Pure and Applied Mathematics Quarterly
Volume 17, Number 3, 971–990, 2021

Pluripotential solutions versus viscosity solutions to complex Monge-Ampère flows

Vincent Guedj, Chinh H. Lu, and Ahmed Zeriahi*

Dedicated to Duong Hong Phong on the occasion of his 65th birthday

Abstract: We compare various notions of weak subsolutions to degenerate complex Monge-Ampère flows, showing that they all coincide. This allows us to show that the viscosity solution coincides with the envelope of pluripotential subsolutions.

Keywords: Parabolic Monge-Ampère equation, pluripotential solution, viscosity solution, Perron envelope.

1. Introduction

A viscosity approach for parabolic complex Monge-Ampère equations (both in local and global contexts) has been developed in [EGZ15, EGZ16, EGZ18, DLT19], while a pluripotential approach has been developed in [GLZ1, GLZ2], which allows to solve these equations with quite degenerate data. The goal of this paper is to compare these two notions, extending the dictionary established in the elliptic case (see [EGZ11, HL13, GLZ17]).

Let Ω be a smooth bounded strictly pseudoconvex domain of $\mathbb{C}^n$. We consider the parabolic complex Monge-Ampère flow in Ω_T

$$(dd^c\varphi_t)^n = e^{\dot{\varphi}_t+F(t,z,\varphi)}g(z)dV(z). \tag{1.1}$$

Here

- $T > 0$ and $\Omega_T =]0,T[\times\Omega$ with parabolic boundary

$$\partial_0\Omega_T := \{0\} \times \Omega \cup [0,T[\times\partial\Omega;$$

arXiv: 1909.07069
Received April 6, 2019.
2010 Mathematics Subject Classification: Primary 53C44, 32W20; secondary 58J35.
*The authors are partially supported by the ANR project GRACK.

- $F : [0,T] \times \Omega \times \mathbb{R} \to \mathbb{R}$ is a continuous function;
- dV denotes the euclidean volume form in $\mathbb{C}^n$;
- $0 \leq g$ is a continuous function on Ω;
- $(t,x) \mapsto \varphi(t,x) = \varphi_t(x)$ is the unknown function and $\dot{\varphi}_t = \partial_t \varphi$ denotes the time derivative of φ.

We assume throughout this article that $h : \partial_0 \Omega_T \to \mathbb{R}$ is a continuous Cauchy-Dirichlet boundary data, i.e.

- h is continuous on $\partial_0 \Omega_T$, and
- h_0 is a continuous plurisubharmonic function in Ω.

We first extend the definition of pluripotential subsolutions proposed in [GLZ1]. This new definition applies to functions which are not necessarily locally Lipschitz in t, it thus allows us to consider (1.1) for less regular data.

We then show that these pluripotential parabolic subsolutions coincide with viscosity subsolutions:

Theorem A. *Assume $\varphi \in \mathcal{P}(\Omega_T)$. The following are equivalent:*

(i) φ is a viscosity subsolution to (1.1)*;*
(ii) φ is a pluripotential subsolution to (1.1)*.*

Here $\mathcal{P}(\Omega_T)$ denotes the set of parabolic potentials, i.e. locally integrable upper semi-continuous functions φ in Ω_T whose slices $\varphi_t = \varphi(t,\cdot)$ are plurisubharmonic in Ω.

The pluripotential parabolic comparison principle [GLZ1, Theorem 6.5] then allows us to conclude that the envelope of pluripotential subsolutions is the unique viscosity solution to (1.1):

Theorem B. *Assume that $g > 0$ is positive almost everywhere in Ω. Then there is a unique viscosity solution to* (1.1) *with boundary value h which coincides with the envelope of all pluripotential subsolutions.*

The techniques developed in the local context allow us to obtain analogous results in the compact setting, comparing viscosity and pluripotential notions for complex Monge-Ampère flows that contain the Kähler-Ricci flow as a particular case. These are briefly discussed in Section 5.

2. Pluripotential subsolutions

Let Ω be a smoothly bounded strongly pseudoconvex domain in $\mathbb{C}^n$. By this we mean there exists a smooth strictly plurisubharmonic function ρ in an open neighborhood of $\bar{\Omega}$ such that $\Omega = \{\rho < 0\}$ and $d\rho \neq 0$ on $\partial\Omega$.

Definition 2.1. The set of parabolic potentials $\mathcal{P}(\Omega_T)$ consists of upper semicontinuous functions $u : \Omega_T :=]0,T[\times\Omega \longrightarrow [-\infty,+\infty[$ such that $u \in L^1_{\text{loc}}(\Omega_T)$ and $\forall t \in]0,T[$, the slice $u_t : z \mapsto u(t,z)$ is plurisubharmonic in Ω.

Let us stress that – by comparison with [GLZ1] – we do not assume here that the family $\{u(\cdot,z)\ ;\ z \in \Omega\}$ is locally uniformly Lipschitz in $]0,T[$. We nevertheless use the same notation $\mathcal{P}(\Omega_T)$ for the set of parabolic potentials, hoping that no confusion will arise.

A pluripotential subsolution is a parabolic potential φ that satisfies

$$(dd^c\varphi)^n \wedge dt \geq e^{\dot{\varphi}_t + F(t,z,\varphi)} g(z) dV(z) \wedge dt$$

in the weak sense of (positive) measures in Ω_T.

We need to make sense of all these quantities. The LHS is defined as in [GLZ1] by using Bedford-Taylor's theory, the novelty here concerns mainly the RHS as we explain hereafter.

2.1. Defining the LHS

The LHS can be defined by using Bedford-Taylor theory:

Lemma 2.2. *If $u \in \mathcal{P}(\Omega_T) \cap L^\infty_{\text{loc}}(\Omega_T)$ then $dt \wedge (dd^c u_t)^n$ is well-defined as a positive Borel measure in Ω_T.*

Proof. Fix χ a test function in Ω_T with support contained in $J \times D \Subset \Omega_T$. We regularize u by taking sup convolution: for $(t,z) \in J \times D$ we set

$$u^j(t,z) := \sup\{u(s,z) - j^2(t-s)^2\ ;\ s \in]0,T[\,\}.$$

The functions u^j decrease pointwise to u on $J \times D$ (by upper semicontinuity of u). Since $t \mapsto u^j$ is continuous, it follows from [GLZ1, Lemma 2.1] that the function

$$t \mapsto \int_\Omega \chi(t,z)(dd^c u_t^j)^n$$

is continuous in t. It follows from [BT82] that

$$\lim_{j\to+\infty} \int_\Omega \chi(t,z)(dd^c u_t^j)^n = \int_\Omega \chi(t,z)(dd^c u_t)^n.$$

Taking limits as $j \to +\infty$ we obtain that $t \mapsto \int_\Omega \chi(t,z)(dd^c u_t)^n$ is a bounded Borel measurable function in $]0,T[$. The Chern-Levine-Nirenberg inequalities yield

$$\left| \int_{\Omega_T} \chi(t,z) dt \wedge (dd^c u_t)^n \right| \leq C(J,D,u) \sup_{\Omega_T} |\chi|,$$

where $C(J, D, u) > 0$ is a constant. It thus follows that the distribution $dt \wedge (dd^c u_t)^n$ extends as a positive Borel measure in Ω_T. □

2.2. Defining the RHS

For each $u \in \mathcal{P}(\Omega_T)$, we define $g\partial_t u$ as a distribution on Ω_T by setting

$$\langle g\partial_t u, \chi\rangle := -\int_\Omega \int_0^T \partial_t \chi(t,z) u(t,z) g(z) dt dz,$$

for all test functions $\chi \in \mathcal{C}^\infty(\Omega_T)$ with compact support.

To define pluripotential subsolutions, we wish to interpret the RHS as a supremum of (signed) Radon measures, setting

$$e^{\dot\varphi_t + F(t,z,\varphi)} g = g \sup_{a>0} \{a(\partial_t \varphi + F(t,z,\varphi_t(z)) - a\log a + a\}.$$

This relies on the following observation:

Lemma 2.3. *Let T be a positive measure in an open set $D \subset \mathbb{R}^N$, f a bounded measurable function on D, and $0 \leq g \in L^p(D)$. If, for all $a > 0$,*

$$T \geq g(af + a - a\log a)\lambda_N,$$

in the sense of measures, then $T \geq e^f g$ in the sense of measures in D.

Here λ_N denotes the Lebesgue measure in D.

Proof. We first assume that $g \geq b > 0$ on D. Replacing T with T/g we can assume that $g \equiv 1$. We regularize T by using non-negative mollifiers, setting $T_\varepsilon := T \star \rho_\varepsilon$. Then for all $a > 0$

$$T_\varepsilon \geq af \star \rho_\varepsilon + a - a\log a,$$

pointwise on D. Taking the supremum over $a > 0$ we obtain

$$T_\varepsilon \geq e^{f \star \rho_\varepsilon}$$

pointwise on D. The inequality thus also holds in the sense of measures. Letting $\varepsilon \to 0$ yields the conclusion.

We now remove the positivity condition on g. Since f is bounded, for each $\varepsilon > 0$ we can find $c(\varepsilon) > 0, A > 0$ such that, for all $a \in]0, A[$,

$$T + \varepsilon\lambda_N \geq (g + c(\varepsilon))(af - a\log a + a)\lambda_N,$$

It follows from the first step and the fact that f is bounded (so that the supremum can be restricted to $a \in]0, A[$) that

$$T + \varepsilon\lambda_N \geq (g + c(\varepsilon))e^f \lambda_N$$

in the sense of measures on D. The conclusion follows by letting $\varepsilon \to 0$. □

This analysis motivates the following:

Definition 2.4. Let $u \in \mathcal{P}(\Omega_T) \cap L^\infty_{\mathrm{loc}}(\Omega_T)$. Then u is a pluripotential subsolution to (1.1) if for all constants $a > 0$,

$$(dd^c\varphi)^n \wedge dt \geq g(a(\partial_t\varphi + F(t, z, \varphi_t(z)) - a\log a + a)\, dV(z) \wedge dt$$

in the sense of distribution in Ω_T.

If $u \in \mathcal{P}(\Omega_T) \cap L^\infty_{loc}(\Omega_T)$ is locally uniformly semi-concave in $t \in]0, T[$, then by Lemma 2.3 u is a pluripotential subsolution to (1.1) iff

$$(dd^c u_t)^n \geq e^{\partial_t^+ u + F(t,z,u_t)} g dV,$$

in the sense of Radon measures in Ω. Here ∂_t^+ is the right derivative defined pointwise in Ω_T (thanks to the semi-concavity property of $t \mapsto u(t, z)$). The above definition thus coincides with the one given in [GLZ1].

Decreasing limits of pluripotential subsolutions are again subsolutions as the following result shows:

Lemma 2.5. *Let (u^j) be a sequence of pluripotential subsolutions to (1.1) which decreases to $u \in \mathcal{P}(\Omega_T) \cap L^\infty_{\mathrm{loc}}(\Omega_T)$. Then u is a pluripotential subsolution to (1.1).*

Proof. It follows from [BT82] that the Radon measures $(dd^c u^j)^n \wedge dt$ weakly converge to $(dd^c u)^n \wedge dt$. On the other hand for each $a > 0$

$$g(a(\partial_t u^j + F) + a - a\log a) \to g(a(\partial_t u + F) + a - a\log a)$$

in the weak sense of distributions in Ω_T. This completes the proof. □

Let us emphasize that in Definition 2.4 we do not ask subsolutions to be locally uniformly Lipschitz in t while the definition given in [GLZ1] does assume this regularity. We observe below that the envelopes of subsolutions in both senses do coincide.

Proposition 2.6. *Assume that the data (F, h, g, u_0) satisfy the assumption of [GLZ1]. Let U be the upper envelope of pluripotential subsolutions to (1.1)* *in the sense of Definition 2.4, and $\tilde{U}$ be the envelope of subsolutions to (1.1) in the sense of [GLZ1]. Then $U = \tilde{U}$.*

Proof. By definition we have $\tilde{U} \leq U$. Fix u a pluripotential subsolution to (1.1) in the sense of Definition 2.4. We regularize u by taking convolution (see [GLZ1])

$$u^{\varepsilon}(t,z) := \int_{\mathbb{R}} u(st,z)\chi((s-1)/\varepsilon)ds,$$

where χ is a cut-off function. Then $u^{\varepsilon} - c(\varepsilon)(t+1)$ is a pluripotential subsolution to (1.1) with data (F,h,g,u_0), where $c(\varepsilon) \to 0$ as $\varepsilon \to 0$. Hence $u^{\varepsilon} - O(\varepsilon)(t+1) \leq \tilde{U}$. Letting $\varepsilon \to 0$ we arrive at $u \leq \tilde{U}$, hence $U \leq \tilde{U}$. □

3. Viscosity vs pluripotential subsolutions

3.1. Viscosity concepts

We now recall the corresponding viscosity notions introduced in [EGZ15].

Definition 3.1. Given $u : \Omega_T \to \mathbb{R}$ an u.s.c. bounded function and $(t_0,x_0) \in X_T$, q is a differential test from above for u at (t_0,x_0) if

- $q \in \mathcal{C}^{1,2}$ in a small neighborhood V_0 of (t_0,x_0);
- $u \leq q$ in V_0 and $u(t_0,x_0) = q(t_0,x_0)$.

Definition 3.2. An u.s.c. bounded function $u : \Omega_T \to \mathbb{R}$ is a *viscosity subsolution* to (1.1) if for all $(t_0,x_0) \in \Omega_T$ and all differential tests q from above,

$$(dd^c q_{t_0}(x_0))^n \geq e^{\dot{q}_{t_0}(x_0)+F(t_0,x_0,u(t_0,x_0))} g(x_0)dV(x_0).$$

Here are few basic facts about viscosity subsolutions:

- a $\mathcal{C}^{1,2}$-smooth function is a viscosity subsolution iff it is psh and a classical subsolution;
- if u_1,u_2 are viscosity subsolutions, then so is $\max(u_1,u_2)$;
- if $(u_\alpha)_{\alpha\in A}$ is a family of subsolutions which is locally uniformly bounded from above, then $\varphi := (\sup\{u_\alpha \,;\, \alpha \in A\})^*$ is a subsolution;
- If u is a subsolution to $(1.1)_g$ then it is also a subsolution to $(1.1)_f$ with g replaced by f, as long as $0 \leq f \leq g$.
- u is a subsolution to (1.1) with $g \equiv 0$ iff u_t is psh for all t.

Definition 3.3. A bounded l.s.c. function $u : \Omega_T \to \mathbb{R}$ is a viscosity supersolution to (1.1) if for all $(t_0,z_0) \in \Omega_T$ and all differential tests q from below,

$$(dd^c q_{t_0}(x_0))^n_+ \leq e^{\dot{q}_{t_0}(x_0)+F(t_0,x_0,u(t_0,x_0))} g(x_0)dV(x_0).$$

Here, for a real $(1,1)$-form α we define α_+ to be α if it is semipositive and 0 otherwise.

Definition 3.4. A function u is a viscosity solution to (1.1) if it is both a viscosity subsolution and a viscosity supersolution to (1.1).

Note in particular that viscosity solutions are continuous functions.

In viscosity theory it is convenient to define the notion of relaxed upper and lower limits of a family of functions. Let $\phi^\epsilon : (E,d) \to \mathbb{R}$, $\epsilon > 0$ be a family of locally uniformly bounded functions on a metric space (E,d). We set

$$\begin{aligned}\underline{\phi}(x) = \liminf{}_* \phi^\epsilon(x) &:= \liminf_{(\epsilon,y)\to(0,x)} \phi^\epsilon(y)\\ \overline{\phi}(x) = \limsup{}^* \phi^\epsilon(x) &:= \limsup_{(\epsilon,y)\to(0,x)} \phi^\epsilon(y).\end{aligned}$$

Observe that $\underline{\phi}$ (resp. $\overline{\phi}$) is lower (resp. upper) semi-continuous on E and $\underline{\phi} \leq (\liminf_{\epsilon\to 0^+} \phi^\epsilon)_*$. If the family is constant and equal to ϕ, $\underline{\phi} = \phi_*$ and $\overline{\phi} = \phi^*$ correpond to the lower and upper semi-continuous regularisations of ϕ respectively.

Lemma 3.5. *Assume that $(F^\epsilon)_{0<\epsilon<\epsilon_0}$ is a family of continuous functions on $]0,T[\times\Omega\times\mathbb{R}$ which converges locally uniformly to F, and let $(g^\epsilon)_{0<\epsilon<\epsilon_0}$ be a family of continuous non negative functions on Ω which converges uniformly to g.*

Assume that for any $0 < \epsilon < \epsilon_0$, $u^\epsilon : \Omega_T \longrightarrow \mathbb{R}$ is a viscosity subsolution (resp. supersolution) to the equation (1.1) for the data (F^ϵ, g^ϵ). Then the function $\overline{u}$ (resp. $\underline{u}$) is a viscosity subsolution (resp. supersolution) to the equation (1.1) *for the data (F,g).*

The proof below is essentially classical (see [DI04]) but we give a complete account for the reader's convenience.

Proof. We prove the statement for supersolutions. The dual arguments work for subsolutions.

Let q be a lower test function for $\underline{u}$ at $\zeta_0 := (t_0,z_0) \in]0,T[\times\Omega$. Fix $r > 0$ such that $D_r := [t_0 - r, t_0 + r] \times \bar{B}(z_0,r) \subset \Omega$. By definition there exists a sequence $(\zeta_j)_{j\in\mathbb{N}}$ in D_r converging to ζ_0 and a sequence $(\varepsilon_j)_{j\in\mathbb{N}}$ decreasing to 0 such that $\lim_{j\to+\infty} u^{\varepsilon_j}(\zeta_j) = \underline{u}(\zeta_0)$.

Fix $\delta > 0$ and set

$$p(z) := q(t,z) - u^{\varepsilon_j}(t,z) - \delta(|z - z_0|^2 + (t - t_0)^2),\ z \in D_r.$$

For each $j \in \mathbb{N}$ let $w_j := (t_j, z_j)$ be a point in D_r such that $p(w_j) = \max_{D_r} p$. We have

$$q(\zeta_j) - u^{\varepsilon_j}(\zeta_j) - \delta|\zeta_j - \zeta_0|^2 = p(\zeta_j) \le p(w_j) = q(w_j) - u^{\varepsilon_j}(w_j) - \delta|w_j - \zeta_0|^2.$$

Taking a subsequence if necessary we can assume that $w_j \to w_0 \in D_r$. Then letting $j \to +\infty$ and taking into account the fact that

$$\liminf_{j \to +\infty} u^{\varepsilon_j}(w_j) \ge \underline{u}(w_0),$$

we obtain

$$q(\zeta_0) - \underline{u}(\zeta_0) \le q(w_0) - \underline{u}(w_0) - \delta|w_0 - \zeta_0|^2.$$

This implies that $\zeta_0 = w_0$, since q is a lower test function for $\underline{u}$ at ζ_0. Hence the sequence (w_j) converges to ζ_0 and then for j large enough w_j is in the interior of D_r. By definition of w_j, it follows that for j large enough, the function $q_j(t,z) := q(t,z) - \delta(|z - z_0|^2 + (t - t_0)^2)$ is a lower test function for u^{ε_j} at the point w_j. Since u^{ε_j} is a supersolution to the equation (1.1) for the data $(F^{\epsilon_j}, g^{\epsilon_j})$, it follows that at the point $w_j = (t_j, z_j)$ we have

$$(dd^c q - \delta\beta)_+^n \le e^{\partial_t q(t_j,z_j) - 2\delta(t_j - t_0) + F^{\varepsilon_j}(t_j, z_j, q(t_j,z_j))} g^{\varepsilon_j}(z_j) dV, \tag{3.1}$$

where $\beta = dd^c|z|^2$ is the standard Kähler form on $\mathbb{C}^n$.

We want to prove that at $\zeta_0 = (t_0, z_0)$ we have

$$(dd^c q)_+^n \le e^{\partial_t q + F(t_0, z_0, q(t_0,z_0))} g(z_0) dV.$$

If $dd^c q(z_0)$ has an eigenvalue ≤ 0 then $(dd^c q)_+^n(z_0) = 0$ and the inequality is trivial. If $dd^c q(z_0) > 0$ then letting $j \to +\infty$ and then $\delta \to 0$ in (3.1) we arrive at the desired inequality. □

3.2. Comparison of subsolutions

The main result of this note provides an identification between viscosity and pluripotential subsolutions:

Theorem 3.6. *Let $u \in \mathcal{P}(\Omega_T) \cap L^\infty_{\mathrm{loc}}(\Omega_T)$. The following are equivalent:*

(i) u is a viscosity subsolution to (1.1);
(ii) u is a pluripotential subsolution to (1.1).

The proof relies on corresponding results in the elliptic case, as well as on the parabolic comparison principle established in [GLZ1, Theorem 6.5].

Proof. We first prove $(i) \Longrightarrow (ii)$. Assume u is a viscosity subsolution to (1.1). Fix $J_1 \Subset J_2 \Subset]0,T[$ compact subintervals. We are going to prove that u is a pluripotential subsolution to (1.1) in $J_1 \times \Omega$.

We regularize u by taking the sup-convolution with respect to the t-variable: for $\varepsilon > 0$ small enough we define

$$u_\varepsilon(t,z) := \sup \left\{ u(t',z) - \frac{1}{2\varepsilon^2}(t-t')^2 \; ; \; t' \in J_2 \right\}.$$

The function u_ε is semi-convex in $t \in J_1$, upper semicontinuous in z. We claim that

$$(dd^c u_\varepsilon)^n \geq e^{\partial_t u_\varepsilon + F_\varepsilon(t,z,u_\varepsilon)} g dV,$$

in the viscosity sense where

$$F_\varepsilon(t,z,r) := \inf \left\{ F(t+s,z,r) \; ; \; |s| \leq C\varepsilon \right\},$$

for a uniform constant $C > 0$ depending on $\sup_{J_2 \times \Omega} |u|$. The argument is classical but we recall it for the reader's convenience. Let q be a differential test from above for u_ε at $(t_0,z_0) \in J_1 \times \Omega$ and let $s_0 \in J_2$ be such that

$$u_\varepsilon(t_0,z_0) = u(s_0,z_0) - \frac{1}{2\varepsilon^2}(s_0-t_0)^2.$$

Then $|t_0 - s_0| \leq C\varepsilon$. Consider the function q_ε defined by

$$q_\varepsilon(t,z) := q(t+t_0-s_0) + \frac{1}{2\varepsilon^2}(s_0-t_0)^2.$$

Then $q_\varepsilon(s_0,z_0) = u(s_0,z_0)$, and for all $(t,z) \in J_1 \times \Omega$,

$$q_\varepsilon(t,z) \quad \geq \quad u_\varepsilon(t+t_0-s_0) + \frac{1}{2\varepsilon^2}(s_0-t_0)^2 \geq u(t).$$

In other words, q_ε is a differential test from above for u at (s_0,z_0). Hence

$$(dd^c q_\varepsilon)^n(s_0,z_0) \geq e^{\partial_t q_\varepsilon(s_0,z_0) + F(s_0,z_0,q_\varepsilon(s_0,z_0))} g(z_0) dV.$$

Since F is increasing in r and $q_\varepsilon(s_0,z_0) \geq q(t_0,z_0)$ we obtain

$$\begin{aligned} (dd^c q)^n(t_0,z_0) \quad & \geq \quad e^{\partial_t q(t_0,z_0) + F(s_0,z_0,q(t_0,z_0))} g(z_0) dV \\ & \geq \quad e^{\partial_t q(t_0,z_0) + F_\varepsilon(t_0,z_0,q(t_0,z_0))} g(z_0) dV, \end{aligned}$$

as claimed.

Let $\partial_t^- u_\varepsilon$ denote the left derivative in t of u_ε. Since $\partial_t^- u_\varepsilon + F_\varepsilon$ is bounded, by considering $u_\varepsilon + \delta|z|^2$ and letting $\delta \to 0$, we can assume that $g \geq c > 0$ is strictly positive in Ω. The function

$$(t,z) \mapsto G(t,z) = e^{\partial_t^- u_\varepsilon(t,z) + F_\varepsilon(t,z,u_\varepsilon(t,z))} g(z),$$

is lower semicontinuous in Ω_T. It can be approximated from below by a sequence of positive continuous functions (G_j). By definition of viscosity subsolutions (applied to u_ε) we have

$$(dd^c u_\varepsilon)^n \geq G_j dV \tag{3.2}$$

in the parabolic viscosity sense. Since G_j is continuous, we can thus invoke [EGZ15, Proposition 3.6] to conclude that (3.2) holds in the elliptic viscosity sense for each $t \in J_1$ fixed. It then follows from [EGZ11, Proposition 1.5] that (3.2) holds in the elliptic pluripotential sense for each $t \in J_1$ fixed. Now, [GLZ1, Proposition 3.2] ensures that u_ε is a parabolic pluripotential subsolution to (1.1). Since u_ε decreases to u, Lemma 2.5 insures that u is a pluripotential subsolution to (1.1).

We now prove $(ii) \Longrightarrow (i)$. Assume that u is a pluripotential subsolution to (1.1). Fix $(t_0, z_0) \in \Omega_T$ and q a differential test from above defined in a neighborhood $J \times U \Subset]0,T[\times\Omega$ of (t_0,z_0). We need to prove that

$$(dd^c q)^n(t_0,z_0) \geq e^{\partial_t q(t_0,z_0) + F(t_0,z_0,q(t_0,z_0))} g(z_0) dV. \tag{3.3}$$

It follows from [EGZ11] that $dd^c q$ is semipositive at (t_0,z_0). If $g(z_0) = 0$ the inequality follows from the elliptic theory (see [EGZ11]). Since g is continuous up to shrinking U, we can assume that $g > 0$ in U.

Assume by contradiction that (3.3) does not hold. Then, by continuity of the functions involved, there exists $\varepsilon, r, \delta > 0$ small enough such that

$$(dd^c q + \varepsilon dd^c |z|^2)^n < e^{\partial_t q(t,z) + F(t,z,q(t,z)) - \delta} g(z) dV$$

holds in the classical sense in $[t_0 - r, t_0 + r] \times B(z_0, r)$. Consider the function

$$v(t,z) := q(t,z) + \gamma(|z - z_0|^2 - r^2 + t_0 - t),$$

for $(t,z) \in [t_0 - r, t_0] \times B(z_0, r)$. For γ small enough one can check that

$$\begin{aligned}(dd^c v)^n &\leq e^{\partial_t q(t,z) + F(t,z,q(t,z)) - \delta} g(z) dV \\ &\leq e^{\partial_t v + F(t,z,v + \gamma r^2 + \gamma(t - t_0)) + \gamma - \delta} g(z) dV \\ &\leq e^{\partial_t v + F(t,z,v)} g(z) dV,\end{aligned}$$

hence v is a supersolution to (1.1) in $]t_0 - r, t_0[\times B(z_0, r)$. We next compare v and u on the parabolic boundary of $]t_0 - r, t_0[\times B(z_0, r)$. For all $z \in B(z_0, r)$ we have

$$v(t_0 - r, z) \geq q(t_0 - r, z) + \gamma(r - r^2) \geq q(t_0 - r, z) \geq u(t_0 - r, z),$$

if $r < 1$. For all $t \in [t_0 - r, t_0], \zeta \in \partial B(z_0, r)$ we have

$$v(t, \zeta) = q(t, \zeta) + \gamma(t_0 - t) \geq u(t, \zeta).$$

If u is locally uniformly Lipschitz in t, it follows from [GLZ1, Theorem 6.5] that $u \leq v$ in $[t_0 - r, t_0] \times B(z_0, r)$. This yields a contradiction as

$$v(t_0, z_0) = q(t_0, z_0) - \gamma r^2 < u(t_0, z_0).$$

We finally remove the Lipschitz assumption on u. For each $\varepsilon > 0$ we define u_ε by

$$u_\varepsilon(t, z) := \int_{\mathbb{R}} u(st, z)\chi((s - 1)/\varepsilon) ds,$$

where χ is a cut-off function. Let F_j be a family of smooth functions which increases to F. Then u is a pluripotential subsolution to (1.1) with data F_j. Arguing as in [GLZ1, Theorem 6.5] we can show that $u_\varepsilon - c(\varepsilon)(t + 1)$ is a pluripotential subsolution to (1.1) (with data F_j) which is locally uniformly Lipschitz. Hence, we can apply the first step to show that $u_\varepsilon - c(\varepsilon)(t + 1)$ is a viscosity subsolution to (1.1) with data F_j. Thanks to Lemma 3.5 we can let $\varepsilon \to 0$ and then $j \to +\infty$ to conclude the proof. □

4. Viscosity vs pluripotential (super)solutions

The notion of pluripotential supersolutions has been introduced in [GLZ1]. In case $u \in \mathcal{P}(\Omega_T) \cap L^\infty_{\text{loc}}(\Omega_T)$ is locally uniformly semiconcave, it is a pluripotential supersolution to (1.1) if

$$(dd^c u)^n \wedge dt \leq e^{\partial_t^- u + F(t,z,u)} g dV \wedge dt,$$

in the sense of Radon measures in Ω_T.

As in the viscosity setting, a *pluripotential solution* is a parabolic potential which is both a subsolution and a supersolution.

4.1. Comparison of supersolutions

Theorem 4.1. *Assume $v \in \mathcal{P}(\Omega_T) \cap C(\Omega_T)$ is a pluripotential supersolution to (1.1) which is locally uniformly semi-concave in $t \in]0, T[$. Then v is a viscosity supersolution to (1.1).*

The proof relies on the parabolic pluripotential comparison principle [GLZ1, Theorem 6.5] which requires the extra semi-concavity hypothesis.

Proof. We can assume that $g > 0$. Fix $(t_0, z_0) \in \Omega_T$ and let q be a differntial test from below for v at (t_0, z_0), defined in $J \times U \Subset \Omega_T$. We want to prove that

$$(dd^c q)^n_+(t_0, z_0) \leq e^{\partial_t q(t_0,z_0)+F(t_0,z_0,q(t_0,z_0))} g(z_0) dV. \tag{4.1}$$

Assume, by contradiction, that it is not the case. Then $dd^c q_{t_0}(z_0)$ is semipositive and there is a constant $\delta > 0$ such that

$$(dd^c q_{t_0}(z_0))^n > e^{\partial_t q(t_0,z_0)+F(t_0,z_0,q(t_0,z_0))+2\delta} g(z_0) dV(z_0).$$

Since $g > 0$ and the data is continuous, we can find $r \in]0, 1[$ so small that

$$(dd^c q - \varepsilon dd^c |z|^2)^n \geq e^{\partial_t q(t,z)+F(t,z,q(t,z))+\delta} g(z) dV(z)$$

holds in the classical sense in $[t_0 - r, t_0 + r] \times B(z_0, r)$. Consider the function

$$u(t, z) := q(t, z) - \gamma(|z - z_0|^2 - r^2 + t_0 - t),$$

for $(t, z) \in [t_0 - r, t_0] \times B(z_0, r)$. For γ small enough one can check that

$$\begin{aligned}(dd^c u)^n &\geq e^{\partial_t q(t,z)+F(t,z,q(t,z))+\delta} g(z) dV \\ &\geq e^{\partial_t u - \gamma + F(t,z,u-\gamma r^2+\gamma(t_0-t))-\delta} g(z) dV \\ &\geq e^{\partial_t u+F(t,z,u)} g(z) dV,\end{aligned}$$

hence u is a subsolution to (1.1) in $]t_0 - r, t_0[\times B(z_0, r)$. We next compare v and u on the parabolic boundary of $]t_0 - r, t_0[\times B(z_0, r)$. For all $z \in B(z_0, r)$ we have

$$u(t_0 - r, z) \leq q(t_0 - r, z) + \gamma(r^2 - r) \leq q(t_0 - r, z) \leq v(t_0 - r, z),$$

since $r < 1$. For all $t \in [t_0 - r, t_0], \zeta \in \partial B(z_0, r)$ we have

$$u(t, \zeta) = q(t, \zeta) - \gamma(t_0 - t) \leq v(t, \zeta).$$

Since v is locally uniformly semi-concave, we can invoke [GLZ1, Theorem 6.5] to conclude that $u \leq v$ in $[t_0 - r, t_0] \times B(z_0, r)$. This yields a contradiction since $u(t_0, z_0) = q(t_0, z_0) + \gamma r^2 > v(t_0, z_0)$. □

In the reverse direction we have the following observation:

Theorem 4.2. *Let v be a viscosity supersolution to (1.1) and assume that v is locally uniformly semi-concave in $t \in]0, T[$. Then $P(v)$ is a pluripotential supersolution to (1.1).*

Here $P(v)(t, z) = P(v_t)(z)$ is the slice plurisubharmonic envelope of v: for each t fixed, we set

$$P(v_t)(z) := \sup\{w(z);\ w \leq v_t \ \text{ and } w \text{ plurisubharmonic in } \Omega\},$$

i.e. $P(v)_t := P(v_t)$ is the largest psh function lying below v_t.

Proof. We first observe that $t \mapsto P(v)(t, z)$ is locally uniformly semi-concave. This follows from the fact that $v \mapsto P(v)$ is increasing and concave: assume for simplicity that $t \mapsto v(t, z)$ is uniformly concave, then

$$\frac{v_{t+s} + v_{t-s}}{2} \leq v_t \Rightarrow \frac{P(v_{t+s}) + P(v_{t-s})}{2} \leq P\left(\frac{v_{t+s} + v_{t-s}}{2}\right) \leq P(v_t).$$

Fix $U \Subset \Omega$ and $S \in]0, T[$. Let v^ε denote the inf-convolution of v. Then v^ε increases pointwise to v and $P(v^\varepsilon) \uparrow P(v)$ as $\varepsilon \downarrow 0$. Since $\partial_t P(v^\varepsilon)$ converges a.e. to $\partial_t P(v)$ (see [GLZ1]), it suffices to prove that each $P(v^\varepsilon)$ is a pluripotential supersolution to (1.1). We can thus assume that v is continuous in Ω_T.

The left derivative $\partial_t^- v$ is upper semicontinuous in Ω_T. It follows from [EGZ15, Proposition 3.6] that, for all $t \in]0, T[$, the inequality

$$(dd^c v_t)_+^n \leq e^{\partial_t^- v + F(t, \cdot, v_t)} g dV$$

holds in the viscosity sense in Ω. It thus follows from [GLZ17] that $P(v_t)$ satisfies

$$(dd^c P(v_t))^n \leq e^{\partial_t^- v + F(y, \cdot, P(v_t))} g dV$$

in the pluripotential sense. Set

$$E = \{(t, z) \in \Omega_T,\ \partial_t^+ v(t, z) = \partial_t^- v(t, z) \ \& \ \partial_t^+ P(v)(t, z) = \partial_t^- P(v)(t, z)\}.$$

Then $\Omega_T \setminus E$ has zero Lebesgue measure. If $(t, z) \in E \cap \{P(v_t) = v_t\}$ then $\partial_t^- P(v)(t, z) = \partial_t^- v(t, z)$. Therefore,

$$(dd^c P(v))^n \wedge dt \leq e^{\partial_t P(v) + F(t,z,P(v))} g dV(z) \wedge dt$$

holds in the pluripotential sense in Ω_T. □

4.2. Viscosity comparison principle

The following stability estimate follows directly from the viscosity comparison principle established in [EGZ15, Theorem B].

Lemma 4.3. *Assume u is a bounded viscosity subsolution to (1.1) with data F and v is a bounded viscosity supersolution to (1.1) with data G. Then*

$$\sup_{\Omega_T}(u - v) \leq \sup_{\partial_0 \Omega_T}(u^* - v_*)_+ + T\|(G - F)_+\|,$$

where $\|(F - G)_+\| := \max_{[0,T] \times \bar{\Omega} \times [-C_0, +C_0]}(F - G)_+$ and $C_0 > 0$ is a uniform bound on $|u|$ and $|v|$ in Ω_T.

Proof. Set

$$M_1 := \sup_{\partial_0 \Omega_T}(u^* - v_*)_+, \; M_2 := \|(G - F)_+\|,$$

and $\tilde{u} := u - M_1 - M_2 t$. Then $\tilde{u}^* \leq v^*$ on $\partial_0 \Omega_T$. It follows directly from the definition of viscosity subsolutions that $\tilde{u}$ is a viscosity subsolution to (1.1) with data G since $F + (G - F)_+ \geq G$. It thus follows from [EGZ15, Theorem B] that $\tilde{u} \leq v$, giving the desired estimate. □

Corollary 4.4. *Assume that $F^j \to F$ locally uniformly in $\Omega_T \times \mathbb{R}$. Let h^j be a sequence of parabolic boundary data converging locally uniformly to a parabolic boundary datum h on $\partial_0 \Omega$.*

Let ϕ^j be the unique viscosity solution to the Cauchy Dirichlet problem for the data (F^j, g, h^j). Then $(\phi^j)_{j \in \mathbb{N}}$ converges locally uniformly in Ω_T to a continuous function ϕ which is the unique viscosity solution to the Cauchy-Dirichlet problem of the equation (1.1) for the data (F, g, h).

Proof. By the viscosity comparison principle (Lemma 4.3) we have for $j, k \in \mathbb{N}$, for any $0 < S < T$,

$$\sup_{\bar{\Omega}_S} |\phi_j - \phi_k| \leq \sup_{\partial_0 \Omega_S} |h_j - h_k| + S\|F^j - F^k\|_{\bar{\Omega}_S \times L},$$

where $L \subset \mathbb{R}$ is a compact set containing the values of ϕ^j, $j \in \mathbb{N}$, on the compact set $\bar{\Omega}_S$. It follows that (ϕ_j) is a Cauchy sequence for the norm of the uniform convergence on each $\bar{\Omega}_S$. Then the sequence has a limit which is a continuous function $\phi : [0, T[\times \bar{\Omega}$. By Lemma 3.5, the function ϕ is a solution to the equation (1.1) for the data (F, g, h). Set

$$\alpha_j := \sup_{\partial_0 \Omega_S} |h_j - h_k| + S\|F^j - F^k\|_{\bar{\Omega}_S \times L}.$$

Then $\alpha_j \to 0$ and for $j >> 1$ we have

$$\phi_j - \alpha_j \leq \phi \leq \phi_j + \alpha_j,$$

in Ω_S. From this inequality it follows that the boundary values of ϕ coincide with h on $\partial_0 \Omega_S$. Letting $S \to T$, we see that ϕ is the unique solution to the equation (1.1) for the data (F, g, h). □

4.3. Viscosity vs pluripotential solutions

If h does not depend on t, it was shown in [EGZ15] that there exists a unique viscosity solution to (1.1) with boundary value h. This is the Perron envelope of all viscosity subsolutions with boundary value h.

This result has been recently extended by Do-Le-Tô [DLT19] to boundary data that are time-dependent. Combining viscosity and pluripotential techniques we provide an alternative proof of this existence result:

Theorem 4.5. *The Perron envelope of viscosity subsolutions to* (1.1) *with boundary value h is the unique viscosity solution to* (1.1) *with boundary value h. It coincides with the envelope of all pluripotential subsolutions to* (1.1) *with boundary value h.*

Proof. We first assume that the data (h, F) satisfiy the assumptions of [GLZ1]. Let U be the envelope of all pluripotential subsolutions to (1.1) with boundary value h, and V be the Perron envelope of viscosity subsolutions to (1.1) with boundary value h. Theorem 3.6 ensures that $U = V$. By Proposition 2.6 and [GLZ1], $U \in \mathcal{C}(\Omega_T)$ is a pluripotential solution to (1.1) which is locally uniformly semi-concave. It then follows from Theorem 4.1 that U is a viscosity supersolution to (1.1), hence U is a viscosity solution to (1.1). Lemma 4.3 ensures that U is the unique viscosity solution to (1.1) with boundary value h.

We now treat the general case. Let (h_j, F_j) be approximants of (h, F) which satisfy the assumptions in [GLZ1], and let U_j be the envelope of pluripotential subsolutions to (1.1) with data (h_j, F_j). Then U_j is a pluripotential solution to (1.1) which is locally uniformly semiconcave. The previous step ensures that U_j is a viscosity solution to (1.1) with data (h_j, F_j). By stability of viscosity solutions (see Lemma 4.3), U_j uniformly converges to U and $U = h$ on $\partial_0\Omega_T$. By Corollary 4.4, U is a solution to the equation (1.1) in Ω_T. Hence U is a solution to the Cauchy-Dirichlet problem for (1.1) in Ω_T with boundary values h.

Uniqueness follows from the viscosity comparison principle in Lemma 4.3 (see [EGZ15, Theorem B]). □

5. Compact Kähler manifolds

The techniques developed in the local context allow us to obtain analogous results in the compact setting.

We consider the following complex Monge-Ampère flow

$$(\omega_t + dd^c\varphi_t)^n = e^{\dot{\varphi}_t + F(t,x,\varphi_t)} g dV, \tag{5.1}$$

where X is a compact Kähler manifold of dimension n and

1. $X_T :=]0, T[\times X$ with $T > 0$;
2. $0 < g$ is a continuous function on X;
3. $t \mapsto \omega(t, x)$ is a smooth family of closed semi-positive $(1, 1)$-forms such that $\theta(x) \leq \omega_t(x) \leq \Theta$, where θ is a closed semi-positive big form, and Θ is a Kähler form;
4. $(t, x, r) \mapsto F(t, x, r)$ is continuous in $[0, T[\times X \times \mathbb{R}$, increasing in r;
5. $\varphi : [0, T[\times X \to \mathbb{R}$ is the unknown function, with $\varphi_t := \varphi(t, \cdot)$.

Let φ_0 be a bounded ω_0-psh function on X which is continuous in Ω, the ample locus of $\{\theta\}$.

Definition 5.1. The set $\mathcal{P}(X_T, \omega_t)$ of parabolic potentials consists of functions $u : X_T \to \mathbb{R} \cup \{-\infty\}$ such that

- u is upper semi-continuous on X_T and $u \in L^1_{\text{loc}}(X_T)$;
- for each $t \in]0, T[$, the function $u_t := u(t, \cdot)$ is ω_t-psh on X.

Definition 5.2. A parabolic potential $u \in \mathcal{P}(X_T, \omega_t) \cap L^\infty(X_T)$ is a pluripotential subsolution to (5.1) if for all constant $a > 0$,

$$(\omega_t + dd^c u_t)^n \wedge dt \geq g(a(\partial_t\varphi + F(t, z, u_t(z)) - a \log a + a) \, dV(z) \wedge dt$$

holds in the sense of distribution in X_T.

If $u \in \mathcal{P}(X_T, \omega_t) \cap L^\infty(X_T)$ is locally uniformly Lipschitz in t then our definition coincides with that of [GLZ2].

Theorem 5.3. *Let U (respectively V) be the envelope of all pluripotential (respectively viscosity) subsolutions u to (5.1) such that $\limsup_{t\to 0} u_t \le \varphi_0$. Then $U = V$ is the unique viscosity solution to (5.1) starting from φ_0.*

The last condition in the theorem means that $\lim_{t\to 0^+} U_t = \varphi_0$ locally uniformly in $\Omega := \mathrm{Amp}(\{\theta\})$, the ample locus of the class $\{\theta\}$.

Proof. The equivalence of pluripotential and viscosity subsolutions for a given parabolic potential $u \in \mathcal{P}(X_T) \cap L^\infty(X_T)$ follows from Theorem 3.6, since being a pluripotential (resp. viscosity) subsolution is a local property. It follows in particular that $U = V$ on X_T.

We approximate F uniformly by a sequence of data F^j which satisfy the assumptions in [GLZ2] (one can e.g. take the convolution with a smoothing kernel in t, r). We approximate ω_t by $\omega_t^j := \omega_t + 2^{-j}\Theta$. Then ω^j also satisfies the assumptions in [GLZ2]. Let U^j be the envelope of pluripotential subsolutions to (1.1) with data $(F^j, \omega^j, \varphi_0)$. By [GLZ2] and the proof of Proposition 2.6, U^j is locally uniformly semi-concave in t, $\lim_{t\to 0^+} U_t^j = \varphi_0$, for all j, and U^j is a pluripotential solution to (1.1) with data (F^j, ω^j). By continuity of φ_0 in Ω and [GLZ2, Proposition 2.2], we infer that U_t^j locally uniformly converges to φ_0 in Ω.

The proof of Theorem 4.1 shows that U^j is a viscosity solution to (5.1) in Ω. We now prove that U^j locally uniformly converges to U on Ω_T. If we can do this then $U \in \mathcal{C}(\Omega_T)$ is a viscosity solution to (5.1) (thanks to Lemma 3.5), and $\lim_{t\to 0^+} U_t = \varphi_0$ locally uniformly in Ω.

In the arguments below we use $\varepsilon(j)$ to denote various positive constants which tend to 0 as $j \to +\infty$.

Since $\omega \le \omega^j$, the function $U - \varepsilon(j)t$ is a pluripotential subsolution to (5.1) with datum (F^j, ω^j), hence

$$U - \varepsilon(j)t \le U^j. \tag{5.2}$$

To obtain the other bound we fix $\rho \in \mathrm{PSH}(X, \theta) \cap L^\infty(X)$, $\sup_X \rho = 0$, such that

$$(\theta + dd^c\rho)^n = 2^n e^{c_1} g dV,$$

for some constant $c_1 \in \mathbb{R}$. The existence of ρ follows from [EGZ09]. Let $\psi \le 0$ be a θ-psh function which is smooth in Ω and satisfies

$$\theta + dd^c\psi \ge 2c_0\Theta,$$

for some positive fixed constant c_0.

Set for $j \in \mathbb{N}$,

$$W^j := (1-\lambda_j)U^j + \lambda_j \frac{\rho+\psi}{2}, \quad \text{with} \quad \lambda_j := \frac{2^{-j}}{2^{-j}+c_0}.$$

Given this choice of λ_j, a direct computation shows that

$$\begin{aligned} \omega_t + dd^c W^j &\geq (1-\lambda_j)(\omega_t + dd^c U_t^j) + \lambda_j(\omega_t + dd^c((\rho+\psi)/2)) \\ &\geq (1-\lambda_j)(\omega_t^j + dd^c U_t^j) + \lambda_j(\theta + dd^c\rho)/2 \geq 0. \end{aligned}$$

Hence, applying [GLZ2, Lemma 3.15] we obtain

$$\begin{aligned} (\omega_t + dd^c W^j)^n &\geq e^{(1-\lambda_j)(\partial_t U_t^j + F^j(t,x,U^j)) + \lambda_j c_1} g dV \\ &\geq e^{\partial_t W^j + F(t,x,W^j) - \varepsilon'(j)} g dV, \end{aligned}$$

in the weak sense on Ω, where $\varepsilon'(j) \to 0$.

It thus follows that $W^j - \varepsilon(j)t$ is a pluripotential subsolution to the equation (5.1) on Ω_T with datum (F,ω). Observe that W^j is not bounded on X. Since, for C large enough $u := \rho + nt\log t - Ct - C$ is a bounded pluripotential subsolution to the equation (5.1) in X_T with datum (F,ω), it follows that $\tilde{W}^j := \sup\{W^j - \varepsilon(j)t, u\}$ is a bounded subsolution to the equation (5.1) on X_T. Since $W^j(t,x) \leq U^j(t,x) + \varepsilon_j''$ where $\varepsilon_j'' \to 0$, and $\lim_{t\to 0} U^j(t,x) = \varphi_0(x)$ for any $x \in X$, it follows that

$$\tilde{W}^j - \varepsilon_j'' \leq U, \text{ in } X_T. \tag{5.3}$$

From (5.2) and (5.3) we conclude that U^j locally uniformly converges to U on X_T.

The uniqueness follows from [To19]. □

Acknowledgement

We thank the referee for useful comments which improve the presentation of the paper.

References

[BT76] Eric Bedford and B. A. Taylor, *The Dirichlet problem for a complex Monge-Ampère equation*, Invent. Math. **37** (1976), no. 1, 1–44.

[BT82] Eric Bedford and B. A. Taylor, *A new capacity for plurisubharmonic functions*, Acta Math. **149** (1982), no. 1-2, 1–40.

[DI04] J. Droniou, C. Imbert, *Solutions de viscosité et solutions variationnelles pour EDP non linéaires*, Cours de D.E.A, Université de Montpellier, 2004.

[DLT19] Hoang-Son Do, Giang Le, and Tat Dat To, *Viscosity solutions to parabolic complex Monge-Ampère equations*, Calc. Var. **59** (2020), 45. arXiv:1905.11818.

[EGZ09] P. Eyssidieux, V. Guedj, and A. Zeriahi, *Singular Kähler-Einstein metrics*, J. Amer. Math. Soc. **22** (2009), 607–639.

[EGZ11] Philippe Eyssidieux, Vincent Guedj, and Ahmed Zeriahi, *Viscosity solutions to degenerate complex Monge-Ampère equations*, Comm. Pure Appl. Math. **64** (2011), no. 8, 1059–1094.

[EGZ15] Philippe Eyssidieux, Vincent Guedj, and Ahmed Zeriahi, *Weak solutions to degenerate complex Monge-Ampère flows I*, Math. Ann. **362** (2015), no. 3-4, 931–963.

[EGZ16] Philippe Eyssidieux, Vincent Guedj, and Ahmed Zeriahi, *Weak solutions to degenerate complex Monge-Ampère flows II*, Adv. Math. **293** (2016), 37–80.

[EGZ17] Philippe Eyssidieux, Vincent Guedj, and Ahmed Zeriahi, *Corrigendum: Viscosity solutions to complex Monge-Ampère equations*, Comm. Pure Appl. Math. **70** (2017), no. 5, 815–821.

[EGZ18] Philippe Eyssidieux, Vincent Guedj, and Ahmed Zeriahi, *Convergence of weak Kähler-Ricci flows on minimal models of positive Kodaira dimension*, Comm. Math. Phys. **357** (2018), no. 3, 1179–1214.

[GLZ17] Vincent Guedj, Chinh H. Lu, and Ahmed Zeriahi, *Plurisubharmonic envelopes and supersolutions*, J. Diff. Geom. **113** (2019), no. 2, 273–313. arXiv:1703.05254.

[GLZ1] Vincent Guedj, Chinh H. Lu, and Ahmed Zeriahi, *The pluripotential Cauchy-Dirichlet problem for complex Monge-Ampère flows*, to appear in Ann. Sc. ENS. arXiv:1810.02122.

[GLZ2] Vincent Guedj, Chinh H. Lu, and Ahmed Zeriahi, *Pluripotential Kähler-Ricci flows*, Geometry & Topology **24** (2020), no. 3, 1225–1296. arXiv:1810.02121.

[HL13] F. Reese Harvey and H. Blaine Lawson, Jr., *The equivalence of viscosity and distributional subsolutions for convex subequations—a strong Bellman principle*, Bull. Braz. Math. Soc. (N.S.) **44** (2013), no. 4, 621–652.

[To19] Tat Dat To, *Convergence of the weak Kähler-Ricci flow on manifolds of general type*, International Mathematics Research Notices (2019), rnz256. arXiv:1905.01276.

Vincent Guedj
Institut de Mathématiques de Toulouse
Université de Toulouse, CNRS
UPS, 118 route de Narbonne
31062 Toulouse cedex 09
France
E-mail: vincent.guedj@math.univ-toulouse.fr

Chinh H. Lu
Laboratoire de Mathématiques d'Orsay
Université Paris-Saclay, CNRS
91405 Orsay Cedex
France
E-mail: hoang-chinh.lu@universite-paris-saclay.fr

Ahmed Zeriahi
Institut de Mathématiques de Toulouse
Université de Toulouse, CNRS
UPS, 118 route de Narbonne
31062 Toulouse cedex 09
France
E-mail: ahmed.zeriahi@math.univ-toulouse.fr