# Chapter 2

# Sobolev spaces

## 2.1 Weak derivatives

In this first paragraph we introduce the notion of weak derivative. This generalizes the notion of differentiability to a class of functions which are not differentiable in the classical sense.

We will sometimes refer to distributions and the notion of derivatives in the sense of distributions, which are assumed to be known. However, we will recall all the required definitions and results to make this chapter self-contained.

### 2.1.1 In dimension one

We begin with the one dimensional case. Let $I$ be an open and non-empty interval of $\mathbb{R}$. The key observation behind the definition of the weak derivative is the integration by parts. For $u \in C^1(I)$ and $\phi \in C_0^\infty(I)$ we have

$$\int_I u'\phi\,dx = -\int_I u\phi'\,dx. \tag{2.1}$$

The right-hand side makes sense even when $u$ is not differentiable. This is how we define the function $u'$ which appears in the left-hand side.

**Définition 2.1.** Let $\Omega$ be an open subset of $\mathbb{R}$ and $u \in L^1_{\mathrm{loc}}(\Omega)$. We say that $v \in L^1_{\mathrm{loc}}(\Omega)$ is a weak derivative of $u$ if

$$\forall \phi \in C_0^\infty(\Omega), \quad -\int_\Omega u\phi'\,dx = \int_\Omega v\phi\,dx. \tag{2.2}$$

Before going further, we observe that the weak derivative of a function in $L^1_{\mathrm{loc}}(I)$ is necessarily unique. This is a consequence of the following classical result of integration.

**Lemma 2.2.** *Let $v \in L^1_{\mathrm{loc}}(I)$ be such that*

$$\forall \phi \in C_0^\infty(I), \quad \int_I v\phi = 0.$$

*Then $v = 0$ almost everywhere on $I$.*

*Last updated: October 2, 2021*

With this lemma we easily see that if $v_1$ and $v_2$ satisfy (2.2) then we have $v_1 = v_2$ almost everywhere. Then if $u \in L^1_{\text{loc}}(I)$ has a weak derivative, it is unique. In this case we denote this weak derivative by $u'$. This is natural since this new definition of the derivative is an extension of the usual one. Indeed, if $u$ is differentiable in the usual sense on $\Omega$, then $u'$ is a weak derivative of $u$ on $\Omega$. The proof of this remark is precisely the integration by parts formula (2.1) on which the definition is based. More generally, we can make the following observation.

*Remark* 2.3. Let $\Omega$ be an open subset of $\mathbb{R}$ and $u \in L^1_{\text{loc}}(\Omega)$. Assume that $u$ has a weak derivative $v \in L^1_{\text{loc}}(\Omega)$. Assume also that $u$ is differentiable in the usual sense in an open subset $\omega$ of $\Omega$. Then $v$ is equal to $u'$ almost everywhere in $\omega$.

As a first non-trivial example, we begin with a function which is close to be differentiable in the usual sense.

**Example 2.4.** *We consider on $\mathbb{R}$ the map $u : x \mapsto |x|$. It is differentiable in the usual sense in $\mathbb{R}^*$ but not in $\mathbb{R}$. A weak derivative of $u$ is given by the function*

$$v : x \mapsto \begin{cases} -1 & \text{if } x < 0, \\ 1 & \text{if } x \geqslant 0. \end{cases}$$

In this example, the function $u$ is differentiable everywhere except at 0. We note that the value of $v$ at 0 is not important since the definition only involves integrals. However, this does not mean that a function which is differentiable everywhere except at one point has a weak derivative.

**Example 2.5.** *We consider on $\mathbb{R}$ the Heaviside function*

$$H : x \mapsto \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x < 0, \end{cases}$$

*Then $u$ has no weak derivative on $\mathbb{R}$. For the proof we consider $v \in L^1_{\text{loc}}(\mathbb{R})$. Let $\phi \in C_0^\infty(\mathbb{R}, [0,1])$ supported in $]-1, 1[$ and such that $\phi(0) = 1$. For $n \in \mathbb{N}^*$ and $x \in \mathbb{R}$ we set $\phi_n(x) = \phi(nx)$. Then $\phi_n \in C_0^\infty(\mathbb{R})$ for all $n \in \mathbb{N}^*$. On the one hand we have by the Dominated Convergence Theorem*

$$\left| \int_{\mathbb{R}} v(x)\phi_n(x)\, dx \right| \leqslant \int_{x=-\frac{1}{n}}^{\frac{1}{n}} |v(x)|\, dx \xrightarrow[n\to+\infty]{} 0.$$

*And on the other hand*

$$-\int_{\mathbb{R}} H(x)\phi_n'(x)\, dx = -\int_0^{+\infty} \phi_n'(x)\, dx = \phi_n(0) = 1.$$

*Then $v$ is not a weak derivative for $H$.*

*Remark* 2.6. The Heaviside function has no weak derivative in $\mathbb{R}$ but it is diffentiable in the usual sense, and hence in the weak sense, in $\mathbb{R}^*$. Its strong (hence weak) derivative in $\mathbb{R}^*$ is just 0.

***Exercise* 1.** For which values of $\alpha \in \mathbb{R}$ does the function $u_\alpha : x \mapsto |x|^\alpha$ have a derivative in the usual sense in $\mathbb{R}$ ? a weak derivative ?

Notice that the weak derivative is just the derivative in the sense of distributions. A function $u \in L^1_{\text{loc}}(I)$ defines a distribution $T_u$ on $I$. This distribution has a derivative $T_u' \in \mathcal{D}'(I)$. Saying that the derivative of $u$ belongs to $L^1_{\text{loc}}(I)$ means that $T_u'$ is the distribution defined by a function in $L^1_{\text{loc}}(I)$. In other words, for some $v \in L^1_{\text{loc}}(I)$ we have $T_u' = T_v$ in $\mathcal{D}'(I)$. The Heaviside function of Example 2.5 has a derivative in the sense of distributions, given by the Dirac distribution $\delta$, but this is not a distribution associated to a function in $L^1_{\text{loc}}$.

Since $u$ always has a derivative $u'$ in the sense of distribution, instead of saying that $u$ has a derivative in $L^1_{\text{loc}}(I)$, we can simply say for short $u' \in L^1_{\text{loc}}(I)$.

***Exercise* 2.** **1.** Let $u_+ \in C^1_0([0,+\infty[)$. For $x \in \mathbb{R}$ we set

$$u_1(x) = \begin{cases} u_+(x) & \text{if } x \geqslant 0, \\ 0 & \text{if } x < 0, \end{cases} \qquad u_2(x) = \begin{cases} u_+(x) & \text{if } x \geqslant 0, \\ u_+(-x) & \text{if } x < 0, \end{cases}$$

and

$$u_3(x) = \begin{cases} u_+(x) & \text{if } x \geqslant 0, \\ -3u_+(-x) + 4u_+(-x/2) & \text{if } x < 0. \end{cases}$$

Are $u_1$, $u_2$ and $u_3$ differentiable ? Do they have a weak derivative in $L^1_{\text{loc}}(\mathbb{R})$ ?

In the following proposition we generalize to this new setting the construction of a primitive known for continuous functions.

**Proposition 2.7.** *Let $I$ be an open interval of $\mathbb{R}$, $w \in L^1_{\text{loc}}(I)$ and $x_0 \in I$. Then the map*

$$u \mapsto \int_{x_0}^x w(s)\,ds$$

*is well defined on $I$, it is continuous, and $w$ is a weak derivative of $u$.*

***Exercise* 3.** Prove Proposition 2.7.

Now we discuss the functions whose derivatives in the weak sense is zero.

**Proposition 2.8.** *Let $u \in L^1_{\text{loc}}(I)$ be such that*

$$\forall \phi \in C^\infty_0(I), \qquad \int_I u\phi'\,dx = 0.$$

*There exists a constant $\alpha$ such that $u = \alpha$ almost everywhere.*

*Proof.* • Let $\phi_0 \in C^\infty_0(I)$ be such that $\int_I \phi_0\,dx = 1$ and $\alpha = \int_I u\phi_0\,dx$. We prove that for all $\phi \in C^\infty_0(I)$ we have $\int_I u\phi = \alpha \int_I u$ almost everywhere.

• Let $\phi \in C^\infty_0(I)$ and $\beta_\phi = \int_I \phi\,dx$. For $x \in I$ we set

$$\psi(x) = \int_{\inf(I)}^x \big(\phi(x) - \beta_\phi \phi_0(x)\big)\,dx.$$

This defines a function $\psi \in C^\infty_0(I)$ such that $\psi' = \phi - \beta_\phi\phi_0$. Then, by assumption on $u$,

$$0 = \int_I u\psi' = \int_I u\phi - \alpha\beta_\phi = \int_I (u-\alpha)\phi\,dx.$$

The conclusion follows by Lemma 2.2. □

With Propositions 2.7 and 2.8 we see that if $u \in L^1_{\text{loc}}(I)$ has a weak derivative in $L^1_{\text{loc}}(I)$ then for $x_0 \in I$ there exists $\alpha \in \mathbb{R}$ such that for almost all $x \in I$ we have

$$u(x) = \alpha + \int_{x_0}^{x} u'(x)\,dx.$$

In particular, $u$ is continuous (in the sense that it is equal almost everywhere to a continuous function). Since there is no continuous function equal almost everywhere to the Heaviside function $H$, we recover the fact that $H$ cannot have a weak derivative on any interval which contains 0. We also observe that if $u \in L^1(\mathbb{R})$ has a continuous weak derivative, then it is of class $C^1$ in $I$ (it is equal almost everywhere to a function of class $C^1$).

Notice that Proposition 2.7 is specific to the dimension 1. In particular the fact that a function which has a derivative in $L^1_{\text{loc}}$ is continuous will not be valid in higher dimension (see Example 2.11 and Exercise 11).

***Exercise 4.*** **1.** Let $\alpha \in \mathbb{R}$. What are the solutions in $L^1_{\text{loc}}(\mathbb{R})$ of the equation $u' + \alpha u = 0$, where the derivative is understood in the weak sense ?
**2.** Same question with the equation $u' + \alpha u = f$, where $f \in L^1_{\text{loc}}(\mathbb{R})$.

We finish this paragraph by the definition of the successive derivatives for a function in $L^1_{\text{loc}}(I)$.

**Définition 2.9.** Let $u \in L^1_{\text{loc}}(I)$ and $k \in \mathbb{N}^*$. We say that $v_k \in L^1_{\text{loc}}(I)$ is a weak derivative of order $k$ of $u$ if

$$\forall \phi \in C_0^\infty(I), \quad (-1)^k \int_I u\phi^{(k)}\,dx = \int_I v_k \phi\,dx.$$

In this case $v_k$ is unique and is denoted by $u^{(k)}$. This is equivalent to saying that the $k$-th derivative of the distribution $T_u$ is the distribution associated to a function in $L^1_{\text{loc}}(I)$, which we denote by $u^{(k)}$.

***Exercise 5.*** Let $u \in L^1_{\text{loc}}(\mathbb{R})$. Prove that $u$ has a weak derivative of order two if and only if it has a weak derivative $u'$ and $u'$ has itself a weak derivative.

***Exercise 6.*** Let $k \geqslant 2$. Do the functions of Example 2.4 and Exercise 1 have $k$ weak derivatives on $\mathbb{R}$ ?

### 2.1.2 Weak derivatives in higher dimension

Let $d \in \mathbb{N}^*$ and let $\Omega$ be an open subset of $\mathbb{R}^d$.

**Définition 2.10.** Let $u \in L^1_{\text{loc}}(\Omega)$ and $j \in \mathbb{N}^d$. We say that $v_d \in L^1_{\text{loc}}(\Omega)$ is a weak derivative of $u$ with respect to $x_j$ if

$$\forall \phi \in C_0^\infty(\Omega), \quad -\int_\Omega u\partial_{x_j}\phi\,dx = \int_\Omega v_j \phi\,dx.$$

In this case $v_j$ is unique and is denoted by $\partial_{x_j} u$.

We begin with the analog of Example 2.4 in higher dimension.

**Example 2.11.** *For $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ we set $u(x) = |x| = \sqrt{x_1^2 + \cdots + x_d^2}$. Then $u$ is of class $C^\infty$ in $\mathbb{R}^d \setminus \{0\}$ and for $j \in [\![1, d]\!]$ and $x \in \mathbb{R}^d \setminus \{0\}$ we have*

$$\partial_j u(x) = \frac{x_j}{|x|}.$$

*This defines a function in $L^1_{\mathrm{loc}}(\mathbb{R}^d)$ (considering that it takes any value at 0). Now let $\phi \in C_0^\infty(\mathbb{R}^d)$ and $\varepsilon > 0$. By the Green formula we have*

$$-\int_{\mathbb{R}^d \setminus B(\varepsilon)} |x| \, \partial_j \phi(x) \, dx = -\int_{|x| = \varepsilon} |x| \, \phi(x) \, \nu_j \, d\sigma(x) + \int_{\mathbb{R}^d \setminus B(\varepsilon)} \frac{x_j}{|x|} \phi(x) \, dx,$$

*where $\nu = (\nu_1, \dots, \nu_d)$ is the exterior derivative to $\mathbb{R}^d \setminus B(\varepsilon)$. Taking the limit $\varepsilon \to 0$ gives*

$$-\int_{\mathbb{R}^d} |x| \, \partial_j \phi(x) \, dx = \int_{\mathbb{R}^d} \frac{x_j}{|x|} \phi(x) \, dx.$$

*This proves that $x \mapsto \frac{x_j}{|x|}$ is a weak derivative of $u$ with respect to $x_j$ in $\mathbb{R}^d$. Thus we can write, in the weak sense,*

$$\nabla u(x) = \frac{x}{|x|}.$$

***Exercise* 7.** Let $\alpha \in \mathbb{R}$ and consider on $\mathbb{R}^d \setminus \{0\}$ the function $u : x \mapsto |x|^\alpha$.
**1.** Check that the grandient of $u$ on $\mathbb{R}^d \setminus \{0\}$ is $x \mapsto \alpha \, |x|^{\alpha - 2} \, x$.
**2.** Can $u$ be extended to a function on $\mathbb{R}^d$ which has derivatives of order 1 in the usual sense ?
**3.** Does $u$ have weak derivatives of order 1 in $L^1_{\mathrm{loc}}(\mathbb{R}^d)$ ?

In applications we often deal with functions which are of classe $C^1$ except at one point, as in Exercise 7. The purpose of the following exercise is to give a general result for this situation.

***Exercise* 8.** Let $d \geqslant 2$. Let $u \in C^1(\mathbb{R}^d \setminus \{0\})$ such that $\nabla u$ (well defined on $\mathbb{R}^d \setminus \{0\}$) is in $L^1_{\mathrm{loc}}(\mathbb{R}^d)$.
**1.** a. Prove that for $\omega \in S(1)$ and $r \in ]0, 1]$ we have

$$|u(r\omega)| \leqslant |u(\omega)| + \int_r^1 |\nabla u(s\omega)| \, ds.$$

b. Deduce that

$$\int_{S(r)} |u(\omega)| \, d\sigma_r(\omega) \leqslant r^{d-1} \int_{S(1)} |u(\omega)| \, d\sigma(\omega) + \int_{B(1) \setminus B(r)} \frac{r^{d-1}}{|x|^{d-1}} |\nabla u(x)| \, dx.$$

**2.** Prove that $u \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$.
**3.** Prove that for $\phi \in C_0^\infty(\mathbb{R}^d)$ we have

$$-\int_{\mathbb{R}^d} u \nabla \phi \, dx = \int_{\mathbb{R}^d} \phi \nabla u \, dx.$$

We recall that the classical notion of differentiability is defined by looking at the limit at each point of the difference quotient. The following result gives a link between this point of view and the weak derivative.

Let $h \in \mathbb{R}^d \setminus \{0\}$ and let $\Omega$ be an open subset of $\mathbb{R}^d$ invariant by translation by $h$. For $u \in L^2(\Omega)$ and we define the diffence quotient $D_h u \in L^2(\Omega)$ by

$$D_h u(x) = \frac{u(x+h) - u(x)}{|h|}. \tag{2.3}$$

Notice that for $u, v \in L^2(\Omega)$ we have

$$\int_{\mathbb{R}^d} (D_h u) v \, dx = \int_{\mathbb{R}^d} u (D_{-h} v) \, dx. \tag{2.4}$$

Moreover, $D_h$ commutes with derivatives: if $\partial_j u \in L^2(\Omega)$ then

$$\partial_j D_h u = D_h \partial_j u \in L^2(\Omega). \tag{2.5}$$

**Proposition 2.12.** *Let $j \in [\![1, d]\!]$ and assume that $\Omega$ is invariant by translation by $te_j$ for all $t \in \mathbb{R}$, where $e_j$ is the $j$-th vector of the canonical basis. Assume that there exists $C > 0$ such that for all $t \in \mathbb{R} \setminus \{0\}$ we have*

$$\left\| D_{te_j} u \right\|_{L^2(\Omega)} \leqslant C.$$

*Then $\partial_j u \in L^2(\Omega)$ and*

$$\left\| \partial_j u \right\|_{L^2(\Omega)} \leqslant C.$$

*Proof.* Let $\phi \in C_0^\infty(\mathbb{R}^d)$. By the dominated convergence theorem we have

$$\left| - \langle u, \partial_j \phi \rangle_{L^2(\mathbb{R}^d)} \right| = \left| \lim_{t \to 0^+} \left\langle u, D_{te_j} \phi \right\rangle_{L^2(\mathbb{R}^d)} \right| = \left| \lim_{t \to 0^+} \left\langle D_{-te_j} u, \phi \right\rangle_{L^2(\mathbb{R}^d)} \right| \leqslant C \left\| \phi \right\|_{L^2(\mathbb{R}^d)}.$$

By the Riesz Theorem there exists $v_j \in L^2(\mathbb{R}^d)$ such that $\|v_j\|_{L^2(\Omega)} \leqslant C$ and

$$\forall \phi \in C_0^\infty(\mathbb{R}^d), \quad - \langle u, \partial_j \phi \rangle_{L^2(\mathbb{R}^d)} = \langle v_j, \phi \rangle_{L^2(\mathbb{R}^d)}.$$

This proves that $\partial_j u = v$. □

*Remark* 2.13. Proposition 2.12 holds $L^2(\Omega)$ replaced by $L^p(\Omega)$ for any $p \in ]1, +\infty[$.

We recall the usual notation for partial derivatives in dimension $d \geqslant 2$. For $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$ we set $|\alpha| = \alpha_1 + \cdots + \alpha_d$.

$$\partial^\alpha = \partial_{x_1}^{\alpha_1} \dots \partial_{x_d}^{\alpha_d}$$

**Définition 2.14.** Let $u \in L^1_{\text{loc}}(\Omega)$ and $\alpha \in \mathbb{N}^d$. We say that $v_\alpha \in L^1_{\text{loc}}(\Omega)$ is a derivative of order $\alpha$ of $u$ if

$$\forall \phi \in \Omega, \quad (-1)^{|\alpha|} \int_\Omega u \partial^\alpha \phi \, dx = \int_\Omega v_\alpha \phi \, dx.$$

In this case $v_\alpha$ is unique and is denoted by $\partial^\alpha u$.

***Exercise* 9.** In dimension $d \geqslant 2$, compute all the second derivatives in the weak sense for the function $x \mapsto |x|$ in $\mathbb{R}^d$.

## 2.2 Sobolev spaces

### 2.2.1 Definition and examples

Let $d \geqslant 1$ and let $\Omega$ be an open subset of $\mathbb{R}^d$.

**Définition 2.15.** For $p \in [1, +\infty]$ and $k \in \mathbb{N}$ we set

$$W^{k,p}(\Omega) = \left\{ u \in L^p(\Omega) \,:\, \partial^\alpha u \in L^p(\Omega) \text{ for all } \alpha \in \mathbb{N}^d \text{ with } |\alpha| \leqslant k \right\},$$

where $\partial^\alpha u$ is the derivative of $u$ in the sense of distributions. In other words, a function $u \in L^p(\Omega)$ belongs to $W^{k,p}(\Omega)$ if it has a weak derivative $\partial^\alpha u$ in $L^p(\Omega)$ for all $\alpha \in \mathbb{N}^d$ with $|\alpha| \leqslant k$.We also set $H^k(\Omega) = W^{k,2}(\Omega)$.

*Remark* 2.16. By the Riesz Theorem and by density of $C_0^\infty(\Omega)$ in $L^2(\Omega)$, a function $u \in L^1_{\text{loc}}(\Omega)$ belongs to $H^k(\Omega)$ is and only if for all $\alpha \in \mathbb{N}^d$ with $|\alpha| \leqslant k$ there exists $C_\alpha > 0$ such that

$$\forall \phi \in C_0^\infty(\Omega), \qquad \left| \int_\Omega u \partial^\alpha \phi \, dx \right| \leqslant C_\alpha \left\| \phi \right\|_{L^2(\Omega)}.$$

**Example 2.17.** *Let $p \in [1, +\infty]$. The map $u : x \mapsto |x|$ is not in $W^{1,p}(\mathbb{R})$, (since it is not in $L^p(\mathbb{R})$), but it is in $W^{1,p}(I)$ for any bounded open interval $I$ of $\mathbb{R}$. It is not in $W^{k,p}(I)$ for $k \geqslant 2$ if $I$ contains 0, since $u$ has no second derivative in the weak sense in a neighborhood of 0.*

**Example 2.18.** *We consider on $]0,1[$ the function $u : x \mapsto x^{-\frac{1}{4}}$. Then $u$ belongs to $L^2(]0,1[)$ but its derivative $u' : x \mapsto -\frac{1}{4}x^{-\frac{5}{4}}$ is not in $L^2(]0,1[)$, so $u$ is not in $H^1(]0,1[)$. We similarly consider $u : x \mapsto x^{-\frac{1}{4}}$ on $]1,+\infty[$. Then $u' \in L^2(]1,+\infty[)$ but $u \notin L^2(]1,+\infty[)$, so $u \notin H^1(]1,+\infty[)$. On the other hand, the function $u \mapsto x^{\frac{3}{4}}$ belongs to $H^1(]0,1[)$ and $x \mapsto x^{-\frac{3}{4}}$ belongs to $H^1(]1,+\infty[)$.*

***Exercise* 10.** Let $p \in [1,+\infty]$ and $\alpha \in \mathbb{R}$. Does the function $x \mapsto x^\alpha$ belongs to $W^{1,p}(]0,1[)$ ? $W^{1,p}(]1,+\infty[)$ ? $W^{1,p}(]0,+\infty[)$ ?

**Example 2.19.** *Let $p \in [1,+\infty[$ and $\beta > 0$. For $x \in B(1)\backslash\{0\}$ we set $u(x) = |x|^{-\beta}$. Then $u \in L^p(B(1))$ if and only if $\beta p < d$. On the other hand $u$ is of class $C^1$ on $B(1)\backslash\{0\}$ and $\nabla u(x) = -\beta\,|x|^{-\beta-2}\,x$ for all $x \in B(1)\backslash\{0\}$. Thus $\nabla u \in L^p(B(1))$ if and only if $(\beta+1)p < d$. This proves (see Exercises 7 and 8) that $u \in W^{1,p}(B(1))$ if and only if $(\beta+1)p < d$.*

***Exercise* 11.** Let $p \in [1,+\infty]$.
**1.** Does the map $x \mapsto \ln(|\ln(|x|)|)$ belong to $W^{1,p}(B(1))$ ?
**2.** Let $\alpha > 0$. Does $x \mapsto |\ln|x||^\alpha$ belong to $W^{1,p}(B(1))$ ?

In the following proposition we give some basic properties for the set $W^{k,p}(\Omega)$. We define $C_0^\infty(\overline{\Omega})$ as the restrictions to $\overline{\Omega}$ of functions in $C_0^\infty(\mathbb{R}^d)$.

**Proposition 2.20.** *Let $p \in [1,+\infty]$, $k \in \mathbb{N}^*$ and $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$ with $|\alpha| \leqslant k$. Let $u \in W^{k,p}(\Omega)$.*

*(i) We have $\partial^\alpha u \in W^{k-|\alpha|,p}(\Omega)$ and for $\beta \in \mathbb{N}^d$ with $|\beta| \leqslant k - |\alpha|$ we have $\partial^\beta(\partial^\alpha u) = \partial^{\alpha+\beta} u$.*

*(ii)* *Let $\omega$ be an open subset of $\Omega$. Then the restriction $u_{|\omega}$ of $u$ on $\omega$ belongs to $W^{k,p}(\omega)$ and $\partial^\alpha(u_\omega) = (\partial^\alpha u)_{|\omega}$.*

*(iii)* *Let $\chi \in C_0^\infty(\overline{\Omega})$. Then $\chi u \in W^{k,p}(\Omega)$ and*

$$\partial^\alpha(\chi u) = \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} \partial^\beta \chi \partial^{\alpha-\beta} u,$$

*where we have set*

$$\binom{\alpha}{\beta} = \frac{\alpha!}{\beta!(\alpha-\beta)!}, \qquad \alpha! = \alpha_1! \dots \alpha_d!.$$

***Exercise* 12.** Prove Proposition 2.20.

***Exercise* 13.** Let $p \in [1, +\infty]$ and $u \in W^{1,p}(\mathbb{R}^d)$. Let $\rho \in C_0^\infty(\mathbb{R}^d)$. We recall that $(\rho * u) \in C^\infty(\mathbb{R}^d)$. Prove that for $j \in [\![1, d]\!]$ we have

$$\partial_j(\rho * u) = \rho * (\partial_j u).$$

Deduce that $(\rho * u) \in W^{1,p}(\mathbb{R}^d)$.

**Définition 2.21.** Let $\Omega$ be an open subset of $\mathbb{R}^d$ and let $u \in L^1_{\text{loc}}(\Omega)$. Let $p \in [1, +\infty]$ and $k \in \mathbb{N}$. We say that $u$ belongs to $W^{k,p}_{\text{loc}}(\Omega)$ if for any $\chi \in C_0^\infty(\Omega)$ we have $\chi u \in W^{k,p}(\Omega)$.

**Example 2.22.** *The function $x \mapsto |x|$ belongs to $W^{1,p}_{\text{loc}}(\mathbb{R})$ for any $p \in [1, +\infty]$ (see Example 2.17).*

### 2.2.2 Norms on the Sobolev spaces

Let $\Omega$ be an open subset of $\mathbb{R}^d$. Let $p \in [1, +\infty]$ and $k \in \mathbb{N}$. For $u \in W^{k,p}(\Omega)$ we set

$$\|u\|_{W^{k,p}(\Omega)} = \left( \sum_{|\alpha| \leqslant k} \|\partial^\alpha u\|^p_{L^p(\Omega)} \right)^{\frac{1}{p}}. \tag{2.6}$$

This defines a norm on $W^{k,p}(\Omega)$. We could also consider the quantity

$$\sum_{|\alpha| \leqslant k} \|\partial^\alpha u\|_{L^p(\Omega)}, \tag{2.7}$$

which defines an equivalent norm on $W^{k,p}(\Omega)$.

On $H^k(\Omega)$ we define an inner product by setting, for $u, v \in H^k(\Omega)$,

$$\langle u, v \rangle_{H^k(\Omega)} = \sum_{|\alpha| \leqslant k} \langle \partial^\alpha u, \partial^\alpha v \rangle_{L^2(\Omega)}. \tag{2.8}$$

The corresponding norm is exactly (2.6) with $p = 2$.

*Remark* 2.23. With the notation of Proposition 2.20, we observe that

$$\|\partial^\alpha u\|_{W^{k-|\alpha|,p}(\Omega)} \leqslant \|u\|_{W^{k,p}(\Omega)},$$

for $\omega \subset \Omega$ we have

$$\|u\|_{W^{k,p}(\omega)} \leqslant \|u\|_{W^{k,p}(\Omega)},$$

and for $\chi \in C_0^\infty(\overline{\Omega})$ there exists $C_\chi > 0$ independant of $u$ such that

$$\|\chi u\|_{W^{k,p}(\Omega)} \leqslant C_\chi \|u\|_{W^{k,p}(\Omega)}.$$

**Theorem 2.24.** *Let $k \in \mathbb{N}$ and $p \in [1, +\infty]$. The Sobolev space $W^{k,p}(\Omega)$, endowed with the norm (2.7) or (2.6), is a Banach space. In particular, $H^k(\Omega)$ with the inner product (2.8) is a Hilbert space.*

*Proof.* Let $(u_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $W^{k,p}(\Omega)$. The sequences $(\partial^\alpha u_n)_{n\in\mathbb{N}}$ for $|\alpha| \leqslant k$ are Cauchy sequences in $L^p(\Omega)$. Since $L^p(\Omega)$ is complete by the Riesz-Fisher theorem, there exist $v_\alpha \in L^p(\Omega)$ for $|\alpha| \leqslant k$ such that $\partial^\alpha u_n$ goes to $v_\alpha$. For $|\alpha| \leqslant k$ and $\phi \in C_0^\infty(\Omega)$ we have

$$(-1)^{|\alpha|} \int_\Omega v_0 \, \partial^\alpha \phi \, dx = (-1)^{|\alpha|} \lim_{n\to+\infty} \int_\Omega u_n \, \partial^\alpha \phi \, dx = \lim_{n\to+\infty} \int_\Omega \partial^\alpha u_n \, \phi \, dx = \int_\Omega v_\alpha \, \phi \, dx.$$

This proves that in the sense of distributions we have $\partial^\alpha v_0 = v_\alpha \in L^p(\Omega)$. Then $v_0 \in W^{k,p}(\Omega)$ and

$$\|u_n - v_0\|^p_{W^{k,p}(\Omega)} = \sum_{|\alpha|\leqslant k} \|\partial^\alpha u_n - v_\alpha\|^2_{L^p(\Omega)} \xrightarrow[n\to+\infty]{} 0.$$

Thus the sequence $(u_n)_{n\in\mathbb{N}}$ has a limit in $W^{k,p}(\Omega)$. This proves that $W^{k,p}(\Omega)$ is complete. □

The proofs of the following two results are omitted (see [Brézis]).

**Theorem 2.25.** *Let $\Omega$ be an open subset of $\mathbb{R}^d$, $p \in [1, +\infty]$ and $k \in \mathbb{N}$.*

(i) *$W^{k,p}(\Omega)$ is reflexive if and only if $p \in ]1, +\infty[$.*

(ii) *$W^{k,p}(\Omega)$ is separable if and only if $p \in [1, +\infty[$.*

*Proof.* We recall that $L^p(\Omega)$ is reflexive if and only if $p \in ]1, \infty[$ (see Section 4.3 in [Brézis]) and separable if and only if $p \in [1, +\infty[$. In particular, $W^{k,p}(\Omega)$ cannot be separable if $p = \infty$.

The map

$$\Phi : \begin{cases} W^{k,p(\Omega)} & \to & \prod_{|\alpha|\leqslant k} L^p(\Omega) \\ u & \mapsto & (\partial^\alpha u)_{|\alpha|\leqslant k} \end{cases}$$

is an isometry from $W^{k,p(\Omega)}$ to a closed subspace of $\prod_{|\alpha|\leqslant k} L^p(\Omega)$. If $p \in ]1, +\infty[$ then $\prod_{|\alpha|\leqslant k} L^p(\Omega)$ is reflexive and so is $W^{k,p}(\Omega)$ (see Proposition 3.20 in [Brézis]). Similarly, if $p \in [1, +\infty[$ then $\prod_{|\alpha|\leqslant k} L^p(\Omega)$ is separable and so is $W^{k,p}(\Omega)$ (see Proposition 3.25 in [Brézis]).

□

### 2.2.3 Characterisation via the Fourier transform

When $\Omega = \mathbb{R}^d$ and $p = 2$ we can use the Fourier transform to give a simple characterisation of $H^k(\mathbb{R}^d)$. Notice that in Definition 2.15 we can see the derivatives of $u$ in the sense of tempered distributions. This means that we can replace $C_0^\infty(\mathbb{R}^d)$ by $\mathcal{S}(\mathbb{R}^d)$ in Definition 2.10.

**Proposition 2.26.** *Let $\alpha \in \mathbb{N}^d$ and $u \in L^2(\mathbb{R}^d)$. Then $\partial^\alpha u \in L^2(\Omega)$ if and only if the map $\xi \mapsto (i\xi)^\alpha \hat{u}(\xi)$ belongs to $L^2(\mathbb{R}^d)$ (and, in this case, it is the Fourier transform of $\partial^\alpha u$). Then, for $k \in \mathbb{N}$, $u \in H^k(\mathbb{R}^d)$ if and only if*

$$\int_{\mathbb{R}^d} \left(1 + |\xi|^2\right)^k |\hat{u}(\xi)|^2 \, d\xi < +\infty. \tag{2.9}$$

*Proof.* Let $u \in L^2(\mathbb{R}^d)$. For $\phi \in \mathcal{S}(\mathbb{R}^d)$ we have

$$\int_{\mathbb{R}^d} (iy)^\alpha \hat{u} \phi \, dy = \int_{\mathbb{R}^d} u \widehat{(iy)^\alpha \phi} \, dy = (-1)^{|\alpha|} \int_{\mathbb{R}^d} u \partial^\alpha \hat{\phi} \, dy. \tag{2.10}$$

Assume that $\partial^\alpha u \in L^2(\mathbb{R}^d)$. Then (2.10) gives

$$\int_{\mathbb{R}^d} (iy)^\alpha \hat{u} \phi \, dy = \int_{\mathbb{R}^d} \partial^\alpha u \hat{\phi} \, dy = \int_{\mathbb{R}^d} \widehat{\partial^\alpha u} \phi \, dy,$$

so the map $y \mapsto (iy)^\alpha \hat{u}(y)$ belongs to $L^2(\mathbb{R}^d)$, and it is the Fourier transform of $\partial^\alpha u$.

Conversely, assume that $y \mapsto (iy)^\alpha \hat{u}(y)$ belongs to $L^2(\mathbb{R}^d)$. By (2.10) applied with $\check{\phi}$ we have

$$\left| \int_{\mathbb{R}^d} u \partial^\alpha \phi \, dy \right| = \left| \int_{\mathbb{R}^d} y^\alpha \hat{u} \check{\phi} \, dy \right| \leqslant \frac{\| y^\alpha \hat{u} \|_{L^2(\mathbb{R}^d)}}{(2\pi)^{\frac{d}{2}}} \| \phi \|_{L^2(\mathbb{R}^d)},$$

so $\partial^\alpha u \in L^2(\mathbb{R}^d)$. □

**Proposition 2.27.** *Let $k \in \mathbb{N}$. The map*

$$(u, v) \mapsto \left( \int_{\mathbb{R}^d} \left( 1 + |\xi|^2 \right)^k \hat{u}(\xi) \overline{\hat{v}(\xi)} \, d\xi \right)^{\frac{1}{2}}$$

*defines a scalar product on $H^s(\mathbb{R}^d)$, and the corresponding norm is equivalent to (2.6) with $p = 2$.*

*Remark* 2.28. If $u \in L^2(\mathbb{R}^d)$ is such that $\Delta u$ belongs to $L^2(\mathbb{R}^d)$, then $u$ belongs to $H^2(\mathbb{R}^d)$. This remark does not hold on a general domain (see Remark 3.18 below). However, on a general $\Omega$ we can at least say that if $u \in H^1(\Omega)$ is such that $\Delta u \in L^2(\Omega)$ then $u \in H^2_{\text{loc}}(\Omega)$ (for any $\chi \in C_0^\infty(\Omega)$ we extend $\chi u$ by 0 on $\mathbb{R}^d$, since $\chi u \in L^2(\mathbb{R}^d)$ and $\Delta(\chi u) \in L^2(\mathbb{R}^d)$ we have $\chi u \in H^2(\mathbb{R}^d)$.

## 2.3 Approximation by smooth functions

In this section we start proving some properties of the Sobolev spaces. The first important property is the density of smooth and compactly supported functions. This will then to extend many properties known for regular functions to functions with only weak derivatives.

Here we mainly discuss the density in the Euclidean space. The density of smooth functions in the general case will be discussed in the following section.

### 2.3.1 In the Euclidean space

We know that for $p \in [1, +\infty[$ the set $C_0^\infty(\Omega)$ of smooth and compactly supported functions on the open set $\Omega$ is dense in $L^p(\Omega)$. In this paragraph we will see in what sense we can similarly approach functions in $W^{k,p}(\Omega)$ by smooth functions.

More precisely, we prove the density of smooth functions in the Sobolev spaces when $\Omega = \mathbb{R}^d$. This will not be the case in general domains. Since the closure of $C_0^\infty(\Omega)$ in $W^{k,p}(\Omega)$ will play an important role in applications, we introduce the following notation.

**Définition 2.29.** For $k \in \mathbb{N}$ and $p \in [1, +\infty[$ we denote by $W_0^{k,p}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{k,p}(\Omega)$. We also set $H_0^k(\Omega) = W_0^{1,2}(\Omega)$.

***Exercise* 14.** For $x \in ]-1,1[$ we set $u(x) = 1$. Prove that for $p \in [1, +\infty]$ there is no sequence $(u_n)_{n\in\mathbb{N}}$ in $C_0^\infty(]-1,1[)$ which goes to $u$ in $W^{1,p}(]-1,1[)$.

As in $L^p(\mathbb{R}^d)$, the proofs will rely on regularization by convolution with a sequence of mollifiers. Let $\rho \in C_0^\infty(\mathbb{R}^d, [0,1])$ be supported in $B(0,1)$ and such that $\int_{\mathbb{R}^d} \rho \, dx = 1$. For $n \in \mathbb{N}^*$ and $x \in \mathbb{R}^d$ we set $\rho_n(x) = n^d \rho(nx)$.

**Lemma 2.30.** *Let $\Omega$ be an open subset of $\mathbb{R}^d$. Let $n \in \mathbb{N}^*$ and let $\omega$ be an open subset of $\Omega$ such that $B\big(x, \frac{1}{n}\big) \subset \Omega$ for all $x \in \omega$. Let $\rho_n \in C_0^\infty(\mathbb{R}^d)$ be as above and let $u \in W^{k,p}(\Omega)$. Then $\rho_n * u \in C^\infty(\mathbb{R}^d) \cap W^{k,p}(\omega)$ and for $|\alpha| \leqslant k$ we have in the weak sense on $\omega$*

$$\partial^\alpha(\rho_n * u) = \rho_n * (\partial^\alpha u).$$

Notice that the lemma applies in particular with $\omega = \Omega = \mathbb{R}^d$.

*Proof.* We prove the case $k = 1$, and the general case follows by induction. Let $j \in [\![1, d]\!]$ and $\phi \in C_0^\infty(\omega)$. We have

$$-\int_\omega (\rho_n * u)(x) \partial_j \phi(x) \, dx = -\int_{B(0,\frac{1}{n})} \rho_n(y) \int_\omega u(x-y) \partial_j \phi(x) \, dx \, dy.$$

For $y \in B\big(0, \frac{1}{n}\big)$ the map $x \mapsto u(x-y)$ belongs to $W^{1,p}(\omega)$, so

$$-\int_\omega (\rho_n * u)(x) \partial_j \phi(x) \, dx = \int_{B(0,\frac{1}{n})} \rho_n(y) \int_\omega \partial_j u(x-y) \phi(x) \, dx \, dy = \int_\omega (\rho_n * \partial_j u)(x) \phi(x) \, dx.$$

The conclusion follows. □

For the following two proofs we also consider $\chi \in C_0^\infty(\mathbb{R}^d)$ supported in the ball $B(0,2)$ and equal to 1 on $B(0,1)$. Then for $m \in \mathbb{N}^*$ and $x \in \mathbb{R}^d$ we set $\chi_m(x) = \chi(\frac{x}{m})$.

**Theorem 2.31.** *Let $p \in [1, +\infty[$. Then $C_0^\infty(\mathbb{R}^d)$ is dense in $W^{k,p}(\mathbb{R}^d)$. In other words, we have $W_0^{k,p}(\mathbb{R}^d) = W^{k,p}(\mathbb{R}^d)$.*

*Proof.* Let $u \in W^{k,p}(\mathbb{R}^d)$ and $\varepsilon > 0$. Let $\alpha \in \mathbb{N}^d$ with $|\alpha| \leqslant k$. By Proposition 2.20 we have $\chi_m u \in W^{k,p}(\mathbb{R}^d)$ for all $m \in \mathbb{N}^*$ and

$$\|\partial^\alpha(\chi_m u) - \partial^\alpha u\|_{L^p(\mathbb{R}^d)} \leqslant \sum_{0 \leqslant \beta \leqslant \alpha} \binom{\alpha}{\beta} \left\|\partial^{\alpha-\beta}(\chi_m - 1) \partial^\beta u\right\|_{L^p(\mathbb{R}^d)}.$$

By the dominated convergence theorem we have for $\beta \leqslant \alpha$

$$\left\|\partial^{\alpha-\beta}(\chi_m - 1)\partial^\beta u\right\|_{L^p(\mathbb{R}^d)}^p \leqslant \left\|\partial^{\alpha-\beta}(\chi_m - 1)\right\|_{L^\infty(\mathbb{R}^d)} \int_{|x| \geqslant m} \left|\partial^\beta u(x)\right|^p \, dx \xrightarrow[m \to +\infty]{} 0,$$

so there exists $m \in \mathbb{N}^*$ such that

$$\|u - \chi_m u\|_{W^{k,p}(\mathbb{R}^d)} \leqslant \frac{\varepsilon}{2}.$$

We set $v = \chi_m u$, and for $n \in \mathbb{N}^*$ we set $v_n = \rho_n * v$. Then $v_n \in C_0^\infty(\mathbb{R}^d)$ and for all $\alpha \in \mathbb{N}^d$ with $|\alpha| \leqslant k$ we have by Lemma 2.30

$$\|\partial^\alpha v_n - \partial^\alpha v\|_{L^p(\mathbb{R}^d)} = \|\rho_n * (\partial^\alpha v) - \partial^\alpha v\|_{L^p(\mathbb{R}^d)} \xrightarrow[n\to+\infty]{} 0.$$

Then, if for $n \in \mathbb{N}^*$ large enough we set $u_\varepsilon = v_n$ we have $u_\varepsilon \in C_0^\infty(\mathbb{R}^d)$ and

$$\|u_\varepsilon - v\|_{W^{k,p}(\mathbb{R}^d)} \leqslant \frac{\varepsilon}{2},$$

so finally

$$\|u_\varepsilon - u\|_{W^{k,p}(\mathbb{R}^d)} \leqslant \varepsilon. \qquad \square$$

*Remark* 2.32. For any $\varepsilon > 0$ the function $u_\varepsilon$ constructed in the previous proof is such that $\|u_\varepsilon\|_{L^\infty(\mathbb{R}^d)} \leqslant \|u\|_{L^\infty(\mathbb{R}^d)}$.

### 2.3.2 A result in general domains

The conclusion of Theorem 2.31 does not hold in a general domain $\Omega$. In other words, $W_0^{k,p}(\Omega) \neq W^{k,p}(\Omega)$ in general (see Exercise 14 and Proposition 2.53 below). However we have the following weaker result of approximation by regular functions on any compact subset of $\Omega$. For a result of approximation on the whole domain $\Omega$ we refer to Theorem 2.39 below.

**Theorem 2.33.** *Let $p \in [1,+\infty[$ and $k \in \mathbb{N}$. Let $\Omega$ be an open subset of $\mathbb{R}^d$. Let $u \in W^{k,p}(\Omega)$. There exists a sequence $(u_n)_{n\in\mathbb{N}}$ in $C_0^\infty(\mathbb{R}^d)$ such that $u_{n|\Omega}$ goes to $u$ in $L^p(\Omega)$ and for any open bounded subset $\omega$ such that $\overline{\omega} \subset \Omega$ we have*

$$\left\|u_{n|\omega} - u_{|\omega}\right\|_{W^{k,p}(\omega)} \xrightarrow[n\to+\infty]{} 0.$$

*Proof.* For $x \in \mathbb{R}^d$ we set

$$v(x) = \begin{cases} u(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \in \mathbb{R}^d \setminus \Omega. \end{cases}$$

Then we set $v_n = \rho_n * v \in C^\infty(\mathbb{R}^d)$ and $u_n = \chi_n v_n$. We have

$$\|u_n - u\|_{L^p(\Omega)} \leqslant \|u_n - v\|_{L^p(\mathbb{R}^d)} \leqslant \|\chi_n(\rho_n * v) - \chi_n v\|_{L^p(\Omega)} + \|\chi_n v - v\|_{L^p(\Omega)} \xrightarrow[n\to+\infty]{} 0.$$

Let $N \in \mathbb{N}$ be so large that $B\big(x, \frac{1}{N}\big) \subset \Omega$ and $\chi_N = 1$ on $B\big(x, \frac{1}{N}\big)$ for all $x \in \omega$. Then for $n \geqslant N$ and $|\alpha| \leqslant k$ we have by Lemma 2.30

$$\|\partial^\alpha(u_n - u)\|_{L^p(\omega)} = \|\partial^\alpha(v_n - v)\|_{L^p(\omega)} = \|\rho_n * (\partial^\alpha v) - \partial^\alpha v\|_{L^p(\omega)} \xrightarrow[n\to+\infty]{} 0.$$

This proves that $\chi v_n$ goes to $u$ in $W^{k,p}(\omega)$. $\square$

### 2.3.3 Examples of properties proved by density

It is not always convenient to prove results about differentiation in the weak sense, and most of the properties of Sobolev spaces are proved by density. We first prove the result for regular functions (smooth, or of class $C^k$ for a property in $W^{k,p}$), and then the general case is deduced by density.

Here we give some examples of results which are already known for regular functions and which can be extended in the suitable Sobolev spaces by density.

We begin with the integration by parts formula.

**Proposition 2.34** (Green Formula without boundary term)**.** *Let $\Omega$ be an open subset of $\mathbb{R}^d$ and $u, v \in H_0^1(\Omega)$. For $j \in [\![1, d]\!]$ we have*

$$\int_\Omega (\partial_j u) v \, dx = -\int_\Omega u(\partial_j v) \, dx.$$

*Proof.* Let $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ be sequences in $C_0^\infty(\Omega)$ which go to $u$ and $v$ in $H^1(\Omega)$. The Green formula for smooth and compactly supported functions gives, for all $n \in \mathbb{N}$,

$$\int_\Omega (\partial_j u_n) v_n \, dx = -\int_\Omega u_n(\partial_j v_n) \, dx.$$

Taking the limit $n \to +\infty$ gives the result. □

We continue with the product of differentiable functions. If $u$ and $v$ are continuously differentiable, then so is the product $uv$. The same result holds for weak derivatives. Notice that in this result and the following we do not take functions in $W_0^{1,p}(\Omega)$. The approximation by regular functions is given by Theorem 2.33.

**Proposition 2.35** (Differentiation of a product)**.** *Let $\Omega$ be an open subset of $\mathbb{R}^d$. Let $p \in [1, +\infty]$ and $u, v \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$. Then $uv \in W^{1,p}(\Omega)$ and, for $j \in [\![1, d]\!]$,*

$$\partial_j(uv) = (\partial_j u)v + u(\partial_j v). \tag{2.11}$$

*Proof.* Assume that $p < +\infty$. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $C_0^\infty(\mathbb{R}^d)$ as given by Theorem 2.33. After extraction of a subsequence if necessary, we can assume that $u_n(x)$ tends to $u(x)$ for almost all $x \in \mathbb{R}^d$. By Remark 2.32, we can also assume that $\|u_n\|_{L^\infty(\mathbb{R}^d)} \leqslant \|u\|_{L^\infty(\mathbb{R}^d)}$ for all $n \in \mathbb{N}$. By Proposition 2.20, we have $u_n v \in W^{1,p}(\Omega)$ for all $n \in \mathbb{N}$ and, for $j \in [\![1, d]\!]$ and $\phi \in C_0^\infty(\Omega)$,

$$-\int_\Omega u_n v \partial_j \phi \, dx = -\int_\Omega v \partial_j(u_n \phi) \, dx + \int_\Omega v\phi \partial_j u_n \, dx = \int_\Omega \big((\partial_j u_n)v + u_n(\partial_j v)\big)\phi \, dx.$$

The limit $n \to +\infty$ yields (2.11). In particular $\partial_j(uv) \in L^p(\Omega)$, and the proof is complete if $p < +\infty$.

Now assume that $p = +\infty$. Then $uv$ and $(\partial_j u)v + u(\partial_j v)$ are in $L^\infty(\Omega)$. Let $\phi \in C_0^\infty(\Omega)$. Let $\chi \in C_0^\infty(\mathbb{R}^d)$ be equal to 1 on a neighborhood of $\operatorname{supp}(\phi)$. Then $\chi u$ and $\chi v$ are in $W^{1,p}(\Omega)$ for any $p \in [1, +\infty[$ so

$$\begin{aligned} -\int_\Omega uv\partial_j\phi \, dx &= -\int_\Omega \chi u \, \chi v \partial_j \phi \, dx \\ &= \int_\Omega \big(\partial_j(\chi u)\, \chi v + \chi u \, \partial_j(\chi v)\big)\, \phi \, dx \\ &= \int_\Omega \big((\partial_j u)v + u(\partial_j v)\big)\phi \, dx. \end{aligned}$$

This proves (2.11) and concludes the proof. □

Then we discuss the chain rule, which will be important in particular for changes of variables.

**Proposition 2.36** (Chain rule). *Let $\Omega_1$ and $\Omega_2$ be two open subsets in $\mathbb{R}^d$, and let $\Phi = (\Phi_1, \dots, \Phi_d) : \Omega_1 \to \Omega_2$ be a diffeomorphism of class $C^1$. We assume that $\mathrm{Jac}(\Phi)$ and $\mathrm{Jac}(\Phi^{-1})$ are bounded on $\Omega_1$ and $\Omega_2$, respectively. Let $p \in [1, +\infty]$. Then for $u \in W^{1,p}(\Omega_2)$ we have $u \circ \Phi \in W^{1,p}(\Omega_1)$ and for $j \in [\![1, d]\!]$,*

$$\partial_j (u \circ \Phi) = \sum_{k=1}^{d} \big( (\partial_k u) \circ \Phi \big) \partial_j \Phi_k. \tag{2.12}$$

*In particular there exists $C_\Phi > 0$ such that*

$$\| u \circ \Phi \|_{W^{1,p}(\Omega_2)} \leqslant C_\Phi \| u \|_{W^{1,p}(\Omega_1)} .$$

*Proof.* Assume that $p < +\infty$. With the change of variables $y = \Phi(x)$ we first observe that

$$\| u \circ \Phi \|_{L^p(\Omega_1)}^p = \int_{\Omega_1} |u(\Phi(x))|^p \, dx = \int_{\Omega_2} |u(y)|^p \left| J\Phi^{-1}(y) \right| dy \leqslant \left\| J\Phi^{-1} \right\|_{L^\infty(\Omega_2)} \| u \|_{L^p(\Omega_2)}^p ,$$

so $u \circ \Phi \in L^p(\Omega_2)$. Similarly $\partial_k u \circ \Phi \in L^p(\Omega_1)$ for all $k \in [\![1, d]\!]$. Since $\partial_j \Phi_k$ is bounded, this proves that the right-hand side of (2.12) belongs to $L^p(\Omega_1)$.

Let $(u_n)_{n \in \mathbb{N}}$ be a sequence in $C_0^\infty(\mathbb{R}^d)$ which goes to $u$ in the sense of Theorem 2.33. Let $\psi \in C_0^\infty(\Omega_1)$, $K_1 = \mathrm{supp}(\psi)$ and $K_2 = \Phi(K_1)$. Then $K_2$ is a compact of $\Omega_2$. For $n \in \mathbb{N}$ and $\psi \in C_0^\infty(\mathbb{R}^d)$ we have

$$-\int_{\Omega_1} (u_n \circ \Phi) \partial_j \psi \, dx = \sum_{k=1}^{d} \int_{\Omega_1} (\partial_k u_n \circ \Phi) \partial_j \Phi_k \, \psi \, dx. \tag{2.13}$$

As above we have

$$\| u_n \circ \Phi - u \circ \Phi \|_{L^p(K_1)}^p \leqslant \left\| J\Phi^{-1} \right\|_{L^\infty(\Omega_2)} \| u_n - u \|_{L^p(K_2)}^p \xrightarrow[n \to \infty]{} 0.$$

Similarly, for $j, k \in [\![1, d]\!]$,

$$\begin{aligned} &\| (\partial_j u_n \circ \Phi) \partial_k \Phi_j - (\partial_j u \circ \Phi) \partial_k \Phi_j \|_{L^p(K_1)} \\ &\qquad \leqslant \| \partial_k \Phi_j \|_{L^\infty(K_1)} \| (\partial_j u_n \circ \Phi) - (\partial_j u \circ \Phi) \|_{L^p(K_1)} \xrightarrow[n \to +\infty]{} 0. \end{aligned}$$

We take the limit $n \to +\infty$ in (2.13) and conclude when $p < +\infty$. The case $p = +\infty$ follows as in the proof of Proposition 2.35. □

We know that if $u \in C^1(\mathbb{R}^d)$ has a bounded differential then it is Lipschitz continuous with a Lipschitz constant given by the $L^\infty$ norm of the differential. In the following exercise we consider the case of $W^{1,\infty}$ fonctions.

**Proposition 2.37.** *Let $\Omega$ be an open and convex subset of $\mathbb{R}^d$. Let $u \in W^{1,\infty}(\Omega)$. Then $u$ is equal almost everywhere to a $\| \nabla u \|_{L^\infty(\mathbb{R}^d)}$-Lipschitz (and in particular continuous) function.*

***Exercise* 15.** Let $K$ be a compact and convex subset of $\Omega$. Let $\rho \in C_0^\infty(\mathbb{R}^d, \mathbb{R}_+)$ such that $\int_{\mathbb{R}^d} \rho \, dx = 1$. For $\varepsilon > 0$ and $x \in \mathbb{R}^d$ we set $\rho_\varepsilon(x) = \rho(x/\varepsilon)$.
**1.** Prove that for $\varepsilon > 0$ small enough the convolution $u_\varepsilon = \rho_\varepsilon * u$ is well defined on $K$.
**2.** Prove that there exists a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ going to 0 such that $u_n = u_{\varepsilon_n}$ goes to $u(x)$ as $n$ goes to $+\infty$ for almost all $x \in K$.
**3.** Prove that for all $n \in \mathbb{N}$ we have $\|\nabla u_n\|_{L^\infty(\mathbb{R}^d)} \leqslant \|\nabla u\|_{L^\infty(\mathbb{R}^d)}$.
**4.** Prove that for almost all $x, y \in K$ we have $|u(x) - u(y)| \leqslant \|\nabla u\|_{L^\infty(\mathbb{R}^d)} |x - y|$.
**5.** Prove Proposition 2.37.

We finish this paragraph with a converse of Proposition 2.12 about the differential quotient defined by (2.3).

**Proposition 2.38.** *For $u \in H^1(\mathbb{R}^d)$ and $h \in \mathbb{R}^d \setminus \{0\}$ we have*

$$\|D_h u\|_{L^2(\mathbb{R}^d)} \leqslant \|\nabla u\|_{L^2(\mathbb{R}^d)} .$$

***Exercise* 16.** Prove Proposition 2.38.

***Exercise* 17.** In this exercise we prove that for $u \in H^1(\mathbb{R}^d)$ (real valued) we have $|u| \in H^1(\mathbb{R}^d)$, $\nabla u = 0$ almost everywhere on $u^{-1}(\{0\})$ and $\nabla |u| = \mathsf{sign}(u)\nabla u$ on $u^{-1}(\mathbb{R}^d \setminus \{0\})$.
**1.** Let $G : \mathbb{R} \to \mathbb{R}$ be of class $C^1$, globally Lipschitz and such that $G(0) = 0$.
 a. Show that $G'$ is bounded on $\mathbb{R}$.
 b. Prove that $G \circ u \in H^1(\mathbb{R}^d)$ with $\nabla(G \circ u) = (G' \circ u)\nabla u$.
**2.** For $t \in \mathbb{R}$ we set

$$H_-(t) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x \leqslant 0, \end{cases} \quad \text{and} \quad H_+(t) = \begin{cases} 1 & \text{if } x \geqslant 0, \\ 0 & \text{if } x < 0. \end{cases}$$

For $n \in \mathbb{N}^*$ we set

$$H_n(t) = \begin{cases} 1 & \text{if } t \geqslant \frac{1}{n}, \\ nt & \text{if } 0 \leqslant t \leqslant \frac{1}{n}, \\ 0 & \text{if } t \leqslant 0. \end{cases}$$

Then we set $V_n(t) = \int_{-\infty}^t H_n(s) \, ds$.
 a. Prove that $(V_n \circ u) \in H^1(\mathbb{R}^d)$ with $\nabla(V_n \circ u) = (H_n \circ u)\nabla u$.
 b. For $t \in \mathbb{R}$ we set

$$g(t) = \begin{cases} t & \text{if } t > 0, \\ 0 & \text{if } t < 0. \end{cases}$$

Prove that $(g \circ u) \in H^1(\mathbb{R}^d)$ with $\nabla(g \circ u) = (H_- \circ u)\nabla u$.
 c. Prove that $\nabla(g \circ u) = (H_+ \circ u)\nabla u$.
 d. Deduce that $\nabla u = 0$ almost everywhere on $u^{-1}(\{0\})$.
**3.** Conclude.

## 2.4 Sobolev spaces on domains with boundary

In the previous section we have given some properties of the Sobolev spaces on $\mathbb{R}^d$, or local properties in general domains. In this section we look more carefully at the behavior of functions in Sobolev spaces at the boundary of the domain.

The model case will be the half space

$$\mathbb{R}^d_+ = \left\{ x = (x_1, \dots, x_d) \in \mathbb{R}^d \, : \, x_1 > 0 \right\}.$$

This is the simplest case because the boundary $\partial \mathbb{R}^d_+ = \{0\} \times \mathbb{R}^{d-1}$ is flat. Then, if the open subset $\Omega$ of $\mathbb{R}^d$ is sufficiently regular, the boundary $\partial\Omega$ can be locally straightened out and, with a partition of unity and a change of variables for each part, the problem on $\Omega$ is reduced to a problem far from the boundary (where we can apply the results on $\mathbb{R}^d$) and a finite number of problems on $\mathbb{R}^d_+$.

It is the purpose of this section to make these ideas clearer and to deduce some results for the Sobolev spaces on bounded subsets.

### 2.4.1 Regular domains

Let $k \in \mathbb{N}^* \cup \{\infty\}$. We recall that an open subset $\Omega$ of $\mathbb{R}^d$ is said to be of class $C^k$ if for any $w \in \partial\Omega$ there exist an orthonormal basis $\beta = (\beta_1, \dots, \beta_d)$ of $\mathbb{R}^d$, an open subset $\mathcal{O}$ of $\mathbb{R}^{d-1}$, $a, b \in \mathbb{R}$ with $a < b$ and an application $\varphi : \mathcal{O} \to ]a, b[$ of class $C^k$ such that $\mathcal{U}$ defined by

$$\mathcal{U} = \left\{ \sum_{j=1}^d x_j \beta_j, (x_2, \dots, x_d) \in \mathcal{O}, x_1 \in ]a, b[ \right\}$$

is a neighborhood of $w$ in $\mathbb{R}^d$ and

$$\Omega \cap \mathcal{U} = \left\{ \sum_{j=1}^d x_j \beta_j, x' = (x_2, \dots, x_d) \in \mathcal{O}, x_1 \in \left]\varphi(x'), b\right[ \right\}.$$

In particular, in $\mathcal{U}$ the boundary $\partial\Omega$ is the graph of $\varphi$ in the basis $\beta$. We can always construct the basis $\beta$ with the vectors of the canonical basis $(e_1, \dots, e_d)$, possibly in a different order. For $x' = (x_2, \dots, x_d) \in \mathcal{O}$ we set

$$\tilde{\varphi}(x') = \varphi(x')\beta_1 + \sum_{j=2}^d x_j \beta_j.$$

Then $\partial\Omega \cap \mathcal{U}$ is also the image of $\mathcal{O}$ by $\tilde{\varphi}$.

Given $w \in \partial\Omega \cap \mathcal{V}$ and $x' = (x_2, \dots, x_d) \in \mathcal{O}$ such that $w = \tilde{\phi}(x')$, the outward normal derivative to $\Omega$ at point $w$ is defined by

$$\nu(w) = \frac{-\beta_1 + \sum_{j=2}^d \partial_j \varphi(x') \beta_j}{\sqrt{1 + |\nabla \varphi(x')|^2}}. \tag{2.14}$$

The is the only vector such that $\nu(w) \perp T_w(\partial\Omega)$, $|\nu(w)| = 1$ and, for some $t_0 > 0$,

$$w + t\nu(w) \begin{cases} \in \Omega, & \forall t \in ]-t_0, 0[, \\ \notin \Omega, & \forall t \in [0, t_0[. \end{cases}$$

We define on $\partial\Omega$ the topology and the corresponding Borel $\sigma$-algebra inherited from the usual structure on $\mathbb{R}^d$. We define the Lebesgue measure of a Borel set $B \in \partial\Omega \cap \mathcal{U}$ as follows:

$$\sigma(B) = \int_{\mathcal{O}} \mathbb{1}_B(\tilde{\varphi}(x)) \sqrt{1 + |\nabla \varphi(x')|^2} \, dx'.$$

Thus, if $f$ is an integrable function on $\partial\Omega \cap \mathcal{U}$ we have

$$\int_{\partial\Omega\cap\mathcal{U}} f\,d\sigma = \int_{\mathcal{O}} f(\tilde{\varphi}(x))\sqrt{1 + |\nabla\varphi(x')|^2}\,dx'.$$

Then we can define Lebesgue spaces on $\partial\Omega$ as on any measure space.

For $x = \sum_{j=1}^d x_j\beta_j \in \mathcal{U}$ we set

$$\Phi(x) = \big(x_1 - \varphi(x_2, \dots, x_d)\big)e_1 + \sum_{j=2}^d x_j e_j.$$

Then $\Phi$ is of classe $C^k$ and it is injective. So it defines a bijection on its image denoted by $\mathcal{W}$. Then $\mathcal{W}$ is open in $\mathbb{R}^d$ and the inverse $\Phi^{-1}$ of $\Phi$ is of class $C^k$ on $\mathcal{W}$ ($\Phi$ defines a diffeomorphism of class $C^k$ from $\mathcal{U}$ to $\mathcal{W}$). Moreover we have

$$\Phi(\mathcal{U} \cap \Omega) = \mathcal{W} \cap \mathbb{R}^d_+.$$

Notice also that for $x' \in \mathcal{O}$ we have $\Phi(\tilde{\varphi}(x')) = (0, x')$, and then $\mathcal{W}_j \cap \partial\mathbb{R}^d_+ = \{0\} \times \mathcal{O}$.

The interest of this change of variables is to transform a function supported in $\Omega \cap \mathcal{U}$ to a function on $\mathbb{R}^d_+$, where the properties of Sobolev spaces are easier.

Notice that if $\Omega$ is bounded then its boundary $\partial\Omega$ is compact. This is not necessary but it will simplify the discussion (an unbounded open subset can also have a compact boundary, but we will not consider this situation here).

Now let $\Omega$ be a bounded open subset of $\mathbb{R}^d$ of class $C^k$ for some $k \geqslant 1$. There exist $N \in \mathbb{N}^*$, open subsets $\mathcal{U}_1, \dots, \mathcal{U}_N, \mathcal{W}_1, \dots, \mathcal{W}_N$ of $\mathbb{R}^d$ and diffeomorphisms $\Phi_j : \mathcal{U}_j \to \mathcal{W}_j$ of class $C^k$ such that $\partial\Omega \subset \bigcup_{j=1}^N \mathcal{U}_j$ and for all $j \in [\![1, N]\!]$ we have $\Phi_j(\Omega \cap \mathcal{U}_j) = \mathbb{R}^d_+ \cap \mathcal{W}_j$.

If we set $\Omega = \mathcal{U}_0$ then $\bigcup_{j=0}^N \mathcal{U}_j$ is an open cover of $\overline{\Omega}$. We consider a corresponding partition of unity $(\chi_j)_{0\leqslant j\leqslant N}$ ($\chi_j \in C_0^\infty(\mathbb{R}^d, [0,1])$ is supported in $\mathcal{U}_j$ for all $j \in [\![0, N]\!]$ and $\sum_{j=0}^N \chi_j = 1$ on $\overline{\Omega}$).

For $u \in W^{1,p}(\Omega)$ we set $u_j = \chi_j u$ for all $j \in [\![0, N]\!]$. Then $u = \sum_{j=0}^N u_j$, $u_j \in W^{1,p}(\Omega)$ for all $j \in [\![0, N]\!]$, $u_0$ is supported in a compact subset of $\Omega$, and $u_j$ is supported in a compact subset of $\overline{\Omega} \cap \mathcal{U}_j$ for all $j \in [\![1, N]\!]$. In particular, the extension of $u_0$ by 0 on $\mathbb{R}^d$ is in $W^{1,p}(\mathbb{R}^d)$, and $(u_j \circ \Phi^{-1})$ belongs to $W^{1,p}(\mathbb{R}^d_+ \cap \mathcal{W}_j)$ (and can be extended by 0 to a function in $W^{1,p}(\mathbb{R}^d_+)$) for all $j \in [\![1, N]\!]$.

We will use this setting to prove results for Sobolev spaces on $\Omega$.

### 2.4.2 Approximation by smooth functions (continued)

We recall that given an open subset $\Omega$ of $\mathbb{R}^d$, we denote by $C_0^\infty(\overline{\Omega})$ the set of restrictions to $\Omega$ of functions in $C_0^\infty(\mathbb{R}^d)$.

**Theorem 2.39.** *Let $p \in [1, +\infty[$ and $k \in \mathbb{N}$. Let $\Omega$ be equal to $\mathbb{R}^d_+$ or be a bounded open subset of $\mathbb{R}^d$. Let $u \in W^{k,p}(\mathbb{R}^d_+)$. There exists a sequence $(u_n)_{n\in\mathbb{N}}$ of functions in $C_0^\infty(\overline{\Omega})$ such that*

$$\|u_n - u\|_{W^{k,p}(\Omega)} \xrightarrow[n\to+\infty]{} 0.$$

*Proof.* We prove the case $\Omega$ bounded. The case $\Omega = \mathbb{R}^d_+$ is more direct and is left as an exercice. Let $w \in \partial\Omega$. We use the notation of Paragraph 2.4.1. Let $u \in W^{1,p}(\Omega)$ be supported in $\overline{\Omega} \cap \mathcal{U}$. We denote by $\tilde{u}$ the extension of $u$ by 0 on $\mathbb{R}^d$. For $\tau > 0$ we set

$$\mathcal{U}_\tau = \left\{ \sum_{j=1}^d x_j \beta_j, (x_2, \dots, x_d) \in \mathcal{O}, x_1 \in ]a, b - \tau[ \right\}.$$

There exists $\tau_0 > 0$ such that $\operatorname{supp}(u) \subset \mathcal{U}_{\tau_0}$. For $\tau \in ]0, \tau_0]$ and $x \in \mathcal{U}_\tau$ we set

$$u_\tau(x) = u(x + \tau\beta_1).$$

We extend $u_\tau$ by 0 on $\mathcal{U} \backslash \mathcal{U}_\tau$. The restriction of $u_\tau$ to $\mathcal{U} \cap \Omega$ is in $W^{k,p}(\mathcal{U} \cap \Omega)$ and the derivatives of $u_\tau$ up to order $k$ are the translations of the corresponding derivatives of $u$. More precisely, for $|\alpha| \leqslant k$ we denote by $v^\alpha$ the extension of $\partial^\alpha u$ by 0 outside $\Omega \cap \mathcal{U}$. Then $\partial^\alpha u_\tau$ coincides with $v^\alpha_\tau$ on $\Omega \cap \mathcal{U}$. Then by continuity in $L^p(\mathbb{R}^d)$ of the translation we have

$$\|u_\tau - u\|^p_{W^{k,p}(\mathcal{U}\cap\Omega)} = \sum_{|\alpha| \leqslant k} \|\partial^\alpha u_\tau - \partial^\alpha u\|^p_{L^p(\mathcal{U}\cap\Omega)} \leqslant \sum_{|\alpha| \leqslant k} \|v^\alpha_\tau - v^\alpha\|^p_{L^p(\mathbb{R}^d)} \xrightarrow[\tau \to 0]{} 0.$$

Now let $\tau \in ]0, \tau_0]$ be fixed. There exists $\eta_0 > 0$ if we set

$$\mathcal{V} = \bigcup_{x \in \operatorname{supp}(u_\tau) \cap \Omega} B(x, \eta_0),$$

then for all $y \in \mathcal{V}$ we have $y + \tau\beta_1 \in \mathcal{U} \cap \Omega$. Let $\rho \in C^\infty_0(\mathbb{R}^d, [0, 1])$ be supported in $B(0, 1)$ and such that $\int_{\mathbb{R}^d} \rho = 1$. For $\eta \in ]0, \eta_0]$ and $x \in \mathbb{R}^d$ we set $\rho_\eta(x) = \eta^{-d}\rho(x/\eta)$. For $\eta \in ]0, \eta_0]$ we set $u^\eta_\tau = \rho_\eta * u_\tau$. Its restriction to $\mathcal{U} \cap \Omega$ belongs to $C^\infty_0(\overline{\Omega})$. Since $u_\tau \in W^{k,p}(\mathcal{V})$ we can prove as in the proof of Theorem 2.31 that

$$\|u^\eta_\tau - u_\tau\|_{W^{k,p}(\mathcal{U}\cap\Omega)} \xrightarrow[\eta \to 0]{} 0.$$

It remains to chose $\tau > 0$ small enough and then $\eta > 0$ small enough to conclude. □

***Exercise* 18.** Let $u \in H^1(\mathbb{R}^d_+)$. Prove that for $j \in [\![2, d]\!]$ and $t \neq 0$ we have

$$\left\|D_{te_j} u\right\|_{L^2} \leqslant \|\nabla u\|_{L^2(\Omega)}.$$

### 2.4.3 Extension

We continue with a result of extension. In order to deduce results in $W^{1,p}(\Omega)$ from results on $W^{1,p}(\mathbb{R}^d)$ it is natural to extend functions in $W^{1,p}(\Omega)$ to functions in $W^{1,p}(\mathbb{R}^d)$ (notice that in the proof of Theorem 2.33 we were able to prove results on $W^{k,p}(\omega)$ for $\omega \subset\subset \Omega$ precisely because we had a function with a nice behavior on a bigger domain).

It is clear, at least in dimension 1, than extending functions by 0 outside $\Omega$ does not always give a function in $W^{1,p}(\mathbb{R}^d)$. However, we have seen in Exercise 2 that in dimension 1 we can indeed extend a function in $H^1(\mathbb{R}^*_+)$ to a function in $H^1(\mathbb{R})$. We generalize this observation to the case of a function in $W^{1,p}(\mathbb{R}^d_+)$ and then, by the argument described above, to the case of a function in $W^{1,p}(\Omega)$ for a regular bounded open subset $\Omega$.

**Proposition 2.40.** *Let $p \in [1,+\infty]$. For $u \in L^p(\mathbb{R}^d_+)$ and $x = (x_1, \dots, x_d) = (x_1, x') \in \mathbb{R}^d$ we set*

$$(Pu)(x) = \begin{cases} u(x_1, x') & \text{if } x_1 > 0, \\ u(-x_1, x') & \text{if } x_1 < 0. \end{cases}$$

*Then $Pu \in L^p(\mathbb{R}^d)$ and $\|Pu\|_{L^p(\mathbb{R}^d)} = 2^{\frac{1}{p}} \|u\|_{L^p(\mathbb{R}^d_+)}$. For $u \in W^{1,p}(\mathbb{R}^d_+)$ we have $Pu \in W^{1,p}(\mathbb{R}^d)$ with*

$$\partial_1(Pu) = \tilde{P}(\partial_1 u) \quad \text{and} \quad \partial_j(Pu) = P(\partial_j u), \quad 2 \leqslant j \leqslant d,$$

*where*

$$(\tilde{P}v)(x) = \begin{cases} v(x_1, x') & \text{if } x_1 > 0, \\ -v(-x_1, x') & \text{if } x_1 < 0. \end{cases}$$

*In particular, $P$ defines a continuous extension from $W^{1,p}(\mathbb{R}^d_+)$ to $W^{1,p}(\mathbb{R}^d)$.*

*Proof.* • We set $\mathbb{R}^d_- = \mathbb{R}^d \setminus \overline{\mathbb{R}^d_+}$. It is easy to see that $\|Pu_{|\mathbb{R}^d_-}\|^p_{L^p(\mathbb{R}^d_-)} = \|u\|^p_{L^p(\mathbb{R}^d_+)}$ if $p < +\infty$, so $Pu \in L^p(\mathbb{R}^d)$ with $\|Pu\|^p_{L^p(\mathbb{R}^d)} = 2\|u\|^p_{L^p(\mathbb{R}^d_+)}$. If $p = +\infty$ we have $\|Pu\|_{L^\infty(\mathbb{R}^d)} = \|u\|_{L^\infty(\mathbb{R}^d_+)}$.

• For $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ we set $\sigma(x) = (-x_1, x_2, \dots, x_d)$. Let $j \in [\![2, d]\!]$. Let $\phi \in C_0^\infty(\mathbb{R}^d)$. If $\phi$ is supported in $\mathbb{R}^d_+$ we have

$$-\int_{\mathbb{R}^d} Pu\, \partial_j \phi \, dx = -\int_{\mathbb{R}^d_+} u\, \partial_j \phi \, dx = \int_{\mathbb{R}^d_+} \partial_j u\, \phi \, dx = \int_{\mathbb{R}^d} P(\partial_j u)\, \phi \, dx.$$

If $\phi$ is supported in $\mathbb{R}^d_-$ then, similarly,

$$\begin{aligned} -\int_{\mathbb{R}^d} Pu\, \partial_j \phi \, dx &= -\int_{\mathbb{R}^d_-} (u \circ \sigma)\, \partial_j \phi \, dx = -\int_{\mathbb{R}^d_+} u\, (\partial_j \phi \circ \sigma) \, dx \\ &= -\int_{\mathbb{R}^d_+} u\, \partial_j (\phi \circ \sigma) \, dx = \int_{\mathbb{R}^d_+} \partial_j u\, (\phi \circ \sigma) \, dx = \int_{\mathbb{R}^d_-} ((\partial_j u \circ \sigma) \phi \, dx \\ &= \int_{\mathbb{R}^d} P(\partial_j u)\, \phi \, dx. \end{aligned}$$

We consider the general case. Let $\chi \in C_0^\infty(\mathbb{R}, [0,1])$ be even, equal to 1 on [-1,1] and supported in ]-2,2[. For $n \in \mathbb{N}$ and $x \in \mathbb{R}^d$ we set $\chi_n(x) = \chi(nx_1)$. Since $(1 - \chi_n)\phi$ is supported outside $\partial \mathbb{R}^d_+$ we have

$$-\int_{\mathbb{R}^d} Pu(1 - \chi_n) \partial_j \phi \, dx = -\int_{\mathbb{R}^d} Pu \partial_j \big((1 - \chi_n)\phi\big) \, dx = \int_{\mathbb{R}^d_+} P(\partial_j u)(1 - \chi_n)\phi \, dx.$$

By the dominated convergence theorem this yields

$$-\int_{\mathbb{R}^d} Pu\, \partial_j \phi \, dx = \int_{\mathbb{R}^d} P(\partial_j u) \phi \, dx.$$

This proves that in the weak sense we have $\partial_j(Pu) = P(\partial_j u)$. In particular $\partial_j(Pu) \in L^p(\mathbb{R}^d)$ with $\|\partial_j(Pu)\|_{L^p(\mathbb{R}^d)} = 2^{\frac{1}{p}} \|\partial_j u\|_{L^p(\mathbb{R}^d_+)}$.

- We proceed similarly for the first partial derivative. We observe that for $\phi \in C_0^\infty(\mathbb{R}^d)$ we now have $\partial_1(\phi \circ \sigma) = -(\partial_1 \phi) \circ \sigma$, so if $\phi$ is supported outside $\partial \mathbb{R}^d_+$ we now have

$$-\int_{\mathbb{R}^d} Pu \, \partial_1 \phi \, dx = \int_{\mathbb{R}^d} \tilde{P}(\partial_1 u) \phi \, dx.$$

On the other hand $(1 - \chi_n)$ does not commute with the partial derivative $\partial_1$. But the additional term is estimated as follows. Let $R > 0$ be such that $\phi$ is supported in $\mathbb{R} \times B_{d-1}(0, R)$ ($B_{d-1}(0, R)$ is the ball of radius $R$ in $\mathbb{R}^{d-1}$). Since $\chi$ is even we have

$$\begin{aligned}
\left| \int_{\mathbb{R}^d} Pu \, \partial_1 \chi_n \, \phi \, dx \right| &= \left| \int_{\mathbb{R}^d_+} u \, \partial_1 \chi_n \, (\phi - \phi \circ \sigma) \, dx \right| \\
&\leqslant n \left\| \chi' \right\|_\infty \int_{x_1=0}^{\frac{2}{n}} \int_{x' \in B_{d-1}(0,R)} |u(x_1, x')| \, |\phi(x_1, x') - \phi(-x_1, x')| \, dx' \, dx_1 \\
&\leqslant 4 \left\| \chi' \right\|_\infty \left\| \partial_1 \phi \right\|_\infty \int_{x_1=0}^{\frac{2}{n}} \int_{x' \in B_{d-1}(0,R)} |u(x_1, x')| \, dx' \, dx_1 \\
&\xrightarrow[n \to +\infty]{} 0.
\end{aligned}$$

The conclusion follows as above. □

**Theorem 2.41.** *Let $\Omega$ be an open bounded subset of class $C^1$ in $\mathbb{R}^d$. Let $p \in [1, +\infty]$. Let $\mathcal{O}$ be an open subset of $\mathbb{R}^d$ such that $\overline{\Omega} \subset \mathcal{O}$. Then there exists a bounded linear operator $P : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^d)$ (which is also bounded for the norm of $L^p(\Omega)$) such that $Pu$ is supported in $\mathcal{O}$ and $(Pu)_{|\Omega} = u$ for all $u \in W^{1,p}(\Omega)$.*

*Proof.* Let $u \in W^{1,p}(\Omega)$. We use the notation introduced in Paragraph 2.4.1. Without loss of generality we can assume that $\mathcal{U}_j \subset \mathcal{O}$ and $\mathcal{W}_j$ is symmetric with respect to $\partial \mathbb{R}^d_+$ for all $j \in [\![1, N]\!]$ (for instance $\mathcal{W}_j$ is a ball centered on $\partial \mathbb{R}^d_+$). We denote by $v_0$ the extension of $u_0$ by 0 on $\mathbb{R}^d$. We have $\|v_0\|_{W^{1,p}(\mathbb{R}^d)} = \|u_0\|_{W^{1,p}(\Omega)}$. Let $j \in [\![1, N]\!]$. We denote by $\tilde{v}_j$ the extension of $u_j \circ \Phi_j^{-1}$ on $\mathcal{W}_j$ given by Proposition 2.40. It is supported in a compact subset of $\mathcal{W}_j$, and $\tilde{v}_j \circ \Phi_j$ is compactly supported in $\mathcal{U}_j$. Then we denote by $v_j$ the extension by 0 of $\tilde{v}_j \circ \Phi_j$ on $\mathbb{R}^d$. By Propositions 2.36 and 2.40 and Remark 2.23 there exist constants $C_\Phi, C_{\Phi^{-1}}, C_P, C_{\chi_j} > 0$ independant of $u$ such that

$$\begin{aligned}
\|v_j\|_{W^{1,p}(\mathbb{R}^d)} = \|\tilde{v}_j \circ \Phi_j\|_{W^{1,p}(\mathcal{U}_j)} &\leqslant C_\Phi \|\tilde{v}_j\|_{W^{1,p}(\mathcal{W}_j)} \leqslant C_P C_\Phi \left\| u_j \circ \Phi^{-1} \right\|_{W^{1,p}(\mathcal{W}_j \cap \mathbb{R}^d_+)} \\
&\leqslant C_P C_\Phi C_{\Phi^{-1}} \|u_j\|_{W^{1,p}(\mathcal{U}_j \cap \Omega)} \leqslant C_P C_\Phi C_{\Phi^{-1}} C_{\chi_j} \|u\|_{W^{1,p}(\Omega)}.
\end{aligned}$$

Finally we set $Pu = \sum_{j=0}^N v_j$, and $Pu \in W^{1,p}(\mathbb{R}^d)$ satisfies all the required properties. □

***Exercise* 19.** Use Theorems 2.31 and 2.41 to give a new proof of Theorem 2.39.

## 2.5 Sobolev Embeddings

In this section we prove some inclusions between Sobolev spaces. The inclusions between Lebesgue spaces are already known. In particular we know that $L^p(\mathbb{R}^d)$ is never included in $L^q(\mathbb{R}^d)$ if $p \neq q$. The purpose here is to prove that if we add information about the derivatives then we get better results. In particular we will prove (continuous) inclusions of the form $W^{1,p}(\Omega) \subset L^q(\Omega)$ for suitable pairs $(p, q)$ or of the form $W^{k,p}(\Omega) \subset C^n$. In this case, this means that we can recover regularity in the usual sense from existence of weak derivatives.

As for Lebesgue spaces, we will get stronger results on a bounded domain $\Omega$. In this case we will prove compact inclusions. For instance, $H^1(\Omega)$ is compactly embedded in $L^2(\Omega)$. This means that if a sequence of functions in $H^1(\Omega)$ is bounded, then it has a convergent subsequence for the $L^2(\Omega)$ norm. This result will be of great importance for the analysis of PDEs. We will already use this fact in the following section (see the proof of Theorem 2.57).

### 2.5.1 Some basic results

We begin with a result in dimension 1. We have already said in Proposition 2.7 that the primitive of a function in $L^p(I)$ is continuous. We can deduce that a function in $W^{1,p}(I)$ is continuous. We can actually say slightly more.

**Proposition 2.42.** *Let $I$ be an interval of $\mathbb{R}$. Let $p \in [1, +\infty]$ and $u \in W^{1,p}(I)$. Then $u$ is equal almost everywhere to a function $\tilde{u}$ on $I$ such that, for $x, y \in I$,*

$$\tilde{u}(y) - \tilde{u}(x) = \int_x^y u'(s)\, ds.$$

*In particular $\tilde{u}$ is continuous. If $p > 1$ then $\tilde{u}$ is even $\frac{p-1}{p}$-Hölder continuous on $I$ (when $p = +\infty$ this means that $\tilde{u}$ is Lipschitz continuous). Moreover, if $I$ is not bounded and if $p \in ]1, +\infty[$ then $\tilde{u}$ goes to 0 at infinity. Finally, for any $p \in [1, +\infty]$, $\tilde{u}$ is bounded and hence $u \in L^\infty(I)$.*

*Proof.* We fix $x_0 \in I$. For $x \in I$ we set

$$v(x) = \int_{x_0}^x u'(s)\, ds.$$

This makes sense since $u' \in L^p(I) \subset L^1_{\text{loc}}(I)$. Then, by Proposition 2.7, $v$ is continuous and its derivative in the sense of distributions is $u'$. By Proposition 2.8, there exists a constant $\alpha$ such that $u - v = \alpha$ almost everywhere. We set $\tilde{u} = v + \alpha$.

For $x, y \in I$ we have

$$\tilde{u}(y) - \tilde{u}(x) = v(y) - v(x) = \int_x^y u'(s)\, ds.$$

If $p = 1$ then for some $x_0 \in I$ we have $|\tilde{u}(y)| \leqslant |\tilde{u}(x_0)| + \|u'\|_{L^1(I)}$.

If $p = +\infty$ then $|\tilde{u}(y) - \tilde{u}(x)| \leqslant |y - x| \, \|u'\|_{L^\infty(I)}$, so $\tilde{u}$ is $\|u'\|_{L^\infty(I)}$-Lipschitz continuous. If $p \in ]1, +\infty[$, we have by the Hölder inequality

$$|\tilde{u}(y) - \tilde{u}(x)| \leqslant \left| \int_x^y |u'(s)| \, ds \right| \leqslant |y - x|^{\frac{p-1}{p}} \left( \int_I |u'(s)|^p \, ds \right)^{\frac{1}{p}}.$$

This proves that $\tilde{u}$ is $\frac{p-1}{p}$-Hölder continuous, and in particular uniformly continuous. All the statements of the proposition follow. $\square$

**Corollary 2.43.** *Let $I$ be an interval of $\mathbb{R}$, $p \in [1,+\infty]$ and $k \in \mathbb{N}^*$. Let $u \in W^{k,p}(I)$. Then $u \in C^{k-1}(I)$.*

This proves that if we have enough weak regularity, then we can recover some weak regularity. This kind of results will actually depend on the dimension. It is not true in dimension $d \geqslant 2$ that a function in $W^{1,p}(\Omega)$ is continuous. For instance, if $1 \leqslant p < d$ then for $\alpha \in \left]-\frac{d}{p}+1, 0\right[$ the function $x \mapsto |x|^\alpha$ belongs to $W^{1,p}(B(0,1))$ but not to $L^\infty(B(0,1))$.

In any dimension we have the following result on $\mathbb{R}^d$, based on the Fourier point of view (see Section 2.2.3).

**Proposition 2.44.** *Let $k > \frac{d}{2}$ and $u \in H^k(\mathbb{R}^d)$. Then $u$ is continous and goes to 0 at infinity (in the sense that $u$ has a representative which satisfies these properties). In particular it is bounded. More generaly, if $k > n + \frac{d}{2}$ for some $n \in \mathbb{N}$ then $u$ is of class $C^n$.*

*Proof.* By the Cauchy-Schwarz inequality we have

$$\int_{\mathbb{R}^d} |\hat{u}(\xi)|\, d\xi \leqslant \left(\int_{\mathbb{R}^d} \left(1+|\xi|^2\right)^{-k} d\xi\right)^{\frac{1}{2}} \left(\left(1+|\xi|^2\right)^k |\hat{u}(\xi)|^2\, d\xi\right)^{\frac{1}{2}} < +\infty,$$

so $\hat{u} \in L^1(\mathbb{R}^d)$. By inverse Fourier transfor, this implies that $u$ is continuous and goes to 0 at infinity. If $k > n + \frac{d}{2}$ then for all $\alpha \in \mathbb{N}^d$ with $|\alpha| \leqslant n$ we have $\partial^\alpha u \in H^{k-n}(\mathbb{R}^d)$, so $\partial^\alpha u$ is a continuous function. This implies that $u$ is of class $C^n$. □

### 2.5.2 Morrey's inequality

We have seen int the previous paragraph that from weak regularity we can recover differentiability in the usual sense. But Proposition 2.42 only holds in dimension 1 and Proposition 2.44 is only valid in $\mathbb{R}^d$ and for $p = 2$. In this paragraph we prove a more general result.

We recall that for $n \in \mathbb{N}$ and $\theta \in ]0,1]$ we denote by $C^{n,\theta}(\Omega)$ the space of functions of class $C^n$ with bounded derivatives and such that the $n$-th derivatives are Hölder continuous with exponent $\theta$ on $\Omega$. It is endowed with the norm defined by

$$\|u\|_{C^{n,\theta}(\Omega)} = \sum_{|\alpha|\leqslant n} \|\partial^\alpha u\|_{L^\infty(\Omega)} + \sum_{|\alpha|=n} \sup_{x\neq y\in\Omega} \frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|}{|x-y|^\theta}.$$

**Theorem 2.45** (Morrey's inequality)**.** *Let $p \in ]d,+\infty]$. Modulo the choice of a continuous representative we have*

$$W^{1,p}(\mathbb{R}^d) \subset C^{0,1-\frac{d}{p}}(\mathbb{R}^d)$$

*with continuous injection. More precisely, there exists $C > 0$ such that, for $u \in W^{1,p}(\mathbb{R}^d)$,*

$$\|u\|_{L^\infty(\mathbb{R}^d)} \leqslant C \|u\|_{W^{1,p}(\mathbb{R}^d)},$$

*and for almost all $x_1, x_2 \in \mathbb{R}^d$,*

$$|u(x_1) - u(x_2)| \leqslant C |x_1 - x_2|^{1-\frac{d}{p}} \|\nabla u\|_{L^p(\mathbb{R}^d)}.$$

In dimension 1, we have used the fundamental theorem of calculus to compare $u(x)$ to $u(x_0)$ for some fixed $x_0$. It gave a one-dimensional integral which was controled by the norm of $u'$. In higher dimension we can still write the fundamental theorem of calculus for regular functions but the corresponding one-dimensional integral is not controled by the $d$-dimensional integral which defines the norm of $\nabla u$. The trick in the following proof is to compare $u(x)$ to the mean value of $u$ on an open subset of $\mathbb{R}^d$. This will give a $d$-dimensional integral controled as stated in the theorem.

*Proof.* • The case $p = +\infty$ follows from Proposition 2.37. We consider the case $p \in ]d, +\infty[$.
• We consider $u \in C_0^\infty(\mathbb{R}^d)$. The general case will follow by density. Let $x \in \mathbb{R}^d$ and let $\mathcal{O}$ be an open subset of $\mathbb{R}^d$. We set

$$\delta(x, \mathcal{O}) = \sup_{y \in \mathcal{O}} |y - x| .$$

For $y \in \mathcal{O}$ and $h = (h_1, \dots, h_d) = y - x$ we have

$$\begin{aligned} |u(y) - u(x)| &\leqslant \int_0^1 \left| \frac{d}{dt} u(x + th) \right| dt \\ &\leqslant \int_0^1 \sum_{j=1}^d |h_j| \, |\partial_j u(x + th)| \, dt \\ &\leqslant \delta(x, \mathcal{O}) \sum_{j=1}^d \int_0^1 |\partial_j u(x + th)| \, dt. \end{aligned}$$

For $t \in ]0, 1]$ we set

$$t(\mathcal{O} - x) = \{ t(y - x), y \in \mathcal{O} \} .$$

If we set

$$u_{\mathcal{O}} = \frac{1}{|\mathcal{O}|} \int_{\mathcal{O}} u(y) \, dy,$$

then we have

$$\begin{aligned} |u(x) - u_{\mathcal{O}}| &\leqslant \frac{1}{|\mathcal{O}|} \int_{y \in \mathcal{O}} |u(x) - u(y)| \, dy \\ &\leqslant \frac{\delta(x, \mathcal{O})}{|\mathcal{O}|} \int_{h \in (\mathcal{O} - x)} \int_0^1 \sum_{j=1}^d |\partial_j u(x + th)| \, dt \, dh \\ &\leqslant \frac{\delta(x, \mathcal{O})}{|\mathcal{O}|} \int_0^1 \frac{1}{t^d} \sum_{j=1}^d \int_{\eta \in t(\mathcal{O} - x)} |\partial_j u(x + \eta)| \, d\eta \, dt. \end{aligned}$$

By the Hölder inequality we have for $t \in [0, 1]$

$$\begin{aligned} \sum_{j=1}^d \int_{t(\mathcal{O} - x)} |\partial_j u(x + \eta)| \, d\eta &\leqslant \sum_{j=1}^d \left( \int_{t(\mathcal{O} - x)} |\partial_j u(x_1 + \eta)|^p \, d\eta \right)^{\frac{1}{p}} |t(\mathcal{O} - x)|^{\frac{p-1}{p}} \\ &\leqslant t^{\frac{d(p-1)}{p}} \, |\mathcal{O}|^{\frac{p-1}{p}} \, \|\nabla u\|_{L^p(\mathbb{R}^d)} , \end{aligned}$$

so

$$|u(x) - u_{\mathcal{O}}| \leqslant \delta(x, \mathcal{O}) \, |\mathcal{O}|^{-\frac{1}{p}} \, \|\nabla u\|_{L^p(\mathbb{R}^d)} \int_0^1 t^{-\frac{d}{p}} \, dt = \frac{\delta(x, \mathcal{O}) \, |\mathcal{O}|^{-\frac{1}{p}} \, \|\nabla u\|_{L^p(\mathbb{R}^d)}}{1 - \frac{d}{p}}. \tag{2.15}$$

- Now let $x_1, x_2 \in \mathbb{R}^d$ and let $\mathcal{O}$ be the open ball with diameter $[x_1, x_2]$. We have $\delta(x_1, \mathcal{O}) = \delta(x_2, \mathcal{O}) = |x_1 - x_2|$ and $|\mathcal{O}| = \frac{c_d}{2^d} |x_1 - x_2|^d$ where $c_d$ is the size of the unit ball in $\mathbb{R}^d$. Thus

$$|u(x_1) - u(x_2)| \leqslant |u(x_1) - u_{\mathcal{O}}| + |u(x_2) - u_{\mathcal{O}}| \leqslant \frac{2^{1+\frac{d}{p}} c_d^{-\frac{1}{p}}}{1 - \frac{d}{p}} |x_1 - x_2|^{1-\frac{d}{p}} \|\nabla u\|_{L^p(\mathbb{R}^d)}.$$

This gives the second statement. Now for $x \in \mathbb{R}^d$ we apply (2.15) with $\mathcal{O} = B(x, 1)$, the ball of center $x$ and radius 1. The Hölder inequality gives

$$|u_{\mathcal{O}}| \leqslant c_d^{-\frac{1}{p}} \|u\|_{L^p(\mathbb{R}^d)},$$

so

$$|u(x)| \leqslant c_d^{-\frac{1}{p}} \left( \|u\|_{L^p(\mathbb{R}^d)} + \frac{1}{1 - \frac{d}{p}} \|\nabla u\|_{L^p(\mathbb{R}^d)} \right).$$

This completes the proof. □

***Exercise* 20.** Find $u \in W^{1,d}(\mathbb{R}^d)$ such that $u \notin L^\infty(\mathbb{R}^d)$.

### 2.5.3 Gagliardo-Nirenberg Inequality

In this paragraph we consider the case $p \leqslant d$. This is particularly interesting for the common case $p = 2$. We have seen that in this case a function in $W^{1,p}(\mathbb{R}^d)$ is not necessarily continuous or bounded. The purpose of the next result is to show that a function in $W^{1,p}(\mathbb{R}^d)$ is now in $L^q(\mathbb{R}^d)$ for some suitable exponent $q$. This kind of results is also of crucial importance in applications.

Assume that there exists $q \in [1, +\infty[$ and $C > 0$ such that

$$\forall v \in C_0^\infty(\mathbb{R}^d), \quad \|v\|_{L^q(\mathbb{R}^d)} \leqslant C \|\nabla v\|_{L^p(\mathbb{R}^d)}. \tag{2.16}$$

Let $u \in C_0^\infty(\mathbb{R}^d) \setminus \{0\}$. For $\lambda > 0$ and $x \in \mathbb{R}^d$ we set $u_\lambda(x) = u(\lambda x)$. Then for all $\lambda > 0$ we have

$$\lambda^{-\frac{d}{q}} \|u\|_{L^q(\mathbb{R}^d)} = \|u_\lambda\|_{L^q(\mathbb{R}^d)} \leqslant C \|\nabla u_\lambda\|_{L^p(\mathbb{R}^d)} = C\lambda^{1-\frac{d}{p}} \|\nabla u\|_{L^p(\mathbb{R}^d)}.$$

Letting $\lambda$ go to 0 or to $+\infty$ we see that we necessarily have

$$-\frac{d}{q} = 1 - \frac{d}{p}. \tag{2.17}$$

In the following theorem we prove that if (2.17) holds then we indeed have (2.16). For $p \in [1, d[$ we define $p^* \in [1, +\infty[$ by

$$p^* = \frac{pd}{d-p}, \quad \frac{1}{p^*} = \frac{1}{p} - \frac{1}{d}. \tag{2.18}$$

Notice that we have $p^* > p$ and $p^* \to +\infty$ if $p \to d$.

**Theorem 2.46** (Gagliardo-Nirenberg-Sobolev inequality)**.** *Let $p \in [1, d[$ and let $p^*$ be defined by (2.18). There exists $C > 0$ such that for all $u \in C_0^1(\mathbb{R}^d)$ we have*

$$\|u\|_{L^{p^*}(\mathbb{R}^d)} \leqslant C \|\nabla u\|_{L^p(\mathbb{R}^d)}.$$

*Proof.* • Let $u \in C_0^1(\mathbb{R}^d)$. For $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ and $j \in [\![1, d]\!]$ we have

$$|u(x)| = \left| \int_{-\infty}^{x_j} \partial_j u(x_1, \dots, x_{j-1}, t, x_{j+1}, \dots, x_d)\, dt \right| \leqslant v_j(\tilde{x}_j)^{d-1}$$

where $\tilde{x}_j = (x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_d)$ and

$$v_j(\tilde{x}_j) = \left( \int_{\mathbb{R}} |\nabla u(x_1, \dots, x_{j-1}, t, x_{j+1}, \dots, x_d)|\, dt \right)^{\frac{1}{d-1}}.$$

This gives

$$|u(x)|^{\frac{d}{d-1}} \leqslant \prod_{j=1}^{d} v_j(\tilde{x}_j).$$

Now we prove by induction on $d \geqslant 2$ that if we set

$$v : x \in \mathbb{R}^d \mapsto \prod_{j=1}^{d} v_j(\tilde{x}_j),$$

then we have

$$\|v\|_{L^1(\mathbb{R}^d)} \leqslant \prod_{j=1}^{d} \|\tilde{v}_j\|_{L^{d-1}(\mathbb{R}^{d-1})}. \tag{2.19}$$

The case $d = 2$ is easy. Assume that (2.19) is true up to the dimension $d - 1$ for some $d \geqslant 3$. We fix $x_1 \in \mathbb{R}$ and see $v$ as a function of $x' = (x_2, \dots, x_d)$. By the Hölder inequality we have

$$\int_{\mathbb{R}^{d-1}} v(x_1, x')\, dx' \leqslant \|\tilde{v}_1\|_{L^{d-1}(\mathbb{R}^{d-1})} \left( \int_{\mathbb{R}^{d-1}} \prod_{j=2}^{d} \tilde{v}_j(x_1, \tilde{x}'_j)^{\frac{d-1}{d-2}}\, dx_2 \dots dx_d \right)^{\frac{d-2}{d-1}},$$

where for $j \in [\![2, d]\!]$ we have set $\tilde{x}'_j = (x_2, \dots, x_{j-1}, x_{j+1}, \dots, x_d)$. The induction assumption gives

$$\int_{\mathbb{R}^{d-1}} \prod_{j=2}^{d} \tilde{v}_j(x_1, \tilde{x}'_j)^{\frac{d-1}{d-2}}\, dx' \leqslant \prod_{j=2}^{d} \left( \int_{\mathbb{R}^{d-2}} \tilde{v}_j(x_1, \tilde{x}'_j)^{d-1}\, d\tilde{x}'_j \right)^{\frac{1}{d-2}}$$

and hence

$$\int_{\mathbb{R}^{d-1}} v(x_1, x')\, dx' \leqslant \|\tilde{v}_1\|_{L^{d-1}(\mathbb{R}^{d-1})} \prod_{j=2}^{d} \left( \int_{\mathbb{R}^{d-2}} \tilde{v}_j(x_1, \tilde{x}'_j)^{d-1}\, d\tilde{x}'_j \right)^{\frac{1}{d-1}}.$$

After integration over $x_1 \in \mathbb{R}$ we get, by the Hölder inequality,

$$\int_{\mathbb{R}^d} v(x)\, dx \leqslant \|\tilde{v}_1\|_{L^{d-1}(\mathbb{R}^{d-1})} \prod_{j=2}^{d} \left( \int_{\mathbb{R}^{d-1}} \tilde{v}_j(\tilde{x}_j)^{d-1}\, d\tilde{x}_j \right)^{\frac{1}{d-1}}.$$

This is (2.19). We deduce

$$\int_{\mathbb{R}^d} |u(x)|^{\frac{d}{d-1}}\, dx \leqslant \left( \int_{\mathbb{R}^d} |\nabla u(x)|\, dx \right)^{\frac{d}{d-1}},$$

which gives the result for $u \in C_0^1(\mathbb{R}^d)$ when $p = 1$.
• Let $\gamma > 1$. The case $p = 1$ applied to $|u|^{\gamma-1} u$ (still in $C_0^1(\mathbb{R}^d)$, with gradient $\gamma |u|^{\gamma-1} \nabla u$) gives

$$\left( \int_{\mathbb{R}^d} |u|^{\frac{\gamma d}{d-1}} \right)^{\frac{d-1}{d}} \leqslant \gamma \int_{\mathbb{R}^d} |u|^{\gamma-1} |\nabla u| \, dx \leqslant \gamma \left( \int_{\mathbb{R}^d} |u|^{\frac{(\gamma-1)d}{d-1}} \, dx \right)^{\frac{p-1}{p}} \left( \int_{\mathbb{R}^d} |\nabla u|^p \, dx \right)^{\frac{1}{p}}. \tag{2.20}$$

If we choose

$$\gamma = \frac{p(d-1)}{d-p} > 1$$

we have

$$\frac{\gamma d}{d-1} = \frac{(\gamma-1)p}{p-1} = \frac{dp}{d-p} = p^*,$$

and the conclusion follows for $u \in C_0^1(\mathbb{R}^d)$. The general case $u \in W^{1,p}(\mathbb{R}^d)$ follows by density. □

In Theorem 2.46 we have only used the fact that $\nabla u \in L^p(\mathbb{R}^d)$. If $u$ is also in $L^p(\mathbb{R}^d)$ we have better conclusions. We know that $L^p(\mathbb{R}^d) \cap L^{p^*}(\mathbb{R}^d) \subset L^q(\mathbb{R}^d)$ (with continuous inclusion) for any $q \in [p, p^*]$. This is the first statement of the following theorem. The second statement is about the limit case $p = d$. Notice that Theorem 2.46 does not hold with $p = d$ and $p^* = +\infty$ (see Exercise 20), but for $u \in W^{1,d}(\mathbb{R}^d)$ we have a result similar to the case $p < d$.

**Theorem 2.47.** (i) *Let $p \in [1, d[$. Then for all $q \in [p, p^*]$ we have $W^{1,p}(\mathbb{R}^d) \subset L^q(\mathbb{R}^d)$ with continuous injection.*

(ii) *For all $q \in [d, +\infty[$ we have $W^{1,d}(\mathbb{R}^d) \subset L^q(\mathbb{R}^d)$ with continuous injection.*

*Proof.* We prove the second statement. We prove by induction on $\gamma \geqslant d-1$ that for $q \in \left[d, \frac{\gamma d}{d-1}\right]$ there exists $C_q > 0$ such that, for all $u \in C_c^1(\mathbb{R}^d)$,

$$\|u\|_{L^q(\mathbb{R}^d)} \leqslant C_q \|u\|_{W^{1,d}(\mathbb{R}^d)}. \tag{2.21}$$

The result will follow by density. (2.21) is clear when $\gamma = d-1$. We assume that it is proved up to $\gamma - 1$ for some $\gamma \geqslant d$. Let $u \in C_c^1(\mathbb{R}^d)$. We use estimate (2.20) from the previous proof with $p = d$. With the induction assumption this gives

$$\|u\|^{\gamma}_{L^{\frac{\gamma d}{d-1}}(\mathbb{R}^d)} \leqslant \gamma \|u\|^{\gamma-1}_{L^{\frac{(\gamma-1)d}{d-1}}(\mathbb{R}^d)} \|\nabla u\|_{L^d(\mathbb{R}^d)} \leqslant \gamma C^{\gamma-1}_{\frac{(\gamma-1)d}{d-1}} \|u\|^{\gamma}_{W^{1,d}(\mathbb{R}^d)}.$$

This gives (2.21) for $q = \frac{\gamma d}{d-1}$. The case $q \in \left[d, \frac{\gamma d}{d-1}\right]$ follows since $u$ belongs to $L^d(\mathbb{R}^d)$. □

Theorems 2.45 and 2.46 only concern functions with one derivative in $L^p(\mathbb{R}^d)$. Iterating these results we see that we get better results if we have more derivatives in $L^p(\mathbb{R}^d)$.

For $m \in \mathbb{N}$ such that $mp < d$ we define $p_m^*$ by

$$\frac{1}{p_m^*} = \frac{1}{p} - \frac{m}{d}, \quad p_m^* = \frac{pd}{d-mp}.$$

**Corollary 2.48.** *Let $k \in \mathbb{N}^*$ and $p \in [1, +\infty]$.*

(i) *Assume that* $kp < d$. *Then for* $q \in [p, p_k^*]$ *we have* $W^{k,p}(\mathbb{R}^d) \subset L^q(\mathbb{R}^d)$ *with continuous injection.*

(ii) *Assume that* $kp = d$. *Then for all* $q \in [p, +\infty[$ *we have* $W^{k,p}(\mathbb{R}^d) \subset L^q(\mathbb{R}^d)$ *with continuous injection.*

(iii) *Assume that* $kp > d$. *Then for all* $q \in [p, +\infty]$ *we have* $W^{k,p}(\mathbb{R}^d) \subset L^q(\mathbb{R}^d)$ *with continuous injection. Moreover, modulo the choice of a continuous representative we have* $W^{k,p}(\mathbb{R}^d) \subset C^{n,\theta}(\mathbb{R}^d)$ *with continuous injection, where* $n \in \mathbb{N}$ *and* $\theta \in ]0,1[$ *are such that* $n + \theta \leqslant k - d/p$.

***Exercise* 21.** Prove Corollary 2.48.

### 2.5.4 Sobolev embeddings on a bounded domain

So far we have only proved results about $W^{1,p}(\mathbb{R}^d)$. Our purpose in this paragraph is to prove analogous results for Sobolev spaces on a bounded open subset $\Omega$. For this, we will use the extension operator of Theorem 2.41 to deduce inequalities on $\Omega$ from their analogs on $\mathbb{R}^d$.

However, as said in introduction, we will get better results on $\Omega$. For instance we recall that $L^p(\Omega) \subset L^q(\Omega)$ if $p > q$. This will automatically improve the result of Theorem 2.46 (in particular the discussion before Theorem 2.46 is not valid on a bounded domain).

Another very important difference between the case of $\mathbb{R}^d$ and the case of a bounded domain is that some inclusions will be not only continuous but also compact.

The results of Theorem 2.47 and 2.45 are extended to bounded domains as follows.

**Theorem 2.49.** *Let* $\Omega$ *be a bounded open subset of class* $C^1$ *in* $\mathbb{R}^d$. *Let* $p \in [1, +\infty]$. *Then we have the following compact inclusions.*

(i) *If* $p < d$ *then for all* $q \in [1, p^*[$ *we have* $W^{1,p}(\Omega) \subset\subset L^q(\Omega)$.

(ii) *For all* $q \in [d, +\infty[$ *we have* $W^{1,d}(\Omega) \subset\subset L^q(\Omega)$.

(iii) *If* $p > d$ *then we have* $W^{1,p}(\Omega) \subset\subset C^0(\overline{\Omega})$.

*In particular we always have* $W^{1,p}(\Omega) \subset\subset L^p(\Omega)$.

*Proof of Theorem 2.49.* • We begin with the last case. By the extension Theorem 2.41, we can see functions in $W^{1,p}(\Omega)$ as functions in $W^{1,p}(\mathbb{R}^d)$ supported in some fixed compact of $\mathbb{R}^d$. If $p < +\infty$, the conclusion follows from the Morrey inequality (Theorem 2.45) and the Ascoli-Arzelá Theorem A.3. Since $W^{1,+\infty}(\Omega)$ is continuously embedded in $W^{1,p}(\Omega)$ for any $p \in ]d, +\infty[$, it is also compactly embedded in $C^0(\overline{\Omega})$.
• Assume that (i) is proved and let $q \in [d, +\infty[$. Then there exists $p \in [1, d[$ such that $q < p^*$. Then we have

$$W^{1,d}(\Omega) \subset W^{1,p}(\Omega) \subset\subset L^q(\Omega),$$

where the first inclusion is continuous (since $\Omega$ is bounded) and the second is compact by (i). Thus it only remains to prove (i).
• Let $q \in [1, p^*[$. We consider a sequence $(u_n)_{n\in\mathbb{N}}$ bounded in $W^{1,p}(\Omega)$. As above, we identify this sequence with a sequence (still denoted by $(u_n)_{n\in\mathbb{N}}$) bounded in $W^{1,p}(\mathbb{R}^d)$ such that the functions $u_n$ are supported in the same bounded open subset $\mathcal{U}$. Let

$\rho \in C_0^\infty(\mathbb{R}^d, \mathbb{R}_+)$ be supported in the unit ball and such that $\int_{\mathbb{R}^d} \rho = 1$. For $\varepsilon > 0$ and $x \in \mathbb{R}^d$ we set $\rho_\varepsilon = \frac{1}{\varepsilon^d}\rho\big(\frac{x}{\varepsilon}\big)$, and then $u_n^\varepsilon = \rho_\varepsilon * u_n \in C^\infty(\mathbb{R}^d)$. Let $\varepsilon > 0$. For $n \in \mathbb{N}$ and $x \in \mathbb{R}^d$ we have

$$|u_n^\varepsilon(x)| \leqslant \|\rho_\varepsilon\|_{L^\infty(\mathbb{R}^d)} \|u_n\|_{L^1(\mathcal{U})}$$

and

$$|\nabla u_n^\varepsilon(x)| \leqslant \|\nabla\rho_\varepsilon\|_{L^\infty(\mathbb{R}^d)} \|u_n\|_{L^1(\mathcal{U})},$$

so the sequence $(u_n^\varepsilon)_{n\in\mathbb{N}}$ is bounded in $C^0(\mathbb{R}^d)$ and uniformly equicontinuous. Moreover the functions $u_n^\varepsilon$ are supported in a common bounded set $\mathcal{V}$ of $\mathbb{R}^d$, so by the Ascoli-Arzelá Theorem A.3 there exists a subsequence $(u_{n_k}^\varepsilon)_{k\in\mathbb{N}}$ which converges uniformly in $\mathcal{V}$ and hence in $\mathcal{U}$. This gives

$$\limsup_{j,k\to+\infty} \left\| u_{n_j}^\varepsilon - u_{n_k}^\varepsilon \right\|_{L^q(\mathcal{U})} = 0.$$

- We already know that $u_n^\varepsilon$ goes to $u_n$ as $\varepsilon \to 0$ in $L^q(\mathcal{U})$ for all $n \in \mathbb{N}$. We prove that this convergence is uniform with respect to $n$. Let $v \in C_0^1(\mathbb{R}^d)$ be supported in $\mathcal{U}$. For $\varepsilon > 0$ we set $v_\varepsilon = \rho_\varepsilon * v$. Then for $x \in \mathbb{R}^d$ we have

$$v_\varepsilon(x) - v(x) = \int_{B(0,1)} \rho(y)\big(v(x-\varepsilon y) - v(x)\big)\,dy = -\varepsilon \int_{B(0,1)} \rho(y) \int_0^1 \nabla v(x-\varepsilon t y)\cdot y \,dt\,dy,$$

and hence

$$\begin{aligned} \|v_\varepsilon - v\|_{L^1(\mathcal{U})} &= \int_{\mathcal{U}} |v_\varepsilon(x) - v(x)|\,dx \leqslant \varepsilon \int_{B(0,1)} \rho(y) \int_0^1 \int_{\mathcal{U}} |\nabla v(x-\varepsilon t y)|\,dx\,dt\,dy \\ &\leqslant \varepsilon \|\nabla v\|_{L^1(\mathcal{U})}. \end{aligned} \tag{2.22}$$

By density, the same estimate holds for any $v \in W^{1,p}(\mathbb{R}^d)$ supported in $\mathcal{U}$ (note that if $v_m \in C^1(\mathbb{R}^d)$ goes to $v$ in $W^{1,p}(\mathbb{R}^d)$ then $\rho_\varepsilon * v_n$ goes to $\rho_\varepsilon * v$ in $L^1(\mathbb{R}^d)$). Let $\theta \in ]0,1[$ be such that

$$\frac{1}{q} = \theta + \frac{1-\theta}{p^*}.$$

By (2.22) applied with $v = u_n$ and the Gagliardo-Nirenberg inquality (Theorem 2.46 there exists $C > 0$ independant on $u$, $n$ or $\varepsilon$ such that

$$\|u_n^\varepsilon - u_n\|_{L^q(\mathcal{U})} \leqslant \|u_n^\varepsilon - u_n\|_{L^1(\mathcal{U})}^\theta \|u_n^\varepsilon - u_n\|_{L^{p^*}(\mathbb{R}^d)}^{1-\theta} \leqslant C\varepsilon^\theta \|\nabla u_n\|_{L^p(\mathcal{U})}.$$

This proves that $u_n^\varepsilon$ goes to $u_n$ in $L^q(\mathcal{U})$ as $\varepsilon \to 0$ uniformly with respect to $n \in \mathbb{N}$. Then for any $\eta > 0$ we get

$$\limsup_{j,k\to+\infty} \left\| u_{n_j} - u_{n_k} \right\|_{L^q(\mathcal{U})} \leqslant \eta.$$

Using a standard diagonal argument, we obtain a subsequence which goes to 0 in $L^q(\mathcal{U})$ and hence in $L^q(\Omega)$. □

***Exercise 22.*** Let $p \in [1, d[$. Prove that we have the continuous inclusion $W^{1,p}(B(0,1)) \subset L^{p^*}(B(0,1))$, but that this inclusion is not compact.

## 2.6 Traces

We recall that functions in the Sobolev spaces are not really functions, but equivalence classes of functions pairwise almost everywhere equal. In particular, for $u$ in some Sobolev space $W^{k,p}(\Omega)$, it does not make sense to consider the value of $u$ at some point $x_0 \in \Omega$.

We have seen in Proposition 2.42 that, in dimension 1, an element $u$ of $W^{1,p}(I)$ has a continuous reprentative $\tilde{u}$. It is reasonnable to consider $\tilde{u}(x_0)$ as the value of $u$ at $x_0$. Indeed, if $\tilde{v}$ is another representative of $u$ then $\tilde{v}(x_0)$ can be far from $\tilde{u}(x_0)$, but for almost all $x \in I$ "close to $x_0$" then $\tilde{v}(x)$ is equal to $\tilde{u}(x)$ and hence "close to $\tilde{u}(x_0)$".

However, this possible definition only works in dimension 1, since in higher dimension an element of $W^{1,p}(\Omega)$ does not necessarily have a continuous representative.

In applications, it is not crucial to give the value of a function at a point, but we are interested in what happens at the boundary of the domain. This will be important for instance for integration by parts (Green formula in higher dimension), where the value of the function at the boundary appears. For regular domains, the boundary is a submanifold of dimension $(d-1)$. This is still of dimension 0 for the Lebesgue measure on $\Omega$, but if $d \geqslant 2$ this is in some sense "bigger" than a point.

Our purpose in this section is the following. Given a regular open subset $\Omega$ of $\mathbb{R}^d$ and $u \in W^{1,p}(\Omega)$, we want to give a natural sense to the restriction of $u$ on the boundary $\partial\Omega$, in such a way that if $u$ belongs to $C^0(\overline{\Omega})$ then the new definition coincides with the usual one.

### 2.6.1 Trace

As explained in the previous section, we begin our analysis with the model case $\Omega = \mathbb{R}^d_+$ and then, using a partition of unity and changes of variables, we will give a more general result.

**Proposition 2.50.** *Let $p \in [1, +\infty[$. There exists $C > 0$ such that for $u \in C_0^\infty(\overline{\mathbb{R}^d_+})$ we have*

$$\|u(0,\cdot)\|^p_{L^p(\mathbb{R}^{d-1})} \leqslant C \, \|u\|^p_{W^{1,p}(\mathbb{R}^d_+)} .$$

For the proof we only have to integrate over $\mathbb{R}^{d-1}$ the one-dimensional case which is very close to Proposition 2.42:

*Proof.* For $x' \in \mathbb{R}^{d-1}$ we have

$$|u(0,x')|^p \leqslant p \int_0^{+\infty} |\partial_{x_1} u(s,x')| \, |u(s,x')|^{p-1} \, ds$$

so, by the Hölder and Young inequalities,

$$\begin{aligned} |u(0,x')|^p &\leqslant p \left( \int_0^{+\infty} |\partial_{x_1} u(s,x')|^p \, ds \right)^{\frac{1}{p}} \left( \int_0^{+\infty} |u(s,x')|^p \, ds \right)^{\frac{p-1}{p}} \\ &\leqslant \int_0^{+\infty} |\partial_{x_1} u(s,x')|^p \, ds + (p-1) \int_0^{+\infty} |u(s,x')|^p \, ds. \end{aligned}$$

After integration over $x' \in \mathbb{R}$ we get

$$\|u(0,\cdot)\|_{L^p(\mathbb{R}^{d-1})}^p \leqslant (p-1)\,\|u\|_{L^p(\mathbb{R}^d_+)}^p + \|\partial_{x_1} u\|_{L^p(\mathbb{R}^d)}^p ,$$

and the conclusion follows. □

**Theorem 2.51.** *Let $\Omega$ be an open subset of $\mathbb{R}^d$ of class $C^1$. Let $p \in [1,+\infty[$. There is a unique bounded linear operator*

$$\gamma_0 : W^{1,p}(\Omega) \to L^p(\partial\Omega)$$

*such that*

$$\forall u \in W^{1,p}(\Omega) \cap C^0(\overline{\Omega}), \quad \gamma_0(u) = u|_{\partial\Omega}.$$

*Proof.* Let $u \in C_0^\infty(\overline{\Omega})$. We use the notation of Paragraph 2.4.1. Let $j \in [\![1,N]\!]$. We have

$$\int_{\partial\Omega\cap\mathcal{U}_j} |u|^p \, d\sigma = \int_{\mathcal{O}_j} |u(\tilde\varphi(x'))|^p \sqrt{1+|\nabla\varphi(x')|^2}\, dx'$$
$$\leqslant C_\varphi \int_{\mathcal{O}_j} |u(\tilde\varphi(x'))|^p \, dx' = C_\varphi \int_{\partial\mathbb{R}^d_+} \left|(u\circ\Phi^{-1})\right|^p \, dx',$$

where $C_\varphi = \sup_{x'\in\mathcal{O}} \sqrt{1+|\nabla\varphi(x')|^2}$ and $(u\circ\Phi^{-1})$ has been extended by 0 on $\mathbb{R}^d_+$. By Propositions 2.50 and 2.36 there exists $C_j > 0$ independant of $u$ such that

$$\int_{\partial\Omega\cap\mathcal{U}_j} |u|^p \, d\sigma \leqslant C_\varphi C \left\|u\circ\Phi^{-1}\right\|_{\mathcal{W}_j\cap\mathbb{R}^d_+}^p \leqslant C_j \|u_j\|_{W^{1,p}(\Omega)}^p .$$

Then,

$$\left\|u_{|\partial\Omega}\right\|_{L^p(\Omega)} \leqslant \sum_{j=1}^N \left\|u_{j|\partial\Omega}\right\|_{L^p(\Omega)} \leqslant \sum_{j=1}^N C_j \|u_j\|_{W^{1,p}(\Omega)} .$$

Finally, there exists $C > 0$ such that for all $u \in C_0^\infty(\overline{\Omega})$ we have

$$\left\|u_{|\partial\Omega}\right\|_{L^p(\partial\Omega)} \leqslant C \|u\|_{W^{1,p}(\Omega)} .$$

Since $C_0^\infty(\overline{\Omega})$ is dense in $W^{1,p}(\Omega)$, the map $u \in C_0^\infty(\overline{\Omega}) \mapsto u_{|\partial\Omega} \in L^p(\partial\Omega)$ extends to a unique continuous map on $W^{1,p}(\Omega)$. Moreover, if $u \in W^{1,p}(\Omega)\cap C^0(\overline{\Omega})$ then the sequence $(u_n)_{n\in\mathbb{N}}$ given by the proof of Theorem 2.39 goes uniformly to $u$ and hence the restriction of $u_n$ goes to the restriction of $u$ uniformly on $\partial\Omega$, and hence in $L^p(\partial\Omega)$. □

***Exercise* 23.** Show that there is no continuous linear map $\gamma : L^2(\mathbb{R}_+^*) \to \mathbb{R}$ such that $\gamma(u) = u(0)$ for all $u \in C^0([0,+\infty[) \cap L^2(\mathbb{R}_+^*)$.

The following notation is motivated by Theorem 2.64 below:

**Définition 2.52.** When $p = 2$ we denote by $H^{1/2}(\partial\Omega)$ the range of $\gamma_0 : H^1(\Omega) \to L^2(\Omega)$.

We do not discuss the properties of $H^{1/2}(\partial\Omega)$ here. However we will use in the following chapter that even if $\gamma_0$ is not surjective, $H^{1/2}(\partial\Omega)$ is dense in $L^2(\partial\Omega)$.

**Proposition 2.53.** *Let $\Omega$ be an open subset of $\mathbb{R}^d$ of class $C^1$. Let $p \in [1, +\infty[$ and $u \in W^{1,p}(\Omega)$. Then we have*

$$\gamma_0(u) = 0 \quad \Longleftrightarrow \quad u \in W_0^{1,p}(\Omega).$$

*Proof.* • Assume that $u \in W_0^{1,p}(\mathbb{R}^d)$. Then there is a sequence $(u_n)_{n\in\mathbb{N}}$ in $C_0^\infty(\Omega)$ going to $u$ in $W^{1,p}(\Omega)$. Since $\gamma_0(u_n) = 0$ for all $n \in \mathbb{N}$ and $\gamma_0$ is continuous, we have $\gamma_0(u) = 0$.
• For the converse, we consider the case $\Omega = \mathbb{R}_+^d$ and $u$ supported in a bounded domain. Then, with a partition of unity and changes of variables as above, we get the general case. So let $u \in W^{1,p}(\mathbb{R}_+^d)$ such that $\gamma_0(u) = 0$. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $C_0^\infty(\overline{\mathbb{R}_+^d})$ which goes to $u$ in $W^{1,p}(\mathbb{R}_+^d)$ (see Theorem 2.39). Let $n \in \mathbb{N}$ and $x_1 > 0$. For $x' \in \mathbb{R}^{d-1}$ we have by the Hölder inequality

$$\begin{aligned}
|u_n(x_1, x')|^p &\leqslant \left( |u_n(0, x')| + \int_0^{x_1} |\nabla u_n(s, x')| \, ds \right)^p \\
&\leqslant 2^{p-1} |u_n(0, x')|^p + 2^{p-1} \left( \int_0^{x_1} |\nabla u_n(s, x')| \, ds \right)^p \\
&\leqslant 2^{p-1} |u_n(0, x')|^p + 2^{p-1} x_1^{p-1} \int_0^{x_1} |\nabla u_n(s, x')|^p \, ds,
\end{aligned}$$

so for $\varepsilon > 0$

$$\int_0^\varepsilon \int_{\mathbb{R}^{d-1}} |u_n(x_1, x')|^p \, dx' \, dx_1 \leqslant 2^{p-1} \varepsilon \left\| \gamma_0(u_n) \right\|_{L^p(\mathbb{R}_+^d)}^p + 2^{p-1} \varepsilon^p \int_{\mathbb{R}^{d-1}} \int_0^\varepsilon |\nabla u_n(s, x')|^p \, ds \, dx'.$$

Taking the limit $n \to 0$ yields, by continuity of the trace,

$$\|u\|_{L^p(]0,\varepsilon[\times\mathbb{R}^{d-1})}^p \leqslant 2^{p-1} \varepsilon^p \left\| \nabla u \right\|_{L^p(]0,\varepsilon[\times\mathbb{R}^{d-1})}^p. \tag{2.23}$$

Let $\chi \in C^\infty(\mathbb{R}_+, [0, 1])$, equal to 1 on $[0, 1]$ and equal to 0 on $[2, +\infty[$. Then for $n \in \mathbb{N}^*$ and $x = (x_1, \ldots, x_d) \in \mathbb{R}_+^d$ we set $\chi_n(x) = \chi(nx_1)$. For $n \in \mathbb{N}^*$ we set $u_n = (1 - \chi_n)u$, so that $u_n \in C_0^\infty(\mathbb{R}_+^d)$. By the dominated convergence theorem, we have

$$\|u_n - u\|_{L^p(\mathbb{R}_+^d)} = \|\chi_n u\|_{L^p(\mathbb{R}_+^d)} \xrightarrow[n\to+\infty]{} 0.$$

For $n \in \mathbb{N}^*$ we have

$$\nabla(u_n - u) = (1 - \chi_n)\nabla u - u\partial_1\chi_n.$$

The first term goes to 0 in $L^p(\mathbb{R}_+^d)$. For the second term we use (2.23) to write

$$\begin{aligned}
\|u\partial_1\chi_n\|_{L^p}^p &= n^p \int_{x_1=\frac{1}{n}}^{\frac{2}{n}} |\chi'(nx_1)|^p \int_{x'\in\mathbb{R}^{p-1}} |u(x_1, x')|^p \, dx' \, dx_1 \\
&\leqslant 2^{2p-1} \left\| \chi' \right\|_\infty \left\| \nabla u \right\|_{L^p(]0,\frac{2}{n}[\times\mathbb{R}^{d-1})}^p \\
&\xrightarrow[n\to+\infty]{} 0.
\end{aligned}$$

This proves that

$$\|u_n - u\|_{W^{1,p}(\mathbb{R}_+^d)} \xrightarrow[n\to+\infty]{} 0,$$

and hence $u \in W_0^{1,p}(\mathbb{R}_+^d)$. □

***Exercise* 24.** Find an open domain $\Omega$ and $u \in W^{1,\infty}(\Omega)$ such that $u_{|\partial\Omega} = 0$ but $u$ is not in the closure of $C_0^\infty(\Omega)$ in $W^{1,+\infty}(\Omega)$.

***Exercise* 25. 1.** Let $u \in H_0^1(\mathbb{R}_+^2)$. For $(x_1, x_2) \in \mathbb{R}^2$ we set

$$\tilde{u}(x_1, x_2) = \begin{cases} u(x_1, x_2) & \text{if } x_1 > 0, \\ 0 & \text{if } x_1 \leqslant 0. \end{cases}$$

Prove that $\tilde{u} \in H^1(\mathbb{R}^2)$ and give an expression for the derivatives of $\tilde{u}$. In particular, what can we say about $\|\tilde{u}\|_{H^1(\mathbb{R}^2)}$ ?
**2.** Let $\Omega$ be an open subset of $\mathbb{R}^2$. Let $u \in H_0^1(\Omega)$. Prove that the extension of $u$ by 0 on $\mathbb{R}^d$ belongs to $H^1(\mathbb{R}^2)$.

### 2.6.2 Normal derivative

Let $\Omega$ be a bounded open subset of class $C^1$ in $\mathbb{R}^d$. For the rest of this section we only consider the case $p = 2$.

Let $u \in H^2(\Omega)$. For $j \in [\![1, d]\!]$ the derivative $\partial_j u$ belongs to $H^1(\Omega)$ and hence has a trace on $\partial\Omega$. Then we set

$$\gamma_1(u) = \partial_\nu u = \sum_{j=1}^d \gamma_0(\partial_j u)\nu_j \in L^2(\partial\Omega),$$

where $\nu = (\nu_1, \dots, \nu_d)$ is the outward normal derivative (see (2.14)). Notice that if $u$ belongs to $C^1(\overline{\Omega})$ then on $\partial\Omega$ we have

$$\partial_\nu u = \nabla u \cdot \nu.$$

This defines a continuous function $\gamma_1$ from $H^2(\Omega)$ to $L^2(\Omega)$. We can prove (see Theorem 2.64 below for the case $\Omega = \mathbb{R}_+^d$) that

$$\left\{\partial_\nu u, u \in H^2(\Omega)\right\} = H^{1/2}(\Omega).$$

### 2.6.3 Green Formula

As said above, one of the motivations for the definition of the traces is the generalization of the Green Formula to functions which are not regular in the usual sense. The following results are deduced from the regular analogs by density of regular functions and continuity of the traces. For $u \in W^{1,1}(\Omega)$ we can write $\int_{\partial\Omega} u \, d\sigma$ instead of $\int_{\partial\Omega} \gamma_0(u) \, d\sigma$ and $\int_{\partial\Omega} \partial_\nu u \, v \, d\sigma$ instead of $\int_{\partial\Omega} \gamma_1(u)\gamma_0(v) \, d\sigma$.

**Theorem 2.54.** *Let $u, v \in H^1(\Omega)$. Then for $j \in [\![1, d]\!]$ we have*

$$\int_\Omega u \, \partial_j v \, dx = \int_{\partial\Omega} uv \, d\sigma - \int_\Omega \partial_j u \, v \, dx$$

**Theorem 2.55.** *Let $u \in H^2(\Omega)$ and $v \in H^1(\Omega)$. Then we have*

$$-\int_\Omega \Delta u \, v \, dx = -\int_{\partial\Omega} \partial_\nu u \, v \, d\sigma + \int_\Omega \nabla u \cdot \nabla v \, dx.$$

## 2.7 Poincaré Inequality

In Theorem 2.46 we have given an estimate with the norm $\|\nabla u\|_{L^p(\mathbb{R}^d)}$ and not the full norm $\|u\|_{W^{1,p}(\mathbb{R}^d)}$. In application, and in particular for the analysis of second order PDEs, we will often be in the situation where we only control the norm of the gradient of the function and not the function itself.

It turns out that in some particular situations, the norm of the function is in fact controled by the norm of the gradient:

$$\|u\|_{L^p(\Omega)} \leqslant C \|\nabla u\|_{L^p(\Omega)} . \tag{2.24}$$

In this case, $\|\nabla u\|_{L^p(\Omega)}$ defines a norm on $W^{1,p}(\Omega)$, equivalent to $\|u\|_{W^{1,p}(\Omega)}$. An inequality like (2.24) is called a Poincaré inequality. This is the subject of this paragraph.

Before giving precise statements, we notice that a Poincaré inquality cannot hold in a space which contains constant functions. In an unbounded domain, troubles can come from slowly varying functions. For instance on $\mathbb{R}$ we consider for $n \in \mathbb{N}^*$ the function $u_n$ defined by

$$u_n(x) = \begin{cases} 1 - \frac{|x|}{n} & \text{if } |x| \leqslant n, \\ 0 & \text{if } |x| > n. \end{cases}$$

Then we have $\|u\|^2_{L^2(\mathbb{R})} = \frac{2n}{3}$ and $\|u'\|^2_{L^2(\mathbb{R})} = \frac{2}{n}$. A Poincaré inequality cannot hold in $H^1(\mathbb{R})$.

In fact, we have discussed all the problems to prove a Poincaré inequality. Roughly speaking, on a bounded domain, and if we remove constant functions, a Poincaré inquality holds. The first way to remove constant functions is to consider only functions vanishing at the boundary.

We first recall that the property that only constant functions (up to equality almost everywhere) have a zero gradient also holds in higher dimension.

**Proposition 2.56.** *Let $\Omega$ be an open connected subset of $\mathbb{R}^d$. Let $u \in L^1_{\text{loc}}(\Omega)$ be such that $\nabla u = 0$ (in the sense of distributions). Then there exists a constant $\alpha$ such that $u = \alpha$ almost everywhere.*

*Proof.* We proceed by induction on the dimension. The case $d = 1$ is already known. We assume that $d \geqslant 2$ and that the result is known up to the dimension $d - 1$.

It is enough to consider the case $\Omega = \prod_{j=1}^d ]a_j, b_j[$. Let $\chi \in C_0^\infty(]a_1, b_1[)$ be such that $\int_{a_1}^{b_1} \chi(x_1)\, dx_1 = 1$. For $x' \in \Omega' = \prod_{j=2}^d ]a_j, b_j[$ we set

$$v(x') = \int_{a_1}^{b_1} u(x_1, x')\chi(x_1)\, dx_1.$$

This defines a function $v \in L^1_{\text{loc}}(\Omega')$. For $\psi \in C_0^\infty(\Omega')$ and $j \in [\![2, d]\!]$ we have

$$\begin{aligned} -\int_{\Omega'} v(x')\partial_j\psi(x')\, dx' &= -\int_\Omega u(x_1, x')\chi(x_1)\partial_j\psi(x')\, dx_1\, dx' \\ &= -\int_\Omega u(x_1, x')\partial_j\big(\chi(x_1)\psi(x')\big)\, dx_1\, dx' \\ &= 0. \end{aligned}$$

This proves that, in the sense of distributions, we have $\nabla v = 0$ on $\Omega'$ By the induction assumption there exists $\alpha$ such that $v = \alpha$ almost everywhere on $\Omega'$.

Now let $\phi \in C_0^\infty(\Omega)$. For $x = (x_1, x') \in \Omega$ we set

$$\tilde{\phi}(x') = \int_{a_1}^{b_1} \phi(x_1, x')\, dx_1$$

and

$$\zeta(x) = \int_{a_1}^{x_1} \big(\phi(t, x') - \chi(t)\tilde{\phi}(x')\big)\, dt.$$

Then $\zeta \in C_0^\infty(\Omega)$ and $\phi = \partial_{x_1}\zeta + \chi \otimes \tilde{\phi}$, so

$$\int_\Omega u\phi\, dx = \int_{a_1}^{b_1} \int_{\Omega'} u(x_1, x')\chi(x_1)\tilde{\phi}(x')\, dx'\, dx_1 = \int_{\Omega'} v\tilde{\phi}\, dx' = \alpha \int_{\Omega'} \tilde{\phi}\, dx' = \alpha \int_\Omega \phi\, dx.$$

This proves that $u = \alpha$ almost everywhere on $\Omega$. □

Now we can prove the Poincaré inequality.

**Theorem 2.57.** *Let $\Omega$ be an open bounded subset of $\mathbb{R}^d$. Let $p \in [1, +\infty[$. Then there exists $C > 0$ such that*

$$\forall u \in W_0^{1,p}(\Omega), \quad \|u\|_{L^p(\Omega)} \leqslant C \|\nabla u\|_{L^p(\Omega)}.$$

*Proof.* Assume by contradiction that the statement is not true. Then for all $n \in \mathbb{N}$ there exists $u_n \in W_0^{1,p}(\Omega)$ such that

$$\|u_n\|_{L^p(\Omega)} > n \|\nabla u_n\|_{L^p(\Omega)}.$$

Since this inequality can be divided by $\|u_n\|_{L^p(\Omega)}$ (which cannot be 0), we can assume without loss of generality that $\|u_n\|_{L^p(\Omega)} = 1$ for all $n \in \mathbb{N}$. Then

$$\|\nabla u_n\|_{L^p(\Omega)} \xrightarrow[n\to+\infty]{} 0, \tag{2.25}$$

and the sequence $(u_n)_{n\in\mathbb{N}}$ is bounded in $W^{1,p}(\Omega)$. Since $W^{1,p}(\Omega)$ is compactly embedded in $L^p(\Omega)$ (see Theorem 2.49), there exists an increasing sequence $(n_k)_{k\in\mathbb{N}} \in \mathbb{N}^\mathbb{N}$ and $v \in L^p(\Omega)$ such that

$$\|u_{n_k} - v\|_{L^p(\Omega)} \xrightarrow[k\to+\infty]{} 0.$$

With (2.25), this implies that the sequence $(u_{n_k})_{k\in\mathbb{N}}$ is a Cauchy sequence in $W^{1,p}(\Omega)$. Since $W^{1,p}(\Omega)$ is complete (see Theorem 2.24), the sequence $(u_{n_k})_{k\in\mathbb{N}}$ has a limit in $W^{1,p}(\Omega)$. This limit is necessarily $v$. In particular $v$ belongs to $W^{1,p}(\Omega)$, and by (2.25) we have $\nabla v = 0$. By Proposition 2.56, $v$ is constant on each connected component of $\Omega$. Since $u_{n_k}$ belongs to $W_0^{1,p}(\Omega)$ for all $k \in \mathbb{N}$, we also have $v \in W_0^{1,p}(\Omega)$, so $v = 0$, which gives a contradiction with the fact that $\|u_{n_k}\|_{L^2(\Omega)} = 1$ for all $k \in \mathbb{N}$. □

Notice that the proof of Theorem 2.57 does not give any clue about the constant $C$ of the inequality. We now give a similar result, with a more constructive proof. Moreover the open set $\Omega$ is only required to be bounded in one direction. This means that $\Omega$ is included in a strip of the form

$$\Omega \subset \left\{x \in \mathbb{R}^d, x \cdot e \in ]a, b[\right\},$$

for some $e \in \mathbb{R}^d$, $|e| = 1$ and $a, b \in \mathbb{R}$.

**Theorem 2.58** (Poincaré inequality)**.** *Let $\Omega$ be an open subset of $\mathbb{R}^d$, bounded in one direction. Let $p \in [1, +\infty[$. Then there exists $C_\Omega > 0$ such that, for all $u \in W_0^{1,p}(\Omega)$,*

$$\|u\|_{L^p(\Omega)} \leqslant C_\Omega \|\nabla u\|_{L^p(\Omega)} .$$

*For instance, we can take $C_\Omega = (b-a)p$ (this is not optimal).*

*Proof.* • It is enough to prove the estimate for $u \in C_0^\infty(\Omega)$. Then the result will follow by density of $C_0^\infty(\Omega)$ in $W_0^{1,p}(\Omega)$. We can extend $u$ by 0, this gives a function in $C_0^\infty(\mathbb{R}^d)$ supported in $\Omega$. The case $p = +\infty$ follows from the Mean Value Inequality, so we assume that

• We first consider the one-dimensional case. Then there exists $a, b \in \mathbb{R}$ such that $\Omega \subset ]a, b[$. For all $x \in ]a, b[$ we have by the Hölder Inequality

$$|u(x)| \leqslant \int_a^b |u'(s)| \, ds \leqslant \|u'\|_{L^p(a,b)} (b-a)^{\frac{p-1}{p}}.$$

Then

$$\|u\|_{L^p}^p \leqslant (b-a) \|u'\|_{L^p}^p (b-a)^{p-1} = \|u'\|_{L^p}^p (b-a)^p$$

so the result follows in this case.

• Now we consider the general case. Let $(f_1, \dots, f_d)$ be an orthonormal basis of $\mathbb{R}^d$ such that

$$\operatorname{supp}(u) \subset \left\{ y_1 f_1 + y' f' \,:\, y_1 \in ]a, b[, y' \in \mathbb{R}^{d-1} \right\},$$

for some $a, b \in \mathbb{R}$, where for $y' = (y_2, \dots, y_d) \in \mathbb{R}^{d-1}$ we have set $y'f' = \sum_{j=2}^d y_j f_j$. By a change of variables and using the one-dimensional case we can write

$$\begin{aligned}
\int_\Omega |u(x)|^p \, dx &= \int_{y' \in \mathbb{R}^{d-1}} \int_{y_1 = a}^b |u(y_1 f_1 + y' f')|^p \, dy_1 \, dy' \\
&\leqslant (b-a)^p \int_{y' \in \mathbb{R}^{d-1}} \int_{y_1 = a}^b \left| \frac{\partial}{\partial y_1} u(y_1 f_1 + y' f') \right|^p \, dy_1 \, dy' \\
&\leqslant (b-a)^p \int_{y' \in \mathbb{R}^{d-1}} \int_{y_1 = a}^b |\nabla u(y_1 f_1 + y' f')|^p \, dy_1 \, dy' \\
&\leqslant (b-a)^p \int_\Omega |\nabla u(x)|^p \, dx.
\end{aligned}$$

The conclusion follows. □

It can be important in application to have an explicit constant for the Poincaré inequality. Computing the optimal constant for particular sets $\Omega$ requires more work, and we do not discuss this issue here, but we already have an upper bound.

After Theorem 2.58, the interest of the proof given for Theorem 2.57 is not clear. For the proof of Theorem 2.58 we have really used the fact that the function $u$ vanishes at the boundary. While for the proof of Theorem 2.57 we have in fact only used the property that the only constant function is 0. The interest of the proof of Theorem 2.57 is that it can be used in any such situation. For instance, we give the following version of the Poincaré inequality.

For a bounded open subset $\Omega$ we define

$$\widetilde{W}^{1,p}(\Omega) = \left\{ u \in W^{1,p}(\Omega) \, : \, \int_\Omega u \, dx = 0 \right\}. \tag{2.26}$$

Notice that if $\Omega$ is connected then the only function $u \in \widetilde{W}^{1,p}(\Omega)$ such that $\nabla u = 0$ is $u = 0$.

**Theorem 2.59** (Poincaré-Wirtinger inequality)**.** *Let $\Omega$ be an open, connected and bounded subset of $\mathbb{R}^d$. Let $p \in [1, +\infty]$ Then there exists $C > 0$ such that, for all $u \in \widetilde{W}^{1,p}(\Omega)$,*

$$\forall u \in \widetilde{W}^{1,p}(\Omega), \quad \|u\|_{L^p(\Omega)} \leqslant C \, \|\nabla u\|_{L^p(\Omega)} .$$

***Exercise* 26.** Let $\Omega$ be an open, bounded and connected subset of $\mathbb{R}^d$. Let $p \in [1, +\infty]$.
**1.** Prove Theorem 2.59.
**2.** For $u \in W^{1,p}(\Omega)$ we set

$$\mathcal{N}(u) = \|\nabla u\|_{L^p(\Omega)} + \left| \int_\Omega u \, dx \right| .$$

Prove that $\mathcal{N}$ is a norm on $W^{1,p}(\Omega)$, equivalent to the usual one.

## 2.8 Appendices

### 2.8.1 Sobolev spaces of fractional order

The space $W^{k,p}(\Omega)$ is the set of functions in $L^p(\Omega)$ whose derivatives of order up to $k$ are in $L^p(\Omega)$. Then, all along this chapter, $k$ was a non-negative integer. When $\Omega = \mathbb{R}^d$ and $p = 2$ we gave another definition of Sobolev spaces (see Proposition 2.26). We observe that in (2.9) the parameter $k$ has no reason to be an integer. Via the Fourier transform, we can then define Sobolev spaces for any real parameter $k$. In some sense, the space $H^{1/2}(\mathbb{R}^d)$ can be seen as the space of $L^2$ functions such that “half a derivative” is in $L^2$. These spaces turn out to be useful. For instance, we have briefly said that $H^{1/2}(\partial\Omega)$ is the set of traces of functions in $H^1(\Omega)$. When $\Omega = \mathbb{R}^d_+$, a proof is given below (see Proposition 2.50)

**Définition 2.60.** Let $s \in \mathbb{R}$. For $u \in \mathcal{S}'(\mathbb{R}^d)$ such that $\hat{u} \in L^1_{\text{loc}}(\mathbb{R}^d)$ we set

$$\|u\|_{H^s(\mathbb{R}^d)} = \left( \int_{\mathbb{R}^d} \left(1 + |\xi|^2\right)^s |\hat{u}(\xi)|^2 \, d\xi \right)^{\frac{1}{2}}, \tag{2.27}$$

and we denote by $H^s(\mathbb{R}^d)$ the set of tempered distribution $u$ such that $\hat{u} \in L^1_{\text{loc}}(\mathbb{R}^d)$ and $\|u\|_{H^s(\mathbb{R}^d)} < +\infty$.

Notice that when $s \in \mathbb{N}$ this definition of $H^s(\mathbb{R}^d)$ coincides with the usual one.

**Proposition 2.61.** *Let $s \in \mathbb{R}$.*

(i) *$H^s(\mathbb{R}^d)$ is a Hilbert space.*

(ii) *If for $u \in H^{-s}(\mathbb{R}^d)$ and $v \in H^s(\mathbb{R}^d)$ we set*

$$\Phi_u(v) = \int_{\mathbb{R}^d} \overline{\hat{u}(\xi)}\hat{v}(\xi)\,d\xi,$$

*then $\Phi_u$ is a continuous linear form on $H^s(\mathbb{R}^d)$ and the map $u \mapsto \Phi_v$ is a semilinear isometry from $H^{-s}(\mathbb{R}^d)$ to $H^s(\mathbb{R}^d)'$.*

*Proof.* • For $u, v \in H^s(\mathbb{R}^d)$ we set

$$\langle u, v\rangle_{H^s(\mathbb{R}^d)} = \int_{\mathbb{R}^d} \langle\xi\rangle^{2s}\,\overline{\hat{u}(\xi)}\hat{v}(\xi)\,d\xi.$$

This is well defined and this gives an inner product on $H^s(\mathbb{R}^d)$. Moreover the corresponding norm is (2.27). Let $(u_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $H^s(\mathbb{R}^d)$. Then $\langle\cdot\rangle^s\,\hat{u}_n$ is a Cauchy sequence in $L^2(\mathbb{R}^d)$ and has a limit $v \in L^2(\mathbb{R}^d)$. We denote by $u$ the inverse Fourier transform of $\langle\cdot\rangle^{-s} v$. Then $u \in H^s(\mathbb{R}^d)$ and

$$\|u_n - u\|_{H^s(\mathbb{R}^d)} = \|\langle\cdot\rangle^s\,\hat{u}_n - v\|_{L^2(\mathbb{R}^d)} \xrightarrow[n\to+\infty]{} 0.$$

This proves that $H^s(\mathbb{R}^d)$ is complete.

• For $u \in H^{-s}(\mathbb{R}^d)$ and $v \in H^s(\mathbb{R}^d)$ we have

$$\Phi_u(v) = \int_{\mathbb{R}^d} \overline{\langle\xi\rangle^{-s}\,\hat{u}(\xi)}\,\langle\xi\rangle^s\,\hat{v}(\xi)\,d\xi.$$

This proves that $\Phi_u(v)$ is well defined and, by the Cauchy-Schwarz Inequality,

$$|\Phi_u(v)| \leqslant \|u\|_{H^{-s}(\mathbb{R}^d)}\,\|v\|_{H^s(\mathbb{R}^d)}\,.$$

Since $\Phi_u$ is linear, it is a continuous linear form on $H^s(\mathbb{R}^d)$ with

$$\|\Phi_u\|_{H^s(\mathbb{R}^d)'} \leqslant \|u\|_{H^{-s}(\mathbb{R}^d)}\,.$$

Choosing $v = \mathcal{F}^{-1}\,\langle\cdot\rangle^{-2s}\,\mathcal{F}u$ we get $\Phi_u(v) = \|u\|_{H^{-s}(\mathbb{R}^d)}\,\|v\|_{H^s(\mathbb{R}^d)}$, so

$$\|\Phi_u\|_{H^s(\mathbb{R}^d)'} = \|u\|_{H^{-s}(\mathbb{R}^d)}\,.$$

Now let $\Phi \in H^s(\mathbb{R}^d)'$. Then the map $w \mapsto \Phi(\mathcal{F}^{-1}\,\langle\xi\rangle^{-s}\,w)$ is a continuous linear form on $L^2(\mathbb{R}^d)$. By the Riesz Theorem, there exists $\tilde{u} \in L^2(\mathbb{R}^d)$ such that

$$\forall w \in L^2(\mathbb{R}^d), \quad \Phi\big(\mathcal{F}^{-1}\,\langle\xi\rangle^{-s}\,w\big) = \langle\tilde{u}, w\rangle_{L^2(\mathbb{R}^d)}\,.$$

Then for $v \in H^s(\mathbb{R}^d)$ we apply this equality with $w = \langle\xi\rangle^s\,\mathcal{F}v$ to get

$$\Phi(v) = \langle\tilde{u}, \langle\xi\rangle^s\,\mathcal{F}v\rangle_{L^2(\mathbb{R}^d)} = \Phi_u(v),$$

where we have set $u = \mathcal{F}^{-1}\,\langle\cdot\rangle^s\,\tilde{u} \in H^{-s}(\mathbb{R}^d)$. This proves that the map $u \in H^{-s}(\mathbb{R}^d) \mapsto \Phi_u \in H^s(\mathbb{R}^d)'$ is surjective. The proof is complete. □

***Exercise* 27.** Let $s \in ]0,1[$.
**1.** Let $z \in \mathbb{R}^d$. Give an expression of

$$\int_{y\in\mathbb{R}^d} |u(y+z) - u(y)|^2 \, dy$$

in terms of $\hat{u}$.
**2.** Prove that there exists $C > 0$ such that for all $u \in \mathcal{S}(\mathbb{R}^d)$ we have

$$\int_{y\in\mathbb{R}^d} \int_{x\in\mathbb{R}^d} \frac{|u(x) - u(y)|^2}{|x-y|^{d+2s}} \, dx \, dy = C \int_{\xi\in\mathbb{R}^d} |\xi|^{2s} |\hat{u}(\xi)|^2 \, d\xi.$$

**3.** Deduce that the quantity

$$\left( \|u\|^2_{L^2(\mathbb{R}^d)} + \int_{y\in\mathbb{R}^d} \int_{x\in\mathbb{R}^d} \frac{|u(x) - u(y)|^2}{|x-y|^{d+2s}} \, dx \, dy \right)^{\frac{1}{2}}$$

defines a norm on $H^s(\mathbb{R}^d)$, equivalent to the usual one. The interest of this expression is that it no longer use the Fourier transform an can be used to define $H^s(\Omega)$ on more genral domains $\Omega$.

### 2.8.2 Green formula for less regular functions

In this additional paragraph we continue the discussion about traces and the Green formula. In particular we define, via the Green Formula, a normal derivative for functions which are not in $H^2(\Omega)$.

We have denoted by $H^{1/2}(\Omega) \subset L^2(\Omega)$ the range of the trace $\gamma_0$ defined on $H^1(\Omega)$. This is a vector space, which can be endowed with the following norm.

$$\|g\|_{H^{1/2}(\partial\Omega)} = \inf_{\substack{w\in H^1 \\ \gamma_0(w)=g}} \|w\|_{H^1(\Omega)} .$$

We notice that $H^1_g(\Omega) = \{w \in H^1(\Omega) : \gamma_0(w) = g\}$ is a nonempty (by definition of $H^{1/2}(\partial\Omega)$) and closed (since $\gamma_0$ is continuous) affine subspace (since $\gamma_0$ is linear) of the Hilbert space $H^1(\Omega)$, so by the Hilbert projection theorem there exists a unique $R(g) \in H^1_g(\Omega)$ such that

$$\|g\|_{H^{1/2}(\partial\Omega)} = \|R(g)\|_{H^1(\Omega)} .$$

Moreover $R(g)$ is the only solution in $H^1_g(\Omega)$ of

$$\forall v \in H^1_0(\Omega), \quad \langle R(g), v\rangle_{H^1(\Omega)} = 0.$$

From this we can deduce that the application which maps $g \in H^{1/2}(\Omega)$ to $R(g) \in H^1(\Omega)$ is linear, and then that $H^{1/2}(\partial\Omega)$ is a Banach space:

**Proposition 2.62.** $H^{1/2}(\partial\Omega)$ *is a Banach space.*

*Proof.* Let $(g_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $H^{1/2}(\Omega)$. Then $(R(g_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in $H^1(\Omega)$. Since $H^1(\Omega)$ is complete, $R(g_n)$ tends to some $w$ in $H^1(\Omega)$. We set $g = \gamma_0(w) \in H^{1/2}(\partial\Omega)$. Then we have

$$\|g_n - g\|_{H^{1/2}(\partial\Omega)} = \|R(g-g_n)\|_{H^1(\Omega)} = \|R(g) - R(g_n)\|_{H^1(\Omega)} \xrightarrow[n\to+\infty]{} 0.$$

This proves that the sequence $(g_n)_{n\in\mathbb{N}}$ has a limit in $H^{1/2}(\partial\Omega)$, and hence that $H^{1/2}(\partial\Omega)$ is complete. □

We denote by $H^{-1/2}(\partial\Omega)$ the dual of $H^{1/2}(\partial\Omega)$.

**Proposition 2.63.** *Let $u \in H^1(\Omega)$ such that $\Delta u \in L^2(\Omega)$. Then the map*

$$g \in H^{1/2}(\Omega) \mapsto \int_\Omega \big(\Delta u\, v_g + \nabla u \cdot \nabla v_g\big)\, dx, \tag{2.28}$$

*where $v_g \in H^1(\Omega)$ satisfies $\gamma_0(v_g) = g$ is well defined (the definition does not depend on the choice of $v_g$) and defines a continuous linear map on $H^{1/2}(\partial\Omega)$ which we denote by $\partial_\nu u$.*

We recall that in a general domain $\Omega$ the assumptions that $u \in H^1(\Omega)$ and $\Delta u \in L^2(\Omega)$ do not imply that $u \in H^2(\Omega)$.

*Proof.* We first observe that if $v_1$ and $v_2$ in $H^1(\Omega)$ are such that $\gamma_0(w_1) = \gamma_0(v_2) = g$ then $v_1 - v_2$ belongs to $H^1_0(\Omega)$, so there exists a sequence $(\phi_n)_{n\in\mathbb{N}}$ in $C_0^\infty(\Omega)$ which goes to $v_1 - v_2$ in $H^1(\Omega)$. For all $n \in \mathbb{N}$ we have

$$\int_\Omega \big(\Delta u\, \phi_n + \nabla u \cdot \nabla \phi_n\big)\, dx = \langle \Delta u, \phi_n\rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)} + \langle \nabla u, \nabla \phi_n\rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)} = 0,$$

so, taking the limit $n \to +\infty$,

$$\int_\Omega \big(\Delta u\, v_1 + \nabla u \cdot \nabla v_1\big)\, dx = \int_\Omega \big(\Delta u\, v_2 + \nabla u \cdot \nabla v_2\big)\, dx.$$

This proves that the definition in (2.28) does not depend on the choice of $v_g$, and the map $\partial_\nu u$ is well-defined on $H^{1/2}(\partial\Omega)$.

For $g \in H^{1/2}(\partial\Omega)$ we have

$$\left|\int_\Omega \big(\Delta u\, v_g + \nabla u \cdot \nabla v_g\big)\, dx\right| \leqslant \Big(\|\Delta u\|_{L^2(\Omega)} + \|\nabla u\|_{L^2(\Omega)}\Big) \|v_g\|_{H^1(\Omega)},$$

and hence

$$\left|\int_\Omega \big(\Delta u\, v_g + \nabla u \cdot \nabla v_g\big)\, dx\right| \leqslant \Big(\|\Delta u\|_{L^2(\Omega)} + \|\nabla u\|_{L^2(\Omega)}\Big) \|g\|_{H^{1/2}(\partial\Omega)}.$$

This proves that the map $\partial_\nu u$ is continuous on $H^{1/2}(\partial\Omega)$. Since it is also linear, this defines an element of $H^{-1/2}(\partial\Omega)$. □

By definition, we have the following Green formula for $u, v \in H^1(\Omega)$ such that $\Delta u \in L^2$:

$$-\int_\Omega \Delta u\, v\, dx = -\left\langle \partial_\nu u, v \right\rangle_{H^{-1/2}(\partial\Omega), H^{1/2}(\partial\Omega)} + \int_\Omega \nabla u \cdot \nabla v. \tag{2.29}$$

We finish this section about traces by giving a general result on $\Omega = \mathbb{R}^d_+$ by means of the Fourier transform. This will in particular ensure that the two definitions of $H^{1/2}$ on $\mathbb{R}^{d-1} \simeq \partial\mathbb{R}^d_+$ are equivalent, and that the trace on $H^1(\Omega)$ and the normal trace on $H^2(\Omega)$ have the same range.

**Theorem 2.64.** *Let $k \in \mathbb{N}$ and $s > k + \frac{1}{2}$. Then the map*

$$\begin{cases} \mathcal{S}(\mathbb{R}^d) & \to & \mathcal{S}(\mathbb{R}^{d-1}) \\ u & \mapsto & \partial_1^k u(0, \cdot) \end{cases}$$

*has a unique continuous extansion $\gamma_k : H^s(\mathbb{R}^d) \to H^{s-k-\frac{1}{2}}(\mathbb{R}^{d-1})$. Moreover, $\gamma_k$ is surjective and there exists a continuous linear map $R_k : H^{s-k-\frac{1}{2}}(\mathbb{R}^{d-1}) \to H^s(\mathbb{R}^d)$ such that*

$$\gamma_k \circ R_k = \mathrm{Id}_{H^{s-k-\frac{1}{2}}(\mathbb{R}^{d-1})}.$$

*Proof.* • We first observe that for $m \in \mathbb{N}$, $\eta > 0$ and $\sigma > \frac{m+1}{2}$ we have, with the change of variable $t = \sqrt{\eta}\theta$

$$\int_\mathbb{R} t^m \left(\eta + t^2\right)^{-\sigma} dt = \eta^{\frac{m+1}{2} - \sigma} C_{m,\sigma}, \quad \text{where} \quad C_{m,\sigma} = \int_\mathbb{R} \theta^m (1 + \theta^2)^{-\sigma}\, d\theta. \tag{2.30}$$

• Let $\phi \in \mathcal{S}(\mathbb{R}^d)$. For $x' \in \mathbb{R}^{d-1}$ we have by the inversion formula

$$\partial_1^k \phi(0, x') = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^{d-1}} e^{ix' \cdot \xi'} \left( \int_\mathbb{R} (i\xi_1)^k \hat{\phi}(\xi_1, \xi')\, d\xi_1 \right) d\xi',$$

so the Fourier transform (in $\mathbb{R}^{d-1}$) of $\partial_1^k \phi(0, \cdot)$ is given by

$$g : \xi' \mapsto \frac{1}{2\pi} \int_\mathbb{R} (i\xi_1)^k \hat{\phi}(\xi_1, \xi')\, d\xi_1. \tag{2.31}$$

By the Cauchy-Schwarz inequality and (2.30) applied with $\eta = 1 + |\xi'|^2$ we have, for all $\xi' \in \mathbb{R}^{d-1}$,

$$\begin{aligned} 4\pi^2 \left|g(\xi')\right|^2 &\leqslant \left( \int |\hat{\phi}(\xi_1, \xi')|^2 \left(1 + \xi_1^2 + |\xi'|^2\right)^s d\xi_1 \right) \left( \int_\mathbb{R} \xi_1^{2k} \left(1 + \xi_1^2 + |\xi'|^2\right)^{-s} d\xi_1 \right) \\ &\leqslant C_{2k,s} \left(1 + |\xi'|^2\right)^{-(s-k-\frac{1}{2})} \int |\hat{\phi}(\xi_1, \xi')|^2 \left(1 + \xi_1^2 + |\xi'|^2\right)^s d\xi_1. \end{aligned}$$

Multiplying by $\left(1 + |\xi'|^2\right)^{s-k-\frac{1}{2}}$ and integrating over $\xi' \in \mathbb{R}^{d-1}$ gives

$$\left\| (\partial_1^k \phi)(0, \cdot) \right\|^2_{H^{s-k-\frac{1}{2}}(\mathbb{R}^{d-1})} \leqslant \frac{C_{2k,s}}{4\pi^2} \left\|\phi\right\|^2_{H^s(\mathbb{R}^d)}.$$

This proves the first statement of the theorem.

• Now we prove that $\gamma_k$ is surjective with a continuous right inverse. We begin with $v \in \mathcal{S}(\mathbb{R}^{d-1})$. Let $g \in \mathcal{S}(\mathbb{R}^{d-1})$ be the Fourier transform of $v$ on $\mathbb{R}^{d-1}$. The expression (2.31) suggests to find $f$ such that

$$g(\xi') = \frac{1}{2\pi} \int_{\mathbb{R}} (i\xi_1)^k f(\xi_1, \xi')\, d\xi_1. \tag{2.32}$$

Let $N > \frac{1}{2}\big(s - k - \frac{1}{2}\big)$. For $\xi = (\xi_1, \xi') \in \mathbb{R}^d$ we set

$$f(\xi) = \frac{2\pi}{C_{k,N+\frac{1}{2}}} \frac{(-i)^k \big(1 + |\xi'|^2\big)^N}{\big(1 + |\xi|^2\big)^{N+\frac{k}{2}+\frac{1}{2}}} g(\xi').$$

In particular, for all $\xi' \in \mathbb{R}^{d-1}$ the map $\xi_1 \mapsto (-i\xi_1)^k f(\xi_1, \xi')$ is integrable on $\mathbb{R}$ and (2.32) holds by (2.30). Moreover, by (2.30) again we have

$$\begin{aligned} &\int_{\mathbb{R}^d} (1 + |\xi|^2)^s \,|f(\xi)|^2 \, d\xi \\ &\quad = \frac{4\pi^2}{C_{k,N+\frac{1}{2}}^2} \int_{\mathbb{R}^{d-1}} (1 + |\xi'|^2)^{2N} \,|g(\xi')|^2 \left( \int_{\mathbb{R}} (1 + |\xi|^2)^{-(2N+k+1-s)}\, d\xi_1 \right) d\xi \\ &\quad = \frac{4\pi^2 C_{0,2N+k+1-s}}{C_{k,N+\frac{1}{2}}^2} \int_{\mathbb{R}^{d-1}} (1 + |\xi'|^2)^{s-k-\frac{1}{2}} \,|g(\xi')|^2 \, d\xi' \end{aligned}$$

Then if we denote by $u$ the inverse Fourier transform of $f$ we obtain that $u \in H^s(\mathbb{R}^d)$ and

$$\|u\|_{H^s(\mathbb{R}^d)}^2 \leqslant \frac{4\pi^2 C_{0,2N+k+1-s}}{C_{k,N+\frac{1}{2}}^2} \|v\|_{H^{s-k-\frac{1}{2}}(\mathbb{R}^{d-1})}^2. \tag{2.33}$$

Moreover (2.32) ensures that $\gamma_k(u) = v$. Thus we have defined a map $R_k : \mathcal{S}(\mathbb{R}^{d-1}) \to H^s(\mathbb{R}^d)$ such that $\gamma_k \circ R_k = \mathrm{Id}$. By (2.33), $R_k$ extends to a continuous map from $H^{s-k-\frac{1}{2}}(\mathbb{R}^{d-1})$ to $H^s(\mathbb{R}^d)$, and the proof is complete. □

### 2.8.3 The dual of $H_0^1(\Omega)$

**Définition 2.65.** We denote by $H^{-1}(\Omega)$ the dual space of $H_0^1(\Omega)$.

We recall that the dual space of $H_0^1(\Omega)$ is the set of continuous linear forms on $H_0^1(\Omega)$. It is endowed with the norm defined by

$$\|\varphi\|_{H^{-1}(\Omega)} = \sup_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{|\varphi(u)|}{\|u\|_{H_0^1(\Omega)}}.$$

We usually write $\langle \varphi, u \rangle$ (or $\langle \varphi, u \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}$) instead of $\varphi(u)$. Notice that if $H^1(\Omega) \neq H_0^1(\Omega)$ then $H^{-1}(\Omega)$ *is not* the dual space of $H^1(\Omega)$.

We recall that by the Riesz Theorem, we can identify a Hilbert space with its dual. However, in this kind of context we usually already identify $L^2(\Omega)$ with its dual. With this identification we have

$$H_0^1(\Omega) \subset L^2(\Omega) \subset H^{-1}(\Omega),$$

with continuous injections. The first inclusion is clear by definition of the Sovolev space $H_0^1(\Omega)$. Now a function $u \in L^2(\Omega)$ is identified with the continuous linear form on $L^2(\Omega)$ defined by

$$v \mapsto \langle u, v \rangle_{L^2(\Omega)} . \tag{2.34}$$

By restriction, this also defines a continuous linear form on $H_0^1(\Omega)$. In this sense, we can say that $u$ belongs to $H^{-1}(\Omega)$. However, all the elements of $H^{-1}(\Omega)$ cannot be identified with a function in $L^2(\Omega)$. For instance, on $\mathbb{R}$, the Dirac distribution

$$\delta : v \mapsto v(0) \tag{2.35}$$

defines a continuous linear for on $H^1(\mathbb{R}) = H_0^1(\mathbb{R})$, and it is not of the form (2.34) (notice that this example is specific to the dimension 1, a Dirac distribution is not in $H^{-1}(\Omega)$ in dimension $d \geqslant 2$, however with the trace Theorem we can generalize this example in higher dimension, see Exercise 28).

Let $f \in L^2(\Omega)$ and $F \in L^2(\Omega, \mathbb{R}^d)$. Then $\varphi = f - \operatorname{div} F$, where the derivatives are understood in the sense of distributions, also defines a continuous linear form on $H_0^1(\Omega)$ (which is not necessarily in $L^2(\Omega)$). For $v \in H_0^1(\Omega)$ it is given by

$$\varphi(v) = \langle f, v \rangle + \sum_{j=1}^{d} \langle F_j, \partial_j u \rangle .$$

In particular we have

$$\|\varphi\|_{H^{-1}(\Omega)} \leqslant \|f\|_{L^2(\Omega)} + \sum_{j=1}^{d} \|F_j\|_{L^2(\Omega)} . \tag{2.36}$$

In fact, using the Riesz Theorem in $H_0^1(\Omega)$ we see that any $\varphi \in H^{-1}(\Omega)$ can be written in this form with $u \in H_0^1(\Omega)$ and $F = \nabla u$. Moreover, in this case we have an equality in (2.36). See Theorem 5.9.1 in [Evans] (see Exercise 29 for the particular case of the Dirac distribution (2.35)).

***Exercise* 28.** Let $f \in L^2(\mathbb{R})$. Prove that the map

$$v \in C_0^\infty(\mathbb{R}^2) \mapsto \int_{\mathbb{R}} f(x) v(x,0)\, dx$$

extends to a continuous linear form on $H^1(\mathbb{R}^2)$.

***Exercise* 29.** Find $u \in H^1(\mathbb{R})$ such that

$$\forall v \in H^1(\mathbb{R}), \quad v(0) = \int_{\mathbb{R}} uv + \int_{\mathbb{R}} u'v'.$$