Appendix A

Compact Operators

In this appendix we give some general properties about compact operators. We first recall the Ascoli-Arzelà Theorem.

Theorem A.1 (Ascoli-Arzelà Theorem)**.** *Let K be a compact metric space and let $\mathcal{F}$ be a bounded subset of $C(K,\mathbb{R})$. We assume that $\mathcal{F}$ is equicontinuous:*

$$\forall \varepsilon > 0, \exists \delta > 0, \forall f \in \mathcal{F}, \forall x, y \in K, \quad d(x,y) \leqslant \delta \implies |f(x) - f(y)| \leqslant \varepsilon.$$

Then the closure $\overline{\mathcal{F}}$ of $\mathcal{F}$ in $C(K)$ is compact.

A.1 Compact operators

A.1.1 Definition and first properties

Définition A.2. Let X and Y be Banach spaces. A bounded linear operator $T : X \to Y$ is said to be compact if for any bounded sequence $(u_n)_{n\in\mathbb{N}} \in X^{\mathbb{N}}$, the sequence $(Tu_n)_{n\in\mathbb{N}}$ has a convergent subsequence in Y. Equivalently, T is compact if $\overline{T(B_X)}$ is compact in Y, where B_X is the unit ball in X.

Given two Banach spaces X, Y we denote by $\mathcal{K}(X,Y)$ the set of compact operators from X to Y. We also write $\mathcal{K}(X) = \mathcal{K}(X,X)$.

Example A.3. *Finite rank operators are compact.*

Example A.4. *We denote by $(e_n)_{n\in\mathbb{N}^*}$ the canonical basis of $\ell^2(\mathbb{N}^*)$. We consider on $\ell^2(\mathbb{N}^*)$ the linear map A such that $Ae_n = \frac{e_n}{n}$ for all $n \in \mathbb{N}^*$. Then A is compact on $\ell^2(\mathbb{N}^*)$.*

Proposition A.5. *Let X and Y be two Banach spaces.*

(i) *Let $K \in \mathcal{K}(X,Y)$ and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X which converges weakly to some $x \in X$ (i.e. for any $\varphi \in X*$ we have $\varphi(x_n) \to \varphi(x)$). Then $K(x_n)$ converges (in norm) to $K(x)$.*

(ii) *$\mathcal{K}(X,Y)$ is a closed subspace of $\mathcal{L}(X,Y)$.*

(iii) *For $K \in \mathcal{K}(X,Y)$, $B_1 \in \mathcal{B}(X_1,X)$ and $B_2 \in \mathcal{B}(Y,Y_2)$ we have $K \circ B_1 \in \mathcal{K}(X_1,Y)$ and $B_2 \circ K \in \mathcal{K}(X,Y_2)$.*

(iv) *For $K \in \mathcal{K}(X,Y)$ we have $K^* \in \mathcal{K}(Y^*,X^*)$.*

Proof. We prove the first and last statements.

- The sequence $(x_n)_{n\in\mathbb{N}}$ is weakly convergent, so it is bounded in X (see Proposition 3.5.(iii) in [Brézis]). By continuity, a convergent subsequence of $(K(x_n))_{n\in\mathbb{N}}$ necessarily goes to $K(x)$. This implies that $K(x_n)$ goes strongly to $K(x)$.
- Let $(\varphi_n)_{n\in\mathbb{N}}$ be a bounded sequence in Y^*. We denote by B_X the unit ball in X. Since K is compact, $\overline{K(B_X)}$ is a compact metric space, and the functions φ_n, $n \in \mathbb{N}$, are equicontinuous thereon. Then, by the Ascoli-Arzelà Theorem, there exists a subsequence $(\varphi_{n_k})_{k\in\mathbb{N}}$ convergent in $C^0(\overline{K(B_X)})$. We denote by $\varphi \in C^0(\overline{K(B_X)})$ the limit. In particular we have

$$\sup_{\|x\|_X \leqslant 1} |\varphi_{n_k}(K(x)) - \varphi(K(x))| \xrightarrow[k\to+\infty]{} 0.$$

We deduce that $(\varphi_{n_k} \circ K)$ is a Cauchy sequence in X^*. Since X^* is a Banach space, it has a limit in X^*. This proves that $K^* \in \mathcal{K}(Y^*,X^*)$. □

We finish this paragraph with more examples of compact operators.

Let Ω be an open subset of $\mathbb{R}^d$. For $k \in \mathbb{N}$ we denote by $C_b^k(\Omega)$ the set of functions u of class C^k on Ω such that $\partial^\alpha u$ is bounded on Ω for all $|\alpha| \leqslant k$. Then $C_b^k(\Omega)$ is endowed by the norm defines by

$$\|u\|_{C_b^k(\Omega)} = \sum_{|\alpha|\leqslant k} \|\partial^\alpha u\|_{L^\infty(\Omega)}.$$

Proposition A.6. *Let Ω be an open bounded and subset of $\mathbb{R}^d$ and $k \in \mathbb{N}$. Then $C_b^{k+1}(\Omega)$ is compactly embedded in $C_b^k(\Omega)$.*

Proof. Let $(u_n)_{n\in\mathbb{N}}$ be a bounded sequence in $C^{k+1}(\overline{\Omega})$. Let M be such that $\|u_n\|_{C_b^{k+1}} \leqslant M$. Let $\alpha \in \mathbb{N}^d$ with $|\alpha| \leqslant k$ and $j \in [\![1,d]\!]$. Let $x \in \Omega$ and $r > 0$ such that $B(x,r) \subset \Omega$. Since $\|\nabla\partial^\alpha u_n\|_{L^\infty(\Omega)} \leqslant M$, the sequence (u_n) is uniformly Lipschitz in $B(x,r)$. In particular, the sequence $(\partial^\alpha u_n)$ is uniformly equicontinuous on Ω. By the Ascoli-Arzelà Theorem, it has a subsequence which converges to some v_α in $C^0(\Omega)$. Then there exists an increasing sequence (n_k) such that $\partial^\alpha u_{n_k}$ goes to v_α when $n \to +\infty$ for all $|\alpha| \leqslant k$.

Let $\alpha \in \mathbb{N}^d$ with $|\alpha| \leqslant k$. Let $x \in \Omega$. For $t \in \mathbb{R}$ small enough we have

$$\begin{aligned} v_\alpha(x+te_j) - v_\alpha(x) &= \lim_{k\to+\infty} \partial^\alpha u_{n_k}(x+te_j) - \partial^\alpha u_{n_k}(x) \\ &= \lim_{k\to+\infty} \int_0^t \partial^{\alpha+e_j} u_{n_k}(x+se_j)\,ds. \end{aligned}$$

Since the map $s \mapsto \partial^{\alpha+e_j} u_{n_k}(x+se_j)$ converges uniformly to $s \mapsto v_{\alpha+e_j}(x+se_j)$ on $[0,t]$ we get

$$v_\alpha(x+te_j) - v_\alpha(x) = \int_0^t v_{\alpha+e_j}(x+se_j)\,ds.$$

This proves that $\partial_j v_\alpha = v_{\alpha+e_j}$. Finally for all $|\alpha| \leqslant k$ we have $\partial^\alpha v = v_\alpha$ and we have

$$\|u_{n_k} - v\|_{C_b^k(\Omega)} \xrightarrow[k\to+\infty]{} 0.$$

□

***Exercise* 42.** Let Ω be a bounded subset of $\mathbb{R}^d$. Let $k \in \mathbb{N}$ and $\theta \in]0,1[$. We recall that $C^{k,\theta}$ is the set of functions of class C^k whose derivatives are bounded and moreover the derivatives of ordre k are Hölder-continuous of exponent θ. It is endowed with the norm defined by

$$\|u\|_{C^{k,\theta}(\Omega)} = \sum_{\alpha \leqslant k} \|\partial^\alpha u\|_{L^\infty(\Omega)} + \sum_{|\alpha|=k} \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|}{|x-y|^\theta}.$$

Prove that $C^{k,\theta}(\Omega)$ is compactly embedded in $C_b^k(\Omega)$.

Example A.7. *Let $K \in C^0([0,1]^2)$. For $u \in C^0([0,1])$ and $x \in [0,1]$ we set*

$$(Tu)(x) = \int_0^1 K(x,y)u(u)\,dy.$$

Let $M > 0$ and let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $C^0([0,1])$ such that $\|u_n\|_\infty \leqslant M$ for all $n \in \mathbb{N}$. Let $x \in [0,1]$ and $\varepsilon > 0$. Since K is uniformly continuous there exists $\delta > 0$ such that for all $(x_1,y_1),(x_2,y_2) \in [0,1]^2$ we have

$$|x_1 - x_2| + |y_1 - y_2| \leqslant \delta \quad \implies \quad |K(x_1,y_1) - K(x_2,y_2)| \leqslant \frac{\varepsilon}{M}.$$

Then for $n \in \mathbb{N}$ and $x' \in [0,1]$ such that $|x - x'| \leqslant \delta$ we have

$$|(Tu_n)(x) - Tu_n(x')| \leqslant \int_0^1 |K(x,y) - K(x',y)|\,|u_n(y)|\,dy \leqslant \frac{\varepsilon}{.}$$

This proves that the family $(Tu_n)_{n\in\mathbb{N}}$ is equicontinuous. By the Ascoli-Arzelà Theorem it has a convergent subsequence in $C^0([0,1])$, which proves that T is compact on $C^0([0,1])$.

A.2 Fredholm Alternative

We consider a Hilbert space $\mathcal{H}$.

Theorem A.8. *Let $K \in \mathcal{K}(\mathcal{H})$. Then $(\mathrm{Id} - K)$ is injective if and only if it is surjective, and in this case its inverse defines a bounded operator on $\mathcal{H}$. In any case we have*

$$\mathsf{dim}(\mathrm{Ker}(\mathrm{Id} - K)) = \mathsf{dim}(\mathrm{Ker}(\mathrm{Id} - K^*)) < +\infty.$$

Moreover $\mathsf{Ran}(\mathrm{Id} - K)$ is always closed, and in particular

$$\mathsf{Ran}(\mathrm{Id} - K) = \mathrm{Ker}(\mathrm{Id} - K^*)^\perp.$$

Remark A.9. We recall that for any $A \in \mathcal{L}(\mathcal{H})$ we have

$$\overline{\mathsf{Ran}(A)} = \mathrm{Ker}(A^*)^\perp.$$

Proof. • Assume by contradiction that $\mathsf{dim}(\mathrm{Ker}(\mathrm{Id} - K)) = +\infty$. Then we can find a sequence $(u_n)_{n\in\mathbb{N}}$ in $\mathcal{H}$ such that $\langle u_n, u_m\rangle = \delta_{n,m}$ and $Ku_n = u_n$ for all $n,m \in \mathbb{N}$. This is in particular a bounded sequence but, for $n \neq m$,

$$\|Ku_n - Ku_m\|_{\mathcal{H}}^2 = \|u_n - u_m\|_{\mathcal{H}}^2 = 2,$$

so the sequence $(Ku_n)_{n\in\mathbb{N}}$ cannot have a convergent subsequence. This gives a contradiction and prove that $\mathsf{dim}(\mathrm{Ker}(\mathrm{Id}-K)) < +\infty$.

- Then we prove that there exists $\gamma > 0$ such that

$$\forall u \in \mathrm{Ker}(\mathrm{Id}-K)^\perp, \quad \|u - Ku\|_{\mathcal{H}} \geqslant \gamma \|u\|_{\mathcal{H}}. \tag{A.1}$$

If this is not the case, we can find a sequence $(u_n)_{n\in\mathbb{N}}$ in $\mathrm{Ker}(\mathrm{Id}-K)^\perp$ such that $\|u_n\|_{\mathcal{H}} = 1$ and $\|u_n - Ku_n\|_{\mathcal{H}} \leqslant 2^{-n}$ for all $n \in \mathbb{N}$. Since $(u_n)_{n\in\mathbb{N}}$ is bounded, there exists a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ and $u \in \mathcal{H}$ such that u_{n_k} goes weakly to u as $k \to +\infty$. By Proposition A.5, Ku_{n_k} goes to Ku as $k \to +\infty$. Then

$$u_{n_k} = Ku_{n_k} + (u_{n_k} - Ku_{n_k}) \xrightarrow[k\to+\infty]{} Ku.$$

This implies that $u = Ku$, so $u \in \mathrm{Ker}(\mathrm{Id}-K)$. In particular, for all $n \in \mathbb{N}$ we have $\langle u, u_{n_k}\rangle_{\mathcal{H}} = 0$ so, taking the limit, $\|u\|_{\mathcal{H}} = 0$. This gives a contradiction and proves (A.1).

- We deduce from (A.1) that $\mathsf{Ran}(\mathrm{Id}-K)$ is closed in $\mathcal{H}$. Indeed, let $(v_n)_{n\in\mathbb{N}}$ be a sequence in $\mathsf{Ran}(\mathrm{Id}-K)$ which goes to some v in $\mathcal{H}$. Then for all $n \in \mathbb{N}$ there exists $u_n \in \mathrm{Ker}(\mathrm{Id}-K)^\perp$ such that $v_n = (\mathrm{Id}-K)u_n$. By (A.1), $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{H}$, and hence it has a limit $u \in \mathcal{H}$, By continuity, we have $v = (\mathrm{Id}-K)u \in \mathsf{Ran}(\mathrm{Id}-K)$, which proves that $\mathsf{Ran}(\mathrm{Id}-K)$ is closed.

- Now assume that $(\mathrm{Id}-K)$ is injective, and assume by contradiction that $\mathcal{H}_1 = (\mathrm{Id}-K)(\mathcal{H})$ is not equal to $\mathcal{H}$. Since $\mathcal{H}_1$ is closed, it is a Hilbert space with the structure inherited from $\mathcal{H}$, and by restriction, K defines a compact operator on $\mathcal{H}_1$. We set $\mathcal{H}_2 = (\mathrm{Id}-K)(\mathcal{H}_1)$. Then $\mathcal{H}_2$ is closed, and since $(\mathrm{Id}-K)$ is injective, we have $\mathcal{H}_2 \subsetneq \mathcal{H}_1$ (take $u \in \mathcal{H}\backslash\mathcal{H}_1$, then $(\mathrm{Id}-K)u$ belongs to $\mathcal{H}_1\backslash\mathcal{H}_2$). By induction we set $\mathcal{H}_k = (\mathrm{Id}-K)(\mathcal{H}_{k-1})$ for all $k \geqslant 2$. Then $\mathcal{H}_k$ is closed and $\mathcal{H}_{k+1} \subsetneq \mathcal{H}_k$ for all $k \in \mathbb{N}^*$. In particular, for all $k \in \mathbb{N}^*$ we can find $u_k \in \mathcal{H}_k$ such that $\|u_k\|_{\mathcal{H}} = 1$ and $u_k \in \mathcal{H}_{k+1}^\perp$. Then for $k \in \mathbb{N}^*$ and $j > k$ we have

$$Ku_j - Ku_k = -(u_j - Ku_j) + (u_k - Ku_k) + u_j - u_k.$$

Since $-(u_j - Ku_j) + (u_k - Ku_k) + u_j \in \mathcal{H}_{k+1}$ this yields

$$\|Ku_j - Ku_k\| \geqslant 1.$$

This gives a contradiction since K is compact. Thus, if $(\mathrm{Id}-K)$ is injective, then it is also surjective.

- Conversely, assume that $\mathsf{Ran}(\mathrm{Id}-K) = \mathcal{H}$. Then $\mathrm{Ker}(\mathrm{Id}-K^*) = \{0\}$. Since K^* is also compact, we deduce that $(\mathrm{Id}-K^*)$ is surjective, and finally

$$\mathrm{Ker}(\mathrm{Id}-K) = \mathrm{Ker}(\mathrm{Id}-K^{**}) = \mathsf{Ran}(\mathrm{Id}-K^*)^\perp = \{0\}.$$

This proves that $(\mathrm{Id}-K)$ is injective if and only if it is surjective. Moreover, in this case, (A.1) proves that the inverse $(\mathrm{Id}-K)^{-1}$ defines a bounded operator with $\|(\mathrm{Id}-K)^{-1}\|_{\mathcal{L}(\mathcal{H})} \leqslant \gamma^{-1}$.

- It remains to prove that $\mathrm{Ker}(\mathrm{Id}-K)$ and $\mathrm{Ker}(\mathrm{Id}-K^*)$ have the same dimension. Assume by contradiction that $\mathsf{dim}(\mathrm{Ker}(\mathrm{Id}-K)) < \mathsf{dim}(\mathsf{Ran}(\mathrm{Id}-K)^\perp)$. There exists a bounded operator $A : \mathrm{Ker}(\mathrm{Id}-K) \to \mathsf{Ran}(\mathrm{Id}-K)^\perp$ injective but not surjective. We extend A by 0 on $\mathrm{Ker}(\mathrm{Id}-K)^\perp$. This defines an operator A on $\mathcal{H}$ which has a finite dimensional range included in $\mathsf{Ran}(\mathrm{Id} - K)^\perp$. In particular it is compact, and so is $\tilde{K} = K + A$. Let $u \in \mathrm{Ker}(\mathrm{Id} - \tilde{K})$. We have $u - Ku = Au$. Since $u - Ku \in \mathsf{Ran}(\mathrm{Id} - K)$ and $Au \in \mathsf{Ran}(\mathrm{Id} - K)^\perp$, we have $u - Ku = 0$. Therefore $u = 0$ since A is injective on $\mathrm{Ker}(\mathrm{Id} - K)$. Then $(\mathrm{Id} - \tilde{K})$ is injective, and hence surjective. However for $v \in \mathsf{Ran}(\mathrm{Id} - K)^\perp \setminus \mathsf{Ran}(A)$ the equation

$$u - (Ku + Au) = v$$

cannot have a solution. This gives a contradiction and proves that

$$\mathsf{dim}(\mathrm{Ker}(\mathrm{Id} - K)) \geqslant \mathsf{dim}(\mathsf{Ran}(\mathrm{Id} - K)^\perp) = \mathsf{dim}(\mathrm{Ker}(\mathrm{Id} - K^*)).$$

We get the opposite inequality by interchanging the roles of K and K^*, and the proof is complete. □

***Exercise* 43.** Let $K \in \mathcal{L}(\mathcal{H})$. Prove that

$$\mathsf{dim}\left(\bigcup_{k \in \mathbb{N}} \mathrm{Ker}((\mathrm{Id} - K)^k)\right) < +\infty.$$

A.3 Spectral properties

In this section we discuss the spectral properties of a compact operator. We first recall the definition of the spectrum of a general operator.

Let $\mathcal{H}$ be a real (or complex) Hilbert. An operator A on $\mathcal{H}$ is a linear map from a dense subset $\mathcal{D}$ of $\mathcal{H}$ to $\mathcal{H}$. We say that $\mathcal{D}$ is the domain of A.

Let $\lambda \in \mathbb{R}$ (or $\lambda \in \mathbb{C}$). We say that λ is in the resolvent set $\rho(A)$ of A if the operator $(A - \lambda \,\mathrm{Id}) : \mathcal{D} \to \mathcal{H}$ is bijective and if its inverse $(A - \lambda \,\mathrm{Id})^{-1}$ defines a bounded operator on $\mathcal{H}$. We usually write $(A - \lambda)$ instead of $(A - \lambda \,\mathrm{Id})$. The spectrum $\sigma(A)$ of A is the complement of $\rho(A)$ in $\mathbb{R}$ (or $\mathbb{C}$).

We recall that if $\mathcal{H}$ is of finite dimension, a linear map is bijective if and only if it is injective, and in this case the inverse is always continuous, so the spectrum of A is exactly the set of eigenvalues. This is not the case in general.

If λ is an eigenvalue of A, then its geometric multiplicity is

$$\mathsf{dim}\big(\mathrm{Ker}(A - \lambda)\big),$$

and its algebraic multiplicity is

$$\mathsf{dim}\left(\bigcup_{k \in \mathbb{N}} \mathrm{Ker}\left((A - \lambda)^k\right)\right) = \lim_{k \to +\infty} \mathsf{dim}\big(\mathrm{Ker}(A - \lambda)^k\big).$$

In particular, the geometric multiplicity is smaller than or equal to the algebraic multiplicity.

A.3.1 Spectrum of compact operators

For compact operators, we have the following result.

Theorem A.10. *Let $K \in \mathcal{K}(\mathcal{H})$.*

(i) *If $\mathsf{dim}(\mathcal{H}) = +\infty$ then 0 belongs to the spectrum of K.*

(ii) *$\lambda \neq 0$ belongs to the spectrum of K if and only if it is an eigenvalue of K. In this case it is an eigenvalue of finite geometric (and algebraic) multiplicity.*

(iii) *$\sigma(K)\setminus\{0\}$ is finite or is given by a sequence of eigenvalues tending to 0.*

Proof. • Assume that 0 belongs to the resolvent set of K. Then Id is the composition of the compact operator K with the bounded operator K^{-1}, so Id is a compact operator. This implies that $\mathsf{dim}(\mathcal{H}) < +\infty$.
• Let $\lambda \in \mathbb{R}^*$ (or $\mathbb{C}^*$). Then we have $K - \lambda = \lambda(\lambda^{-1}K - \mathrm{Id})$. Since $\lambda^{-1}K$ is compact, Theorem A.8 shows that $(K - \lambda)$ is bijective (with bounded inverse) if and only if it is injective, so λ is in the resolvent set of K if and only if it is not an eigenvalue. Moreover, if λ is an eigenvalue of K we have $\mathsf{dim}(\mathrm{Ker}(K - \lambda)) = \mathsf{dim}(\mathrm{Ker}(\lambda^{-1}K - \mathrm{Id})) < +\infty$. More generally, Exercise 43 shows that 1 is an eigenvalue of finite algebraic multiplicity for $\lambda^{-1}K$.
• Since K is a bounded operator, the set of eigenvalues of K is bounded in $\mathbb{R}$ ($\mathbb{C}$). Assume that $(\lambda_n)_{n\in\mathbb{N}}$ is a sequence of distinct non-zero eigenvalues of K tending to some λ. We prove that $\lambda = 0$. For $n \in \mathbb{N}$ we consider $w_n \in \mathcal{H}\setminus\{0\}$ such that $Kw_n = \lambda_n w_n$. Then for $n \in \mathbb{N}$ we set $\mathcal{H}_n = \mathsf{span}(w_0, \dots, w_{n-1})$ and we consider $u_n \in \mathcal{H}_n$ such that $\|u_n\| = 1$ and $u_n \in \mathcal{H}_{n-1}^{\perp}$ if $n \geqslant 1$. Then for $j \in \mathbb{N}$ and $k > j$ we have

$$\left\| \frac{Ku_k}{\lambda_k} - \frac{Ku_j}{\lambda_j} \right\|_{\mathcal{H}} = \left\| \frac{Ku_k - \lambda_k u_k}{\lambda_k} - \frac{Ku_j - \lambda_j u_j}{\lambda_j} + u_k - u_j \right\|_{\mathcal{H}} \geqslant 1,$$

since $Ku_k - \lambda_k u_k, Ku_j - \lambda_j u_j, u_j \in \mathcal{H}_{k-1}$. If $\lambda \neq 0$ we obtain a contradiction with the compactness of K. □

A.3.2 The case of symmetric operators

Let A be a bounded operator on $\mathcal{H}$. We assume that A is symmetric:

$$\forall \varphi, \psi \in \mathcal{H}, \quad \langle A\varphi, \psi \rangle_{\mathcal{H}} = \langle \varphi, A\psi \rangle_{\mathcal{H}}.$$

In particular, even if $\mathcal{H}$ is a complex Hilbert space, we have $\langle Au, u \rangle \in \mathbb{R}$ for all $u \in \mathcal{H}$. In particular, the eigenvalues of A are real. Moreover, two eigenspaces of A corresponding to two distinct eigenvalues are orthogonal.

Lemma A.11. *Let A be a bounded symmetric operator on $\mathcal{H}$. Let*

$$m = \inf_{\substack{u \in \mathcal{H} \\ \|u\|=1}} \langle Au, u \rangle_{\mathcal{H}} \quad \text{and} \quad M = \sup_{\substack{u \in \mathcal{H} \\ \|u\|=1}} \langle Au, u \rangle_{\mathcal{H}}.$$

Then $\sigma(A) \subset [m, M]$ and $m, M \in \sigma(A)$.

Proof. We consider the case where $\mathcal{H}$ is a real Hilbert space. We prove that $]M, +\infty[\subset \rho(A)$ and that $M \in \sigma(A)$. Let $\lambda > M$. For $u \in \mathcal{H}$ we have

$$\langle \lambda u - Au, u\rangle_{\mathcal{H}} \geqslant (\lambda - M) \|u\|_{\mathcal{H}}^2 .$$

By the Lax-Milgram Theorem, the operator $\lambda - A$ is bijective with bounded inverse on $\mathcal{H}$, so $\lambda \in \rho(A)$.

Now let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{H}$ such that $\|u_n\|_{\mathcal{H}} = 1$ for all $n \in \mathbb{N}$ and

$$\langle Au_n, u_n\rangle \xrightarrow[n\to+\infty]{} M.$$

The quadratic form $u \mapsto \langle (M - A)u, u\rangle$ is non-negative, so by the Cauchy-Schwarz inequality we have for all $u, v \in \mathcal{H}$

$$\left|\langle (M - A)u, v\rangle_{\mathcal{H}}\right|^2 \leqslant \langle (M - A)u, u\rangle_{\mathcal{H}} \langle (M - A)v, v\rangle_{\mathcal{H}}$$

Applied with $u = u_n$ and $v = (M - A)u_n$ this gives

$$\|(M - A)u_n\|_{\mathcal{H}}^2 \leqslant \langle (M - A)u_n, u_n\rangle_{\mathcal{H}} \left\langle (M - A)^3 u_n, (M - A)u_n\right\rangle_{\mathcal{H}} \xrightarrow[n\to+\infty]{} 0.$$

This proves that $M \in \sigma(A)$. □

Theorem A.12. *Let $\mathcal{H}$ be a separable Hilbert space and let K be a compact and symmetric operator on $\mathcal{H}$. Then there exists an orthonormal basis $(e_n)_{n\in\mathbb{N}}$ consisting of eigenvectors of K.*

Proof. Let $(\lambda_n)_{1\leqslant n\leqslant N}$ for $N \in \mathbb{N}\cup\{+\infty\}$ be the sequence of distinct non-zero eigenvalues of K. For $n \in [\![1, N]\!]$ we set $\mathcal{H}_n = \mathrm{Ker}(K - \lambda_n)$. Then we have $\mathsf{dim}(\mathcal{H}_n) \in \mathbb{N}^*$. We also set $\mathcal{H}_0 = \mathrm{Ker}(K)$.

We set $\tilde{\mathcal{H}} = \mathsf{span}\big(\bigcup_{n=0}^{N} \mathcal{H}_n\big)$. We have $K(\tilde{\mathcal{H}}) \subset \tilde{\mathcal{H}}$ and hence $K(\tilde{\mathcal{H}}^\perp) \subset \tilde{\mathcal{H}}^\perp$. Assume by contradiction that $\tilde{\mathcal{H}}^\perp \neq \{0\}$. The restriction of K to $\tilde{\mathcal{H}}^\perp$ is compact and symmetric, and it has no eigenvalue, so its spectrum is included in $\{0\}$. By Lemma A.11, we have $\langle Ku, u\rangle = 0$ for all $u \in \tilde{\mathcal{H}}^\perp$. We deduce that $K = 0$ on $\tilde{\mathcal{H}}^\perp$, and hence $\tilde{\mathcal{H}}^\perp \subset \mathrm{Ker}(K) \subset \tilde{\mathcal{H}}$. This gives a contradiction and proves that $\tilde{\mathcal{H}}^\perp = \{0\}$, so $\tilde{\mathcal{H}}$ is dense.

It only remains to choose an orthonormal basis of each $\mathcal{H}_n$ for $n \in [\![1, N]\!]$, and a countable orthonormal basis of $\mathcal{H}_0$ (it exists since $\mathcal{H}$ is separable). □

A.3.3 Operators with compact resolvent

We finish we operators which are not compact but have a compact resolvent.

Theorem A.13. *Let A be an operator on $\mathcal{H}$ with domain $\mathcal{D}$. Assume that there exists z_0 such that $(A - z_0)$ is bijective and $(A - z_0)^{-1} : \mathcal{H} \to \mathcal{D} \subset \mathcal{H}$ defines a compact operator on $\mathcal{H}$. Then the spectrum of A consists of a discrete set of eigenvalues with finite (geometric and algebraic) multiplicities (in particular the spectrum of A is countable without accumulation points).*

Proof. Let $B = A - z_0 : \mathcal{D} \to \mathcal{H}$. We have $0 \in \rho(B)$ and B^{-1} defines a compact operator on $\mathcal{H}$. Let $\lambda \in \mathbb{C}^*$. Assume that $\lambda \in \rho(B)$. We have

$$B^{-1} - \lambda^{-1} = -\lambda^{-1}(B - \lambda)B^{-1},$$

so $B^{-1} - \lambda^{-1} : \mathcal{H} \to \mathcal{H}$ is invertible, with bounded inverse $(B^{-1} - \lambda^{-1})^{-1} = -B(B - \lambda)^{-1}\lambda = -\lambda - \lambda^2(B - \lambda)^{-1}$. Similarly, on $\mathcal{D}$ we have

$$B - \lambda = -\lambda(B^{-1} - \lambda^{-1})B. \tag{A.2}$$

If $\lambda^{-1} \in \rho(B^{-1})$ then $B - \lambda : \mathcal{D} \to \mathcal{H}$ is invertible and its inverse $(B - \lambda)^{-1} = -B^{-1}(B^{-1} - \lambda^{-1})^{-1}\lambda^{-1}$ defines a bounded operator on $\mathcal{H}$. Thus $\lambda \in \rho(B)$. This proves that the map $\lambda \mapsto \lambda^{-1}$ is a bijection between the spectrum of B and the non-zero spectrum of B^{-1}. In particular, the spectrum of B is discrete. Moreover, if $\lambda \in \sigma(B)$ then $(B^{-1} - \lambda^{-1})$ is not injective. By (A.2), λ is an eigenvalue of B, with finite geometric multiplicity. More precisely, since B and B^{-1} commute, we see that for $k \in \mathbb{N}^*$ we have

$$\operatorname{Ker}\big((B - \lambda)^k\big) = \operatorname{Ker}\big((B^{-1} - \lambda^{-1})^k\big),$$

so the eigenvalues of B have finite algebraic multiplicities. After translation, the operator A shares the same properties and the proof is complete. □