \in]r_0, R_2[$. We have on G
$$\begin{aligned} (A_\mathsf{G} - z)^{-1} &= (A_\mathsf{G} - z)^{-1}(1 - \Pi) \\ &= (A_\mathsf{G} - z)^{-1} - \frac{1}{2i\pi} \int_{\zeta \in \mathcal{C}(z_0,r)} \frac{(A_\mathsf{G} - z)^{-1} - (A_\mathsf{G} - \zeta)^{-1}}{\zeta - z} \, \mathrm{d}\zeta \\ &= \frac{1}{2i\pi} \int_{\zeta \in \mathcal{C}(z_0,r)} \frac{(A_\mathsf{G} - \zeta)^{-1}}{z - \zeta} \, \mathrm{d}\zeta. \end{aligned}$$

This is bounded uniformly in $z \in \rho(A_\mathsf{G}) \cap D(z_0, r_0)$, so

$$\sigma(A_\mathsf{G}) \subset \mathbb{C}\backslash\overline{D}(0, r_0). \tag{2.13}$$

Finally, with Proposition 2.59 and (2.12)-(2.13) we deduce that $\sigma(A_\mathsf{F}) = \sigma(A) \cap D(0, r_0)$ and $\sigma(A_\mathsf{G}) = \sigma(A)\backslash\overline{D}(0, r_0)$. □

Ex. ??, 2.15

Definition 2.62. *Let A be a closed operator on E. Assume that $\lambda \in \mathbb{C}$ is an isolated point in the spectrum of A. Let $r_0 > 0$ such that $\sigma(A) \cap D(\lambda, r) = \{\lambda\}$ and $r \in]0, r_0[$. Then the Riesz Projection of A at λ is*

$$\Pi_\lambda = -\frac{1}{2i\pi} \int_{\mathcal{C}(\lambda,r)} (A - z)^{-1} \, \mathrm{d}z \tag{2.14}$$

(the definition does not depend on the choice of r).

Definition 2.63. *Let λ be an isolated eigenvalue of A. The algebraic multiplicity of λ is $\mathsf{dim}(\mathsf{Ran}(\Pi_\lambda))$, where Π_λ is the Riesz projection at λ.*

Remark 2.64. Since $\ker(A - \lambda) \subset \mathsf{Ran}(\Pi_A(\lambda))$ the geometric multiplicity is not greater than the algebraic multiplicity.

Example 2.65. Let $\alpha, \beta \in \mathbb{C}$ distinct and

$$M = \begin{pmatrix} \alpha & 1 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & \beta & 1 \\ 0 & 0 & 0 & 0 & \beta \end{pmatrix}$$

Then α is an eigenvalue of geometric multiplicity 2. For $z \in \mathbb{C}\backslash\{\alpha, \beta\}$ we have

$$(M - z)^{-1} = \begin{pmatrix} (\alpha - z)^{-1} & -(\alpha - z)^{-2} & 0 & 0 & 0 \\ 0 & (\alpha - z)^{-1} & 0 & 0 & 0 \\ 0 & 0 & (\alpha - z)^{-1} & 0 & 0 \\ 0 & 0 & 0 & (\beta - z)^{-1} & -(\beta - z)^{-2} \\ 0 & 0 & 0 & 0 & (\beta - z)^{-1} \end{pmatrix}.$$

Then for $r \in]0, |\alpha - \beta|\,[$ we have

$$\Pi_\alpha = -\frac{1}{2i\pi} \int_{\mathcal{C}(\alpha,r)} (M - z)^{-1} \, \mathrm{d}z = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

so α has algebraic multplicity 3 and Π_α is the projection of $\mathbb{C}^5$ on $\ker((M - \alpha)^2)$ parallel to $\ker(M - \beta)$.

Proposition 2.66. *We use the notation of Proposition 2.60.*

(i) *Let $\lambda \in D(z_0, r_0)$ and $m \in \mathbb{N}^*$. Then* $\ker((A - \lambda)^m) \subset \mathsf{F}$.

(ii) *Let $\lambda \in \mathbb{C}\backslash\overline{D}(z_0, r_0)$ and $m \in \mathbb{N}^*$. Then* $\ker((A - \lambda)^m) \subset \mathsf{G}$.

Proof. • Let $\varphi \in \mathsf{Dom}(A)$ such that $(A - \lambda)\varphi \in \mathsf{F}$. For $\zeta \in \mathcal{C}(z_0, r_0)$ we have

$$(A - \zeta)^{-1}\varphi = (\lambda - \zeta)^{-1}\varphi - (\lambda - \zeta)^{-1}(A - \zeta)^{-1}(A - \lambda)\varphi,$$

Then

$$\begin{aligned} \Pi\varphi &= -\frac{1}{2i\pi} \int_{\mathcal{C}(z_0,r)} \left((\lambda - \zeta)^{-1}\varphi + (\lambda - \zeta)^{-1}(A - \zeta)^{-1}(A - \lambda)\varphi \right) \mathrm{d}\zeta \\ &= \varphi + \frac{1}{2i\pi} \int_{\mathcal{C}(z_0,r)} (\lambda - \zeta)^{-1}(A - \zeta)^{-1}(A - \lambda)\varphi \, \mathrm{d}\zeta. \end{aligned}$$

Since

$$\forall \zeta \in \mathcal{C}(z_0, r), \quad (A - \lambda)(A - \zeta)^{-1}(1 - \Pi)\varphi = (A - \zeta)^{-1}(1 - \Pi)(A - \lambda)\varphi = 0,$$

we deduce

$$(1 - \Pi)\varphi = (1 - \Pi)^2\varphi = -\frac{1}{2i\pi}\int_{\mathcal{C}(z_0,r)} (\lambda - \zeta)^{-1}(A - \zeta)^{-1}(1 - \Pi)(A - \lambda)\varphi \, \mathrm{d}\zeta = 0.$$

This proves that $\varphi \in \mathsf{F}$. Then we can prove by induction on $m \in \mathbb{N}^*$ that $\ker((A - \lambda)^m) \subset \mathsf{F}$. The second statement is similar. □

Proposition 2.67. *Assume that λ is an isolated point of $\sigma(A)$ such that $\mathsf{Ran}(\Pi_\lambda)$ is of finite dimension $m \in \mathbb{N}^*$. Then λ is an eigenvalue and*

$$\mathsf{Ran}(\Pi_\lambda) = \ker((A - \lambda)^m).$$

Proof. The restriction A_{F} of A to F is an operator on the finite dimensional space F, with $\sigma(A_{\mathsf{E}_\lambda}) = \{\lambda\}$. Then the result follows from the finite dimensional case. □

Remark 2.68. We recall that (see Exercise 1.1)

- an isolated point λ of $\sigma(A)$ is not necessarily an eigenvalue (in this case we have $\mathsf{dim}(\mathsf{Ran}(\Pi_\lambda)) = +\infty$ by Proposition 2.67);
- as isolated eigenvalue of finite geometric multiplicity can have infinite algebraic multiplicity.

Definition 2.69. *Let A be a closed and densely defined operator on E. Let $\lambda \in \mathbb{C}$. We say that λ belongs to the discrete spectrum $\sigma_{\mathsf{disc}}(A)$ of A and λ is an isolated eigenvalue of A with finite algebraic multiplicity. The essential spectrum of A is $\sigma_{\mathsf{ess}}(A) = \sigma(A)\backslash\sigma_{\mathsf{disc}}(A)$*

Proposition 2.70. *Let A be a closed operator on E. $\sigma_{\mathsf{ess}}(A)$ is closed.*

2.6.2 Regularity of the spectrum with respect to a parameter

[Not discussed in class]

Lemma 2.71. *Let Π_1 and Π_2 be two projections on E. Assume that $\|\Pi_2 - \Pi_1\|_{\mathcal{L}(\mathsf{E})} < 1$. Then*

$$\mathsf{dim}(\mathsf{Ran}(\Pi_1)) = \mathsf{dim}(\mathsf{Ran}(\Pi_2)).$$

Proof. Let $\pi : \mathsf{Ran}(\Pi_2) \to \mathsf{Ran}(\Pi_1)$ be the restriction of Π_1 to $\mathsf{Ran}(\Pi_2)$. This is a continuous linear map. For $\varphi \in \ker(\pi)$ we have $\Pi_2(\varphi) = \varphi$ and $\Pi_1(\varphi) = 0$ so

$$\|\varphi\| = \|\Pi_2(\varphi) - \Pi_1(\varphi)\| \leqslant \|\Pi_2 - \Pi_2\| \|\varphi\|,$$

so $\varphi = 0$. This implies that $\mathsf{dim}(\mathsf{Ran}(\Pi_1)) \geqslant \mathsf{dim}(\mathsf{Ran}(\Pi_1))$. Interverting the roles of Π_1 and Π_2 gives the reverse inequality and concludes the proof. □

Proposition 2.72. *Let ω be a connected subset of $\mathbb{C}$. Let $(A_\alpha)_{\alpha \in \mathbb{C}}$ be a family of linear operators on E. Assume that there exists $\lambda_0 \in \mathbb{C}$ and $r_0 > 0$ such that $\mathcal{C}(\lambda_0, r_0) \subset \rho(A_\alpha)$ for all $\alpha \in \omega$. Assume that the map*

$$\begin{cases} \omega \times \mathcal{C}(\lambda_0, r_0) & \to & \mathcal{L}(\mathsf{E}) \\ (\alpha, z) & \mapsto & (A_\alpha - z)^{-1} \end{cases}$$

is continuous.

(i) *We denote by Π_α the Riesz projection of A_α on $\mathcal{C}(\lambda_0, r)$. Then $\mathsf{dim}(\mathsf{Ran}(\Pi_\alpha))$ does not depend on $\alpha \in \omega$.*

(ii) *Assume that $\mathsf{dim}(\mathsf{Ran}(\Pi_\alpha)) = 1$. Then for all $\alpha \in \omega$ the operator A_α has a unique simple eigenvalue λ_α in $D(\lambda_0, r)$. Moreover the maps $\alpha \mapsto \lambda_\alpha$ and $\alpha \mapsto \Pi_\alpha$ are continuous on ω. If moreover $\alpha \mapsto (A_\alpha - z)^{-1}$ is holomorphic on ω for all $z \in \mathcal{C}(\lambda_0, r_0)$, then $\alpha \mapsto \Pi_\alpha$ and $\alpha \mapsto \lambda_\alpha$ are holomorphic.*

Proof. • Let $\alpha_0 \in \omega$. Since $\mathcal{C}(\lambda_0, r)$ is compact, there exists a neighborhood $\mathcal{V}$ of α_0 in ω such that for all $\alpha \in \mathcal{V}$ and $\zeta \in \mathcal{C}(\lambda_0, r)$ we have

$$\left\|(A_\alpha - \zeta)^{-1} - (A_{\alpha_0} - \zeta)^{-1}\right\| \leqslant \frac{1}{2r_0}.$$

Then we have

$$\|\Pi_\alpha - \Pi_{\alpha_0}\| \leqslant \frac{1}{2},$$

and, by Lemma 2.71, $\mathsf{Ran}(\Pi_\alpha) = \mathsf{Ran}(\Pi_{\alpha_0})$ for all $\alpha \in \mathcal{V}$. Then $\mathsf{Ran}(\Pi_\alpha)$ is locally constant, so it is constant on the connected set ω.

• By continuity under the integral sign, we see that Π_α is continuous with respect to α. If $(A_\alpha - \zeta)^{-1}$ is holomorphic with respect to α for all $\zeta \in \mathcal{C}(l_0, r)$, then Π_α is holomorphic by complex differentiation under the integral sign.

• Now assume that $\mathsf{Ran}(\Pi_\alpha) = 1$ for all $\alpha \in \omega$. Let $\alpha_0 \in \omega$ and $\psi \in \mathsf{Ran}(\Pi_{\alpha_0})$ with $\|\psi\| = 1$. Then ψ is an eigenvector corresponding to an eigenvalue $\lambda_{\alpha_0} \in D(\lambda_0, r)$. For $\alpha \in \omega$ we set $\psi_\alpha = \Pi_\alpha \psi$. For α close to α_0 we have $\psi_\alpha \neq 0$. Then ψ_α is an eigenvector of A_α, continuous (holomorphic if the resolvent is holomorphic) with respect to α. Finally we have $(A_\alpha - z)^{-1}\psi_\alpha = (\lambda_\alpha - z)^{-1}\psi_\alpha$. Taking the inner product with ψ gives

$$\left\langle \psi, (A_\alpha - z)^{-1}\psi_\alpha \right\rangle = (\lambda_\alpha - z)^{-1} \left\langle \psi, \psi_\alpha \right\rangle.$$

We have $\langle \psi, \psi_\alpha \rangle = 1$ when $\alpha = \alpha_0$, so this does not vanish on a neighborhood of α_0. This gives

$$(\lambda_\alpha - z)^{-1} = \frac{\left\langle \psi, (A_\alpha - z)^{-1}\psi_\alpha \right\rangle}{\left\langle \psi, \psi_\alpha \right\rangle}.$$

Thus $(\lambda_\alpha - z)^{-1}$ is continuous (holomorphic if the resolvent is holomorphic) for α an a neighborhood of α_0, and so is λ_α. □

Proposition 2.73 (Analytic family of type A)**.** *Let ω be an open subset of $\mathbb{C}$. Let $(A_\alpha)_{\alpha \in \omega}$ be a family of closed operators on E. We assume that*

(i) *the operators A_α, $\alpha \in \omega$, have the same domain $\mathcal{D}$;*

(ii) *for all $\psi \in \mathcal{D}$ the map $\alpha \mapsto A_\alpha \psi \in \mathcal{H}$ is holomorphic on ω.*

Let $\alpha_0 \in \omega$ and $z_0 \in \rho(A_{\alpha_0})$. Then there exists $r > 0$ such that $z \in \rho(A_\alpha)$ for all $\alpha \in D(\alpha_0, r)$ and $z \in D(z_0, r)$ and the map

$$(\alpha, z) \mapsto\mapsto (A_\alpha - z)^{-1}$$

is continuous on $D(\alpha_0, r) \times D(z_0, r)$ and analytic in $D(\alpha_0, r)$ for all $z \in D(z_0, r)$.

Proof. For $\alpha \in \omega$ and $z \in \mathbb{C}$ we have

$$(A_\alpha - z) = \Big(1 + \big(A_\alpha - A_{\alpha_0}\big) - (z - z_0)\big)(A_{\alpha_0} - z_0)^{-1}\Big)(A_{\alpha_0} - z_0)$$

Since $(A_{\alpha_0} - z_0)^{-1}$ maps $\mathcal{H}$ to $\mathcal{D}$, the operators $A_\alpha(A_{\alpha_0} - z_0)^{-1}$ and $A_{\alpha_0}(A_{\alpha_0} - z_0)^{-1}$ are well defined on $\mathcal{H}$. Since they are closed, they are bounded by the closed graph theorem. Then the function $\alpha \mapsto A_\alpha(A_{\alpha_0} - z)^{-1}$ is weakly holomorphic, and hence holomorphic by Proposition A.7. In particular it is continuous, so there exists $r > 0$ such that $\left\|(A_{\alpha_0} - z_0)^{-1}\right\| < 1/(4r)$, $D(\alpha_0, r) \subset \omega$ and for all $\alpha \in D(\alpha_0, r)$ we have

$$\left\|(A_\alpha - A_{\alpha_0})(A_{\alpha_0} - z_0)^{-1}\right\|_{\mathcal{L}(\mathcal{H})} \leqslant \frac{1}{4}.$$

Then the map $(\alpha, z) \mapsto \left(1 + \big(A_\alpha - A_{\alpha_0}\big) - (z - z_0)\big)(A_{\alpha_0} - z_0)^{-1}\right)^{-1}$ is well defined and continuous on $D(\alpha_0, r) \times D(z_0, r)$, and analytic with respect to α for all $z \in D(z_0, r)$. We deduce that the same holds for $\alpha \mapsto (A_\alpha - z)^{-1}$. □

Proposition 2.74 (Analytic family of type B)**.** *Let $\mathcal{V}$ be a Hilbert space continuously and densely embedded in $\mathcal{H}$. Let ω be an open subset of $\mathbb{C}$. Let $z \in \mathbb{C}$. Let $(q_\alpha)_{\alpha\in\omega}$ be a family of continuous forms on $\mathcal{V}$ such that $\varphi \mapsto q_\alpha(\varphi) \in \mathbb{C}$ is analytic for all $\varphi \in \mathcal{V}$. Let $\alpha_0 \in \omega$ and $z_0 \in \mathbb{C}$ such that $q_{\alpha_0} - z_0$ is coercive. Then there exists $r > 0$ such that $q_\alpha - z$ is coercive for all $\alpha \in D(\alpha_0, r)$ and $z \in D(z_0, r)$. For $\alpha \in D(\alpha_0, r)$ we denote by A_α the operator on $\mathcal{H}$ given by the representation theorem (see Theorem 2.52 and Remark 2.53). Then the map*

$$(\alpha, z) \mapsto (A_\alpha - z)^{-1}$$

is continuous on $D(\alpha_0, r) \times D(z_0, r)$ and holomorphic with respect to $\alpha \in D(\alpha_0, r)$ for all $z \in D(z_0, r)$.

Proof. We denote by Q_α the operator in $\mathcal{L}(\mathcal{V}, \mathcal{V}')$ associated with q_α (see (1.8)). For $\alpha \in \omega$ we have in $\mathcal{L}(\mathcal{V}, \mathcal{V}')$

$$(Q_\alpha - z) = \Big(1 + \big((Q_\alpha - Q_{\alpha_0}) - (z - z_0)\big)(Q_{\alpha_0} - z)^{-1}\Big)(Q_{\alpha_0} - z)$$

Since $(Q_{\alpha_0} - z)^{-1}$ maps $\mathcal{V}'$ to $\mathcal{V}$, the operators $Q_\alpha(Q_{\alpha_0} - z)^{-1}$ and $Q_{\alpha_0}(Q_{\alpha_0} - z)^{-1}$ are bounded on $\mathcal{V}'$. Then the function $\alpha \mapsto Q_\alpha(Q_{\alpha_0} - z)^{-1}$ is weakly holomorphic, and hence holomorphic by Proposition A.7. In particular it is continuous, so there exists $r > 0$ such that $\left\|(Q_{\alpha_0} - z_0)^{-1}\right\|_{\mathcal{L}(\mathcal{V}',\mathcal{V})} \leqslant 1/(4r)$, $D(\alpha_0, r) \subset \omega$ and for all $\alpha \in D(\alpha_0, r)$ we have

$$\left\|(Q_\alpha - Q_{\alpha_0})(Q_{\alpha_0} - z)^{-1}\right\|_{\mathcal{L}(\mathcal{V}')} \leqslant \frac{1}{4}.$$

Then the map $(\alpha, z) \mapsto \big(1 + \big((Q_\alpha - Q_{\alpha_0}) - (z - z_0)\big)(Q_{\alpha_0} - z)^{-1}\big)^{-1} \in \mathcal{L}(\mathcal{V}')$ is well defined and continuous on $D(\alpha_0, r) \times D(z_0, r)$, and analytic on $D(\alpha_0, r)$ for all $z \in D(z_0, r)$. We deduce that the same holds for $\alpha \mapsto (Q_\alpha - z)^{-1}$ in $\mathcal{L}(\mathcal{V}', \mathcal{V})$. Since $(Q_\alpha - z)^{-1}$ and $(A_\alpha - z)^{-1}$ coincide on $\mathcal{H}$, the conclusion follows. □

For the perturbation of a double eigenvalue, we refer to Exemple II.1.1 (page 64) in [Kat80]

2.7 Exercises

***Exercise* 2.1.** Let A be a densely defined operator from E to F. Assume that there exists $C > 0$ such that $\|A\varphi\|_{\mathsf{F}} \leqslant C\,\|A\|_{\mathsf{E}}$ for all $\varphi \in \mathsf{Dom}(A)$. Prove that A extends uniquely to a bounded operator $\tilde{A} \in \mathcal{L}(\mathsf{E}, \mathsf{F})$ and that $\|\tilde{A}\|_{\mathcal{L}(\mathsf{E},\mathsf{F})} \leqslant C$.

***Exercise* 2.2.** Prove Proposition 2.23

***Exercise* 2.3.** Let $(\lambda_n)_{n\in\mathbb{N}}$ be a complex sequence. We consider on $\ell^2(\mathbb{N})$ the operator A defined by

$$\mathsf{Dom}(A) = \left\{ u = (u_n)_{n\in\mathbb{N}} \,:\, \sum_{n=0}^{\infty} |\lambda_n|^2 \, |u_n|^2 < +\infty \right\}$$

and, for $u = (u_n)_{n\in\mathbb{N}} \in \mathsf{Dom}(A)$,

$$Au = (\lambda_n u_n)_{n\in\mathbb{N}}.$$

1. Prove that A is closed.
2. What is the spectrum of A.

***Exercise* 2.4.** Let Ω be an open subset of $\mathbb{R}^d$. Let $f : \Omega \to \mathbb{C}$ be a measurable function. We consider the multiplication operator M_f as in example 2.4.
1. Prove that M_f is densely defined.
2. What is the adjoint of M_f ?

***Exercise* 2.5.** We set

$$\mathcal{H} = \left\{ u \in L^2(\mathbb{R}) \,:\, u \text{ is even} \right\}.$$

1. Prove that $\mathcal{H}$ is a Hilbert space.
2. We want to consider on $\mathcal{H}$ the operator defined by $Au = -u''$. What is the natural domain for A (in particular, we want A to be closed) ?
3. Then what is the spectrum of A ?

***Exercise* 2.6.** Let A be a closed and densely defined operator on E. Assume that there exists $C > 0$ such that $\|A\varphi\|_{\mathsf{E}} \leqslant C\|\varphi\|_{\mathsf{E}}$ for all $\varphi \in \mathsf{Dom}(A)$. Prove that $\mathsf{Dom}(A) = \mathcal{H}$ and that $A \in \mathcal{L}(\mathsf{E})$.

***Exercise* 2.7.** We consider on $L^2(\mathbb{C})$ (K is endowed with its usual Lebesgue measure) the operator A defined by $(Au)(y) = yu(y)$ on the domain

$$\mathsf{Dom}(A) = \left\{u \in L^2(\mathbb{C}) \,:\, yu \in L^2(\mathbb{C})\right\}.$$

1. Prove that A is closed.
2. Prove that $\sigma(A) = \mathbb{C}$.

***Exercise* 2.8** (Regular points)**.** Let A be an operator on the Hilbert space $\mathcal{H}$. Let z be a regular point of A (see Proposition 2.34). We denote by $d_A(z) = \mathsf{dim}(\mathsf{Ran}(A - z)^{\perp})$ the defect number of A. We also denote by $\pi(A)$ the set of regular points of A.
1. Prove that $\pi(A)$ is open (more precisely, if $z_0 \in \pi(A)$ and $c_0 > 0$ is the constant given by (2.4), show that $D(z_0, c_{z_0}) \subset \pi(A)$).
2. Assume that A is closable. Prove that the defect number is constant on each connected component of $\pi(A)$.

***Exercise* 2.9.** We consider the operator T from $L^2(\mathbb{R})$ to $\mathbb{C}$ defined by $\mathsf{Dom}(T) = C_0^\infty(\mathbb{R})$ and $T\phi = \phi(0)$ for all $\phi \in \mathsf{Dom}(T)$. Compute the adjoint T^* of T.

***Exercise* 2.10.** Prove Proposition 2.46.

***Exercise* 2.11.** Prove Proposition 2.48.

***Exercise* 2.12.** Prove Proposition 2.48.

***Exercise* 2.13.** Let $\alpha \in \mathbb{C}$. For $\varphi, \psi \in H^1(0,1)$ we set

$$\mathsf{q}_\alpha(\varphi) = \int_0^1 \left|u'(x)\right|^2 \mathrm{d}x + \alpha\left|u(0)\right|^2.$$

1. Prove that the quadratic form q_α is continuous on $H^1(0,1)$.
2. Prove that there exists $\beta \geqslant 0$ such that the form $\mathsf{q}_\alpha + \beta : u \mapsto \mathsf{q}_\alpha(u) + \beta\|u\|^2_{L^2(0,1)}$ is coercive.
3. We denote by A_α the operator on $L^2(0,1)$ associated with the form q_α by the representation theorem (see Remark 2.53). Describe A_α (domain and action on an element of this domain).

***Exercise* 2.14.** Let E_1 and E_2 be two Banach spaces and $\mathsf{E} = \mathsf{E}_1 \oplus \mathsf{E}_2$. Let A_1 and A_2 be two closed operators, on E_1 and E_2 respectively. For $\varphi = \varphi_1 + \varphi_2 \in \mathsf{E}$ we set $A = A_1\varphi_1 + A_2\varphi_2$.
1. Prove that this defines a closed operator A on E.
2. Prove that $\sigma(A) = \sigma(A_1) \cup \sigma(A_2)$.
3. Prove that $\sigma_{\mathsf{p}}(A) = \sigma_{\mathsf{p}}(A_1) \cup \sigma_{\mathsf{p}}(A_2)$.
4. Assume that λ is an isolated eigenvalue of A. Prove that the geometric (algebraic) multiplicity of λ as an eigenvalue of A is the sum of the geometric (algebraic) multiplicities of λ as an eigenvalue of A_1 and A_2.

***Exercise* 2.15.** Let A be a closed operator on E. Let $\lambda \in \sigma_{\mathsf{disc}}(A)$. Let $r_0 > 0$ be such that $D(\lambda, r_0) \cap \sigma(A) = \{\lambda\}$. For $r \in]0, r_0[$ and $n \in \mathbb{Z}$ we set

$$R_n = \frac{1}{2i\pi} \int_{\mathcal{C}(\lambda,r)} \frac{(A-\zeta)^{-1}}{(\zeta-\lambda)^{n+1}} \,\mathrm{d}\zeta.$$

1. Prove that for $n_1, n_2 \in \mathbb{Z}\setminus\{0\}$ we have $R_{n_1}R_{n_2} = -R_{n_1+n_2+1}$.
2. We set $N = -R_{-2}$. Prove that for all $n \geqslant 2$ we have $R_{-n} = -N^{n-1}$.
3. We denote by Π the Riesz projection at λ. Prove that $N\Pi = \Pi N = N$. Deduce that N has finite rank.
4. Prove that for $z \in D(\lambda, r_0)\setminus\{\lambda\}$ we can write $(A - z)^{-1}$ as the Laurent series

$$(A - z)^{-1} = \sum_{n\in\mathbb{Z}} (z - \lambda)^n R_n,$$

and in particular that the power series $\sum_{m\geqslant 0} \rho^n R_{-m}$ is convergent for any $\rho \in \mathbb{C}$.
5. Prove that N is nilpotent and that $R_{-n} = 0$ for n large enough.