Degenerate complex Monge-Ampère equations and singular Kähler-Einstein metrics

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Abstract

These are notes from a series of lectures given by the author at the International Conference on Geometric Analysis, held at ICTP (Trieste, Italy), 25-29 june 2012.

We explain recent developments in the theory of degenerate complex Monge-Ampère equations, with applications towards the Kähler-Einstein equation and the study of the Kähler-Ricci flow on mildly singular projective algebraic varieties.

Contents

Introduction

Complex Monge-Ampère equations have been one of the most powerful tools in Kähler geometry since Aubin and Yau's classical works [Aub78, Yau78], culminating in Yau's solution to the Calabi conjecture. A notable application is the construction of Kähler-Einstein metrics on compact Kähler manifolds.

Whereas their existence on manifolds with trivial or ample canonical class was settled as a corollary of the Calabi conjecture, determining necessary and sufficient conditions on a Fano manifold to carry a Kähler-Einstein metric is still an open problem that attracts a lot of attention.

In recent years, following Tsuji's pioneering work [Tsu88], degenerate complex Monge-Ampère equations have been intensively studied by many authors. In relation to the Minimal Model Program, they led to the construction of singular Kähler-Einstein metrics with zero or negative Ricci curvature [EGZ09] or, more generally, of canonical volume forms on compact Kähler manifolds with nonnegative Kodaira dimension [ST09].

Making sense of and constructing singular Kähler-Einstein metrics on (possibly singular) Fano varieties requires more advanced tools in the study of degenerate complex Monge-Ampère equations. First steps in this direction are [BBGZ09] and [BBEGZ11]. The former combines a variational approach with the finite energy formalism of [GZ07, BEGZ10] to give a new perspective on the classical results of Ding-Tian [Tian]; the latter pushes this approach further and studies the convergence of the normalized Kähler-Ricci flow, generalizing a deep result of Perelman-Tian-Zhu [TZ07].

These lectures serve as an introduction to those recent works [BBGZ09, BBEGZ11]. We survey the theory of Kähler-Einstein metrics/currents in *Lecture 1*, explaining their equivalent formulation in terms of degenerate complex Monge-Ampère equations. We explain the variational approach to solving the latter in *Lecture 2* and apply these results in *Lecture 3* to show how the Kähler-Ricci flow can be used to detect a Kähler-Einstein current.

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Lecture 1

Singular Kähler-Einstein currents

1.1 The Kähler-Einstein equation

1.1.1 The Calabi conjecture

Let X be an n-dimensional compact Kähler manifold and fix ω an arbitrary Kähler form. If we write locally

$$\omega = \sum \omega_{\alpha\beta} \frac{i}{\pi} dz_\alpha \wedge d\overline{z}_\beta,$$

then the Ricci form of ω is

$$\mathrm{Ric}(\omega) := -\sum \frac{\partial^2 \log(\det \omega_{pq})}{\partial z_\alpha \partial \overline{z}_\beta} \frac{i}{\pi} dz_\alpha \wedge d\overline{z}_\beta.$$

Observe that $\mathrm{Ric}(\omega)$ is a closed $(1,1)$-form on X such that for any other Kähler form ω' on X, the following holds globally:

$$\mathrm{Ric}(\omega') = \mathrm{Ric}(\omega) - dd^c \left[\log \omega'^n / \omega^n\right].$$

Here $d = \partial + \overline{\partial}$ and $d^c = (\partial - \overline{\partial})/2i\pi$ are both real operators.

In particular $\mathrm{Ric}(\omega')$ and $\mathrm{Ric}(\omega)$ represent the same cohomology class, which turns out to be $c_1(X)$. Conversely, given η a closed differential form representing $c_1(X)$, Calabi asked in [Cal57] whether one could find a Kähler form ω such that

$$\mathrm{Ric}(\omega) = \eta.$$

This problem, known as the *Calabi conjecture,* remained open for two decades, until Yau [Yau78] solved it positively.

1.1.2 The associated complex Monge-Ampère equation

The Calabi conjecture reduces to solving a complex Monge-Ampère equation as we now explain. Fix $\alpha \in H^{1,1}(X,\mathbb{R})$ a Kähler class. Fix $\eta \in c_1(X)$ and ω a Kähler form in α. Since $\mathrm{Ric}(\omega)$ also represents $c_1(X)$, it follows from the $\partial\overline{\partial}$-lemma that there exists $h \in \mathcal{C}^\infty(X,\mathbb{R})$ such that

$$Ric(\omega) = \eta + dd^c h.$$

We now seek for $\omega_\varphi := \omega + dd^c\varphi$ a new Kähler form in α such that $Ric(\omega_\varphi) = \eta$. Since

$$Ric(\omega_\varphi) = Ric(\omega) - dd^c \log\left(\frac{(\omega + dd^c\varphi)^n}{\omega^n}\right),$$

the equation $Ric(\omega_\varphi) = \eta$ is thus equivalent to

$$dd^c\left\{h - \log\left(\frac{(\omega + dd^c\varphi)^n}{\omega^n}\right)\right\} = 0.$$

The function inside the brackets is pluriharmonic hence constant since X is compact: initially shifting h by a constant, our problem is thus equivalent to solving the complex Monge-Ampère equation

$$(CY) \qquad (\omega + dd^c\varphi)^n = e^h\omega^n.$$

Note that h necessarily satisfies the normalizing condition

$$\int_X e^h\omega^n = \int_X \omega^n = V_\alpha.$$

Theorem 1.1.1 (Yau 78)**.** *The equation (CY) admits a solution $\varphi \in \mathcal{C}^\infty(X,\mathbb{R})$ such that $\omega_\varphi := \omega + dd^c\varphi$ is a Kähler form.*

The solution is unique, up to an additive constant, as was shown by Calabi.

1.1.3 The Kähler-Einstein equation

The following metrics are the main objects of interest for us in this course:

Definition 1.1.2. *A Kähler metric ω is Kähler-Einstein if there exists $\lambda \in \mathbb{R}$ such that*

$$\mathrm{Ric}(\omega) = \lambda\omega.$$

The existence of such a metric requires $c_1(X)$ to have a definite sign, since $c_1(X) \in \mathbb{R}\{\omega\}$. This is always satisfied in dimension $n = 1$, but not in dimension 2: in the simple case where $X = S_1 \times S_2$ is the product of two compact Riemann surfaces, then $c_1(X)$ is proportional to a Kähler class if and only if S_1 and S_2 are of the same type.

Note that $\mathrm{Ric}(\varepsilon\omega) = \mathrm{Ric}(\omega)$ for any $\varepsilon > 0$, hence there are essentially three cases to be considered, $\lambda \in \{-1, 0, +1\}$.

Fix $\lambda \in \mathbb{R}$ such that $\lambda\{\omega\} = c_1(X)$ and $h \in \mathcal{C}^\infty(X, \mathbb{R})$ such that

$$\mathrm{Ric}(\omega) = \lambda\omega + dd^c h.$$

We now seek for a Kähler form $\omega_\varphi := \omega + dd^c\varphi$ such that $Ric(\omega_\varphi) = \lambda\omega_\varphi$. Since

$$Ric(\omega_\varphi) = Ric(\omega) - dd^c \log\left(\frac{(\omega + dd^c\varphi)^n}{\omega^n}\right),$$

the equation $Ric(\omega_\varphi) = \lambda\omega_\varphi$ is thus equivalent to

$$dd^c\left\{h - \lambda\varphi - \log\left(\frac{(\omega + dd^c\varphi)^n}{\omega^n}\right)\right\} = 0.$$

The function inside brackets is pluriharmonic hence constant. Shifting h by a constant, our problem is thus equivalent to solving the complex Monge-Ampère equation

$$(MA)_\lambda \qquad (\omega + dd^c\varphi)^n = e^{-\lambda\varphi + h}\omega^n.$$

This contains the Calabi conjecture as a particular case.

1.2 MMP and singular K-E metrics

Let X be a manifold of general type, i.e. a compact Kähler manifold whose canonical bundle K_X is *big*: this means that there are asymptotically "many" pluricanonical sections (so many that one can birationally embed X into a complex projective space), or, equivalently, that one can find a positive singular current T which dominates a Kähler form and represents $c_1(K_X)$. This is the "right" higher dimensional analogue of hyperbolic Riemann surfaces.

It has been shown recently by Birkar-Cascini-Hacon-McKernan [BCHM10] that the canonical ring

$$R_X = \bigoplus_{k\in\mathbb{N}} H^0(X, K_X^k)$$

is finitely generated, hence $X_{can} = Proj(R_X)$ the canonical model associated to X has only canonical singularities and ample canonical bundle

$$K_{X_{can}} > 0.$$

It is natural to wonder whether there is a unique Kähler-Einstein metric on the regular part of X_{can}, generalizing the classical situation when K_X is ample (in which case $X = X_{can}$).

The Minimal Model Program (MMP for short, or Mori program) studies more generally the problem of classifying higher dimensional algebraic manifolds up to birational equivalence. While the classification of algebraic surfaces (complex dimension two !) has been understood for decades, the situation is incredibly more complicated in (complex) dimension ≥ 3.

It has been realized long ago that one has to enlarge the picture and consider mildly singular varieties. The purpose of this section is to define the corresponding singularities and Kähler-Einstein equations, and to explain how these can be interpreted in terms of degenerate complex Monge-Ampère equations. For more information on the MMP, we refer the reader to [KM].

1.2.1 Canonical and log terminal singularities

The complex analytic varieties X we are considering are *normal*, in particular their singular set X_{sing} is a complex subvariety of complex codimension ≥ 2. We let X_{reg} denote the set of smooth points.

Recall that a normal variety X is $\mathbb{Q}$-*Gorenstein* if its canonical divisor K_X exists as a $\mathbb{Q}$-line bundle, which means that there exists $r \in \mathbb{N}$ and a line bundle L on X such that $L|_{X_{\rm reg}} = rK_{X_{\rm reg}}$.

Let X be a $\mathbb{Q}$-Gorenstein variety and choose a log resolution of X, i.e. a projective birational morphism $\pi : X' \to X$ which is an isomorphism over $X_{\rm reg}$ and whose exceptional divisor $E = \sum_i E_i$ has simple normal crossings. There is a unique collection of rational numbers a_i, called the *discrepancies* of X (with respect to the chosen log resolution) such that

$$K_{X'} \sim_{\mathbb{Q}} \pi^* K_X + \sum_i a_i E_i.$$

Definition 1.2.1. *By definition, X has* log terminal singularities *(resp.* canonical singularities*) if $a_i > -1$ (resp. $a_i \geq 0$) for all i.*

This definition is independent of the choice of a log resolution; this will be a consequence of the following analytic interpretation of log terminal singularities as a *finite volume* condition.

After replacing X with a small open set, we may choose a non-zero section σ of the line bundle rK_X for some $r \in \mathbb{N}^*$. Restricting to X_{reg} we get a smooth positive volume form by setting

$$\mu_\sigma := \left(i^{rn^2}\sigma \wedge \bar{\sigma}\right)^{1/r} \tag{1.2.1}$$

Such measures are called *adapted measures* in [EGZ09]. The key fact is:

Lemma 1.2.2. *Let z_i be a local equation of E_i, defined on a neighborhood $U \subset X'$ of a given point of E. Then*

$$(\pi^*\mu_\sigma)_{U\setminus E} = \prod_i |z_i|^{2a_i} dV$$

for some smooth volume form dV on U.

As a consequence we see that a $\mathbb{Q}$-Gorenstein variety X has log terminal singularities iff every adapted measure μ_σ has locally finite mass near points of X_{sing}. The construction of adapted measures can be globalized as follows: let ϕ be a smooth metric of the $\mathbb{Q}$-line bundle K_X. Then

$$\mu_\phi := \left(\frac{i^{rn^2}\sigma \wedge \bar{\sigma}}{|\sigma|_{r\phi}}\right)^{1/r} \tag{1.2.2}$$

becomes independent of the choice of a local non-zero section σ of rK_X, hence defines a smooth positive volume form on X_{reg}, which has locally finite mass near points of X_{sing} iff X is log terminal.

Example 1.2.3. *Let S be a normal algebraic surface. The following are equivalent:*

1. *S has only canonical singularities.*
2. *S is locally isomorphic to $\mathbb{C}^2/G$, $G \subset SL_2(\mathbb{C})$ a finite subgroup.*
3. *The exceptional divisors of the minimal resolution π_{min} of S, have simple normal crossings, their components are (-2) smooth rational curves, their incidence graphs are of type A-D-E (Du Val singularities).*

The log terminal surface singularities are precisely the singularities of the form $X = \mathbb{C}^2/G$, $G \subset GL_2(\mathbb{C})$ a finite subgroup.

Example 1.2.4. *In higher dimension, quotient singularities are still log terminal. Fix $n > 0$ and let $H \subset \mathbb{CP}^{n+1}$ be a smooth degree d hypersurface. The affine cone over H has only canonical singularities iff $d \leq n+1$. More generally the hypersurface singularities of type $A - D - E$ are canonical.*

In particular, the ordinary double point $x^2 + y^2 + z^2 + t^2 = 0$ has only canonical singularities but it is not a quotient singularity.

1.2.2 Degenerate complex Monge-Ampère equations

Singular Ricci curvature

Let X be a Gorenstein compact Kähler variety. Let ω_X be a Kähler form on X. Its Ricci curvature form is defined as

$$\mathrm{Ric}(\omega_X) := -dd^c\phi = dd^c \log h,$$

where $h = e^{-\phi}$ is a smooth metric of K_X such that $\omega_X^n = \mu_\phi$. We let the reader check that this is equivalent to the previous formulation when X is smooth.

Definition 1.2.5. *A Kähler metric ω_X on a Gorenstein compact Kähler variety X is called a singular Kähler-Einstein metric if*

$$\mathrm{Ric}(\omega_X) = \lambda\omega_X$$

for some $\lambda \in \mathbb{R}$.

As in the smooth case, these can only exist when $c_1(X)$ has a definite sign.

Resolving the singularities

The Kähler-Einstein problems boils down again to solving a complex Monge-Ampère equation,

$$(\omega_X + dd^c\psi)^n = e^{-\lambda\psi}\mu_\phi$$

It is more convenient to work in a desingularization: if $\pi : X' \to X$ is a log resolution of X, the above Monge-Ampère equation transfers on X' as

$$(\omega + dd^c\varphi)^n = e^{-\lambda\varphi}\mu,$$

where

$$\omega = \pi^*\omega_X$$

is semi-positive and *big*, which simply means here that $\int_X \omega^n > 0$, and

$$\mu = \pi^*\mu_\phi = f dV$$

is absolutely continuous, smooth above X_{reg}, with density in L^p for some $p > 1$ (it may have zeroes and/or poles above X_{sing}, depending on the type of singularities). Note that φ is ω-psh if and only if ψ is ω_X-psh, as follows from Zariski's main theorem (the fibers are connected).

These equations are degenerate in two ways: not only is the right hand side measure degenerate, but the reference form ω (or rather its cohomology class) also lacks positivity: this is a source of difficulty, in particular for the regularity theory.

We explain in the sequel how one can try and solve such degenerate complex Monge-Ampère equations.

1.3 Known results

1.3.1 Non-positive curvature

When X is a manifold with $c_1(X) < 0$ (i.e. K_X is ample), it was shown independently by Aubin [Aub78] and Yau [Yau78] that X admits a unique Kähler-Einstein metric. Yau [Yau78] also settled positively the case of manifolds with $c_1(X) = 0$, since then the latter have been called Calabi-Yau manifolds.

These results have been extended to the case of varieties with log terminal singularities in [Tsu88, Sug90, EGZ09, TZha06]:

Theorem 1.3.1. *Let X be a projective variety with log terminal singularities such that K_X is ample. Then K_X contains a unique Kähler-Einstein current with negative Ricci curvature, which is a smooth Kähler form on X_{reg} and has globally bounded potentials.*

It follows therefore from [BCHM10] that a manifold of general type admits a canonical current of negative Ricci curvature. This result has been proved again in [BEGZ10] independently of [BCHM10].

Theorem 1.3.2. *Let X be a compact Kähler variety with log terminal singularities and $K_X \sim 0$. Then each Kähler cohomology class contains a unique Ricci flat Kähler-Einstein current, which is a smooth Kähler form on X_{reg} and has globally bounded potentials.*

It is an important issue to study the behavior of these Kähler-Einstein currents near the singular points of X, a largely **open problem**.

Tosatti has observed in [Tos09] that the resulting metrics in X_{reg} are not complete. It has been proved in [EGZ11] that the potentials of these currents are actually globally continuous (see also [DZ10]). The case of orbifolds (quotient singularities) shows that these potentials cannot be more than globally Hölder continuous.

1.3.2 Positive curvature

We let now X be a $\mathbb{Q}$-Fano variety with log terminal singularity; the Kähler-Einstein problem is then far more delicate.

1.3.2.1 Curves

If $\dim_{\mathbb{C}} X = 1$, X is the Riemann sphere $\mathbb{CP}^1$ and (a suitable multiple of) the Fubini-Study Kähler form is a Kähler-Einstein metric.

1.3.2.2 Surfaces

When $\dim_{\mathbb{C}} X = 2$ it is not always possible to solve $(MA)_1$. In this case X is a *DelPezzo surface*, biholomorphic either to $\mathbb{CP}^1 \times \mathbb{CP}^1$ or $\mathbb{CP}^2$ which both admit the (product) Fubini-Study metric as a Kähler-Einstein metric, or else to X_r, the blow up of $\mathbb{CP}^2$ at r points in general position, $1 \leq r \leq 8$.

Various authors (notably Yau [TY87], Siu [Siu88], Tian [Tian87], Nadel [Nad90]) studied the Kähler-Einstein problem on DelPezzo surfaces in the eighties. The final and difficult step was made by Tian [Tian90] who proved the following:

Theorem 1.3.3. *The DelPezzo surface X_r admits a Kähler-Einstein metric if and only if $r \neq 1, 2$.*

The interested reader will find an up-to-date proof of this result in [Tos12]. He will also find some results in the non-smooth (orbifold) case in [GK07].

1.3.2.3 Higher dimension

The situation becomes much more difficult and largely open in higher dimension, despite many important works (it would be a difficult and long task to give a list of references, we rather refer the reader to the survey notes [Tian, PS10]).

There is a finite but long list (105 families) of Fano threefolds. It is unknown, for most of them, whether they admit or not a Kähler-Einstein metric. Among them, the Mukai-Umemura manifold is particularly interesting: this manifold admits a Kähler-Einstein metric as was shown by Donaldson [Don08], and there are arbitrary small deformations of it which do (resp. do not) admit a Kähler-Einstein metric as shown by Donaldson (resp. Tian [Tian97]).

There are even more families in dimension $n \geq 4$. Those which are *toric* admit a Kähler-Einstein metric if and only if the Futaki invariant vanishes (see [WZ04]), the non-toric case is essentially open and has motivated the **conjecture of Yau-Tian-Donaldson** (see [PS10] for a recent account on this leading conjecture which will not be discussed in these notes).

1.3.2.4 Fano varieties

Ding and Tian have developed in [Ding88, DT92] a variational interpretation of the Kähler-Einstein problem which allowed Tian to produce in [Tian97] an analytic characterization of the existence of these objects on Fano manifolds.

We explain this characterization hereafter and we will generalize it to the context of $\mathbb{Q}$-Fano varieties in Lecture 2, following [BBGZ09, BBEGZ11]:

Theorem 1.3.4. *Let X be $\mathbb{Q}$-Fano variety whose Mabuchi functional is proper. Then there exists a unique Kähler-Einstein current of positive Ricci curvature. It is a smooth Kähler form on X_{reg} and has continuous potentials.*

The Mabuchi functional was initially defined and studied by Mabuchi [Mab86] and Bando [Ban87]. We will recall its definition in Lecture 3.

Bando and Mabuchi have shown in [BM87] that any two Kähler-Einstein metrics on a Fano manifold can be connected by the holomorphic flow of a holomorphic vector field. This result has been generalized recently by Berndtsson [Bern11].

In the sequel we'll make the simplifying assumption that X does not admit non-zero holomorphic vector field, so that it admits a unique Kähler-Einstein metric, if any. Let us stress that the properness of the Mabuchi functional implies that there is no non-zero holomorphic vector field.

1.4 How to solve complex Monge-Ampère equations ?

A naive idea to solve degenerate complex Monge-Ampère equations of the type, say $MA(\varphi) = \mu$, is to treat first the case when μ is a linear combination of Dirac masses, and then proceed by approximation.

This works quite well for the *real* Monge-Ampère equation (see [RT77] or [Gut01, Theorem 1.6.2]), but fails miserably in the complex case: one does not even know, in general, how to treat the case of a single Dirac mass [CG09] ! The difficulty lies in the lack of regularity of quasi-plurisubharmonic functions (qpsh for short). In particular the complex Monge-Ampère operator is not well defined for all qpsh functions, and it is not continuous for the L^1-topology (the natural topology in this context).

We therefore present different methods, starting with the historical continuity method advocated by Calabi [Cal57] to solve the Calabi conjecture, which was successfully completed by Yau [Yau78].

1.4.1 Continuity method

The continuity method is a classical tool to try and solve non linear PDE's. It consists in deforming the PDE of interest into a simpler one for which one already knows the existence of a solution.

For the Calabi conjecture, one can use the following path,

$$(CY)_t, \qquad (\omega + dd^c\varphi_t)^n = \left[te^h + (1-t)\right]\omega^n,$$

where $0 \leq t \leq 1$ and $\varphi_t \in PSH(X, \omega)$ is normalized so that $\int_X \varphi_t \, \omega^n = 0$ (to guarantee uniqueness). The equation of interest corresponds to $t = 1$ while $(CY)_0$ admits the obvious (and unique) solution $\varphi_0 \equiv 0$.

The goal is then to show that the set $S \subset [0, 1]$ of parameters for which there is a (smooth) solution is both open and closed in $[0, 1]$: since $[0, 1]$ is connected and $0 \in S$, it will then follow that $S = [0, 1]$ hence $1 \in S$.

The openness follows by linearizing the equation (this involves the Laplace operator associated to $\omega_t = \omega + dd^c \varphi_t$) and using the inverse function theorem. One then needs to establish various a priori estimates to show that S is closed. The reader will find details e.g. in [Tian].

The situation is a bit more delicate when ω is merely semi-positive and big (the Laplace operator Δ_ω is for instance no longer invertible). One can approximate ω by $\omega + \varepsilon \omega_X$, where ω_X is Kähler and $\varepsilon \searrow 0$ decreases to 0^+.

When the left hand side $\mu = f dV$ is also degenerate, one can regularize it, use Yau's solution to the Calabi conjecture and try and pass to the limit. This is the approach used in [Kol98, GZ07, EGZ09].

When the reference cohomology class is merely big (see [BEGZ10]), it is not clear how to smoothly deform the equation unless the class is also nef. The regularity of solutions is an interesting **open problem** in this general context.

It is thus important to develop alternative soft methods to construct weak solutions to degenerate complex Monge-Ampère equations, as we explain in the forthcoming sections. Techniques from pluripotential theory (e.g. [Kol98, Ceg98, GZ05]) play then a central role. A recent work by Szekelehydi-Tosatti [SzTo09] shows how the Kähler-Ricci flow can also be used to study the regularity properties of weak solutions.

1.4.2 Variational approach

Given $\varphi : X \to \mathbb{R} \cup \{-\infty\}$ an upper semi-continuous function, we say that φ is ω-plurisubharmonic (ω-psh for short) if φ is locally given as the sum of a smooth and a plurisubharmonic function, and $\omega + dd^c \varphi \geq 0$ in the weak sense of currents. We let $PSH(X, \omega)$ denote the set of all ω-psh functions.

For $\varphi \in PSH(X, \omega) \cap \mathcal{C}^\infty(X)$, we set

$$E(\varphi) := \frac{1}{(n+1)V} \sum_{j=0}^{n} \int_X \varphi (\omega + dd^c \varphi)^j \wedge \omega^{n-j}.$$

We let the reader check, by using Stokes formula, that

$$\frac{d}{dt} E(\varphi + tv)_{|t=0} = \int_X v \, MA(\varphi), \text{ where } MA(\varphi) := (\omega + dd^c \varphi)^n / V.$$

The functional E is thus a primitive of the complex Monge-Ampère operator, in particular $\varphi \mapsto E(\varphi)$ is non-decreasing since $E' = MA \geq 0$.

Definition 1.4.1. *We assume here that X is Fano manifold, $\omega \in c_1(X)$ and $h \in \mathcal{C}^\infty(X, \mathbb{R})$ is such that* $\mathrm{Ric}(\omega) = \omega - dd^c h$. *The Ding functional is*

$$\mathcal{F}(\varphi) := E(\varphi) + \log \left[\int_X e^{-\varphi - h} \frac{\omega^n}{V} \right],$$

where $V = \int_X \omega^n = \mathrm{vol}_\omega(X)$.

The Euler-Lagrange equation of the functional $\mathcal{F}$ is obtained by checking that φ is a critical point for $\mathcal{F}$ if and only if it satisfies

$$MA(\varphi) = \frac{e^{-\varphi - h}\omega^n}{\int_X e^{-\varphi - h}\omega^n}$$

so that $\omega + dd^c\varphi$ is Kähler-Einstein.

Observe that $\mathcal{F}(\varphi + C) = \mathcal{F}(\varphi)$, for all $C \in \mathbb{R}$, thus $\mathcal{F}$ actually is a functional acting on the metrics $\omega_\varphi := \omega + dd^c\varphi$. It is natural to try and extremize the functional $\mathcal{F}$. This motivates the following:

Definition 1.4.2. *We say that $\mathcal{F}$ is proper if $\mathcal{F}(\varphi_j) \to -\infty$ whenever $\varphi_j \in PSH(X, \omega) \cap \mathcal{C}^\infty(X)$ is such that $E(\varphi_j) \to -\infty$ and $\int_X \varphi_j \, \omega^n = 0$.*

The importance of this notion was made clear in a series of works by Ding and Tian in the 90's, culminating with the following deep result of [Tian97]:

Theorem 1.4.3. *Let X be a Fano manifold with no holomorphic vector field. There exists a Kähler-Einstein metric if and only if $\mathcal{F}$ is proper.*

A key point in the proof is a Moser-Trudinger type inequality which has been obtained in an optimal form by Phong-Song-Sturm-Weinkove [PSSW08]. Establishing fine Moser-Trudinger inequalities in a singular context is a challenging **open problem** (see [BerBer11] for some results in this direction).

Remark 1.4.4. *It turns out that $\mathcal{F}$ is proper if and only if so is the Mabuchi functional $\mathcal{M}$, as we shall see later on.*

We will explain a partial generalization of this result in Lecture 2 following [BBGZ09]. It has interesting applications, such as the study of the long term behavior of the Kähler-Ricci flow, as we shall explain in Lecture 3.

1.4.3 The Kähler-Ricci flow approach

The Ricci flow is the parabolic evolution equation

$$(KRF) \qquad \frac{\partial \omega_t}{\partial t} = -\mathrm{Ric}(\omega_t) \quad \text{with initial data } \omega_0.$$

When ω_0 is a Kähler form, so is ω_t, for all $t > 0$, hence it is called the Kähler-Ricci flow (KRF for short). We say here just a few words on the behavior of the KRF on Fano manifolds. Much more information are to be found in Lecture 3.

Long time existence

The short time existence is guaranteed by standard parabolic theory: in the Kähler context, this translates into a parabolic scalar equation as we explain below.

It is more convenient to analyze the long time existence by considering the normalized Kähler-Ricci flow. When X has non-negative Kodaira dimension $kod(X) \geq 0$, this has been extensively studied by [Cao85, Tsu88, TZha06, CL06, ST07, ST08, ST09]. We focus here on the Fano case $c_1(X) > 0$ (in particular $kod(X) = -\infty$). The right normalization is then

$$(NKRF) \qquad \frac{\partial \omega_t}{\partial t} = -\mathrm{Ric}(\omega_t) + \omega_t.$$

One passes from (KRF) to (NKRF) by changing $\omega(t)$ in $e^t \omega(1 - e^{-t})$. At the level of cohomology classes,

$$\frac{d\{\omega_t\}}{dt} = -c_1(X) + \{\omega_t\} \in H^{1,1}(X, \mathbb{R})$$

therefore $\{\omega_t\} \equiv c_1(X)$ is constant if we start from $\omega_0 \in c_1(X)$. This justifies the name (normalized KRF) since in this case

$$\mathrm{vol}_{\omega_t}(X) = \mathrm{vol}_{\omega_0}(X) = c_1(X)^n$$

is constant. Note that the volume blows up exponentially fast if we rather start from initial data such that $\{\omega_0\} > c_1(X)$. One has [Cao85]:

Theorem 1.4.5. *Let X be a Fano manifold and $\omega_0 \in c_1(X)$ a Kähler form. Then the normalized Kähler-Ricci flow exists for all $t > 0$.*

The main issue is then whether (ω_t) converges as $t \to +\infty$. Hopefully $\partial \omega_t / \partial t \to 0$ and $\omega_t \to \omega_{KE}$ such that $\mathrm{Ric}(\omega_{KE}) = \omega_{KE}$.

Perelman established in 2003 (seminar talk at MIT, see [SeT08]) uniform estimates for the Ricci deviation along the NKRF, which allowed to prove the following (see [TZ07, PSS07]):

Theorem 1.4.6. *Let X be a Fano manifold and $\omega_0 \in c_1(X)$ an arbitrary Kähler form. If $\mathcal{F}$ is proper, then the normalized Kähler-Ricci flow (ω_t) converges, as $t \to +\infty$, towards the unique Kähler-Einstein metric ω_{KE}.*

The Yau-Tian-Donaldson conjecture aims at giving algebro-geometric interpretation of the analytic properness condition. We refer the reader to [PS10, Don10] and the reference therein.

The situation is much more delicate in the presence of holomorphic vector fields. The convergence of the (NKRF) for instance is an **open problem** on the projective space $\mathbb{CP}^n$, $n \geq 2$ (for $n = 1$, this is already non-trivial and was settled by Hamilton [**?**] and Chow [Chow91]).

1.4.4 Viscosity techniques

A standard PDE approach to second-order degenerate elliptic equations is the method of viscosity solutions, see [CIL] for a survey. This method is local in nature - and solves existence and unicity problems for weak solutions very efficiently.

Whereas the viscosity theory for real Monge-Ampère equations has been developed by P.L. Lions and others (see e.g.[IL]), the complex case hasn't been studied until very recently. There is a viscosity approach to the Dirichlet problem for the complex Monge-Ampère equation on a smooth hyperconvex domain in a Stein manifold in [HL]. This recent article however does not prove any new results for complex Monge-Ampère equations since this case serves there as a motivation to develop a deep generalization of plurisubharmonic functions to Riemannian manifolds with some special geometric structure (e.g. exceptional holonomy group).

There has been some recent interest in adapting viscosity methods to solve degenerate elliptic equations on compact or complete Riemannian manifolds [AFS]. This theory can be applied to complex Monge-Ampère equations only in very restricted cases since it requires the Riemann curvature tensor to be non-negative.

In [EGZ11] we develop the viscosity approach for complex Monge-Ampère equations on compact complex manifolds. Combining pluripotential and viscosity techniques, we are able to show the following:

Theorem 1.4.7. *Let X be a compact complex manifold in the Fujiki class. Let v be a semi-positive probability measure with L^p-density, $p > 1$, and fix $\omega \geq 0$ a smooth closed real semipositive $(1,1)$-form such that $\int_X \omega^n = 1$. The unique locally bounded ω-psh function on X normalized by $\int_X \varphi = 0$ such that its Monge-Ampère measure satisfies $(\omega + dd^c\varphi)^n_{BT} = v$ is continuous.*

A proof of the continuity when X is projective under technical assumptions has been obtained in [DZ10]. Some advantages of our method are:

- it gives an alternative proof of Kolodziej's C^0-Yau theorem which does not depend on [Yau78].
- it allows us to easily produce the unique negatively curved singular Kähler-Einstein metric in the canonical class of a projective manifold of general type, a result obtained first in [EGZ09] by assuming [BCHM10], then in [BEGZ10] by means of asymptotic Zariski decompositions.

An important part of [EGZ11] consists in developing a dictionary between the viscosity notions and the pluripotential ones. The viscosity subsolutions turn out to be coinciding with the pluripotential ones, however the concept of viscosity supersolution has no clear analogue in the pluripotential world. Finding the right analogue (perhaps using the notion of quasi-psh envelope), is an interesting **open question**.

Lecture 2

Variational approach to complex Monge-Ampère equations

Our goal in this second lecture is to explain a variational approach to solve degenerate complex Monge-Ampère equations, as proposed in [BBGZ09].

The context is the following: we work on a compact n-dimensional Kähler manifold (X, ω_X) and fix $\omega \geq 0$ a smooth closed semi-positive $(1,1)$ form which is *big*, i.e. such that

$$V := \mathrm{Vol}_\omega(X) := \int_X \omega^n > 0.$$

A prototype (see Lecture 1) is $\omega = \pi^*\omega_V$, the pull-back of a Kähler form ω_V on a mildly singular variety V, where $\pi : X \to V$ is a resolution of the singularities of V.

In connection with the Kähler-Einstein equation, we are interested in solving degenerate complex Monge-Ampère equations of the form

$$(MA)_\lambda \qquad\qquad MA(\varphi) := e^{\lambda\varphi}\mu,$$

where $\lambda \in \mathbb{R}$ is a real parameter, φ is ω-plurisubharmonic,

$$MA(\varphi) := \frac{1}{V}(\omega + dd^c\varphi)^n$$

is the normalized complex Monge-Ampère operator, and

$$\mu = f\omega_X^n$$

is a probability measure, absolutely continuous with respect to the Lebesgue measure ω_X^n, with density $f \in L^p$, $p > 1$. We set

$$\mathcal{F}_\lambda(\varphi) := E(\varphi) + \frac{1}{\lambda}\log\left[\int_X e^{-\lambda\varphi}d\mu\right],$$

where

$$E(\varphi) = \frac{1}{(n+1)V} \sum_{j=0}^{n} \int_X \varphi(\omega + dd^c\varphi)^j \wedge \omega^{n-j},$$

and

$$\mathcal{F}_0(\varphi) = \lim_{\lambda \to 0} \mathcal{F}_\lambda(\varphi) := E(\varphi) - \int_X \varphi \, d\mu.$$

Our goal is to explain the proof of the following result [BBGZ09]:

Theorem. *If $\mathcal{F}_\lambda$ is proper, then there exists $\varphi \in \mathcal{E}^1(X, \omega)$ an ω-psh function with finite energy which maximizes $\mathcal{F}_\lambda$. Any maximizer of $\mathcal{F}_\lambda$ with finite energy is a critical point of $\mathcal{F}_\lambda$ hence solves $(MA)_\lambda$.*

This result is proved in [BBGZ09] in the more general setting of big cohomology classes. This requires to introduce the pluripotential theory in this context which goes beyond the scope of these lecture notes. We refer the interested reader to [BEGZ10] for the basis of this theory.

We define the properness condition and the finite energy classes later on. It turns out that $\mathcal{F}_\lambda$ is always proper when $\lambda \leq 0$, or when $0 < \lambda << 1$ is small enough. However $\mathcal{F}_\lambda$ is not necessarily proper when say $\lambda = +1$, and it is known that one cannot always solve $(MA)_1$ (see Lecture 1).

We shall say just a few words about the regularity issue: while there are some interesting open problems, it turns out that the solvability of $(MA)_\lambda$ in the finite energy classes $\mathcal{E}^1(X, \omega)$ is equivalent to its solvability in the smooth category (whenever μ is smooth), as follows from the work of Kolodziej [Kol98] and Szekelyhidi-Tosatti in [SzTo09].

We will also briefly discuss the uniqueness issue in the last section.

2.1 Preliminary results

2.1.1 Quasi-plurisubharmonic functions

Recall that a function is quasi-plurisubharmonic if it is locally given as the sum of a smooth and a psh function. In particular quasi-psh functions are upper semi-continuous and L^1-integrable. Quasi-psh functions are actually in L^p for all $p \geq 1$, and the induced topologies are all equivalent. A much stronger integrability property actually holds: Skoda's integrability theorem [Sko72] asserts indeed that $e^{-\varepsilon\varphi} \in L^1(X)$ if $0 < \varepsilon$ is smaller than $2/\nu(\varphi)$, where $\nu(\varphi)$ denotes the maximal logarithmic singularity (Lelong number) of φ on X.

Quasi-plurisubharmonic functions have gradient in L^r for all $r < 2$, but not in L^2 as shown by the local model $\log |z_1|$.

We let $PSH(X,\omega)$ denote the set of all ω-plurisubharmonic functions. These are quasi-psh functions $\varphi : X \to \mathbb{R} \cup \{-\infty\}$ such that

$$\omega + dd^c\varphi \geq 0$$

in the weak sense of currents. The set $PSH(X,\omega)$ is a closed subset of $L^1(X)$, when endowed with the L^1-topology.

We let the reader check that the following maximal Lelong number

$$\nu(\{\omega\}) := \sup\{\nu(\varphi, x) \,|\, x \in X \text{ and } \varphi \in PSH(X,\omega)\} < +\infty$$

is finite and only depends on the cohomology class of ω. More generally, if P is a compact subfamily of $PSH(X,\omega)$, we set

$$\nu(P) := \sup\{\nu(\varphi, x) \,|\, x \in X \text{ and } \varphi \in P\} \leq \nu(\{\omega\}).$$

The uniform version of Skoda's integrability theorem is quite useful [Zer01]:

Theorem 2.1.1. *Let $P \subset PSH(X,\omega)$ be a compact family. If $A < 2/\nu(P)$, then*

$$\sup\left\{\int_X e^{-A\varphi} dV \mid \varphi \in P \text{ and } \sup_X \varphi = 0\right\} < +\infty.$$

2.1.2 The Bedford-Taylor theory

Bedford and Taylor have observed in [BT82] that one can define the complex Monge-Ampère operator

$$MA(\varphi) := \frac{1}{V}(\omega + dd^c\varphi)^n$$

for all *bounded* ω-psh function: they showed that whenever φ_j is a sequence of smooth ω-psh functions locally decreasing to φ, then the smooth probability measures $MA(\varphi_j)$ converge, in the weak sense of Radon measures, towards a unique probability measure that we denote by $MA(\varphi)$.

Example 2.1.2. *There are only few examples for which one can explicitly compute this Monge-Ampère measure. When $X = \mathbb{CP}^n$ is the complex projective space equipped with the Fubini-Study Kähler form, and*

$$\varphi[z] = \max_{0 \leq i \leq n} \log |z_i| - \frac{1}{2} \log \left[\sum_{i=0}^{n} |z_i|^2\right],$$

the reader will check that $V = 1$ and $MA(\varphi)$ is the normalized Lebesgue measure on the torus

$$\mathbb{T} = \{[z] \in \mathbb{CP}^n \,|\, |z_0| = \cdots = |z_n|\}.$$

At the heart of Bedford-Taylor's theory lies the following *maximum principle*: if u, v are bounded ω-plurisubharmonic functions, then

$$(MP) \qquad 1_{\{v<u\}} MA(\max(u,v)) = 1_{\{v<u\}} MA(u).$$

This equality is elementary when u is *continuous*, as the set $\{v < u\}$ is then a Borel open subset of X. When u is merely *bounded*, this set is only open in the plurifine topology. Since Monge-Ampère measures of bounded qpsh functions do not charge pluripolar sets (by the so called Chern-Levine-Nirenberg inequalities), and since u is nevertheless *quasi-continuous*, this gives a heuristic justification for (MP).

We now show how (MP) easily implies the so called *comparison principle*:

Proposition 2.1.3. *Let u, v be bounded ω-plurisubharmonic functions. Then*

$$\int_{\{v<u\}} MA(u) \le \int_{\{v<u\}} MA(v).$$

Proof. It follows from (MP) that

$$\begin{aligned}\int_{\{v<u\}} MA(u) &= \int_{\{v<u\}} MA(\max(u,v)) = 1 - \int_{\{v\ge u\}} MA(\max(u,v)) \\ &\le \int_X MA(v) - \int_{\{v>u\}} MA(\max(u,v)) = \int_{\{v\le u\}} MA(v).\end{aligned}$$

Replacing u bu $u-\varepsilon$ and letting ε decrease to 0 yields the desired result. □

2.1.3 Envelopes

Given a bounded u.s.c. function $u : X \to \mathbb{R}$, one can consider its ω-psh envelope

$$P(u)(x) := \sup\{v(x) \mid v \in PSH(X,\omega) \text{ and } v \le u\}.$$

The reader will check that $P(u)$ is ω-psh (in particular it is u.s.c.) and bounded. It is the largest ω-psh function lying below u.

Proposition 2.1.4. *The Monge-Ampère measure*

$$MA(P(u)) = V^{-1}(\omega + dd^c P(u))^n$$

is supported on the contact set $\{P(u) = u\}$.

The proof uses Choquet's lemma (which insures that the regularized supremum can be realized through an increasing sequence), the continuity property of the Monge-Ampère operator along increasing sequences, and a balayage procedure: on each small "ball" in $\{P(u) < u\}$, one can solve a homogeneous Dirichlet problem and show, by using the comparison principle, that the resulting sequence still increases towards $P(u)$. The details are left as an exercise to the reader.

Let us stress, for later use, that the same result holds when the function $P(u)$ merely has finite energy (i.e. when u and $P(u)$ are not necessarily bounded but not too singular), a notion that we define in the next section.

2.2 Finite energy classes

2.2.1 The class $\mathcal{E}(X, \omega)$

Given $\varphi \in PSH(X, \omega)$, we consider its canonical approximants

$$\varphi_j := \max(\varphi, -j) \in PSH(X, \omega) \cap L^\infty(X).$$

It follows from Bedford-Taylor theory that the measures $MA(\varphi_j)$ are well defined probability measures. Since the φ_j's are decreasing, it is natural to expect that these measures converge (in the weak sense). The following strong monotonicity property holds:

Lemma 2.2.1. *The sequence $\mu_j := \mathbf{1}_{\{\varphi > -j\}} MA(\varphi_j)$ is an increasing sequence of Borel measures.*

Proof. It follows from the maximum principle that

$$1_{\{\varphi_j > -k\}}[\omega + dd^c \varphi_j]^n = 1_{\{\varphi_j > -k\}}[\omega + dd^c \max(\varphi_j, -k)]^n.$$

Now if $j \geq k$, then $(\varphi_j > -k) = (\varphi > -k)$ and $\max(\varphi_j, -k) = \varphi_k$, thus

$$1_{\{\varphi > -k\}}[\omega + dd^c \varphi_j]^n = 1_{\{\varphi > -k\}}[\omega + dd^c \varphi_k]^n. \tag{2.2.1}$$

Observe also that $(\varphi > -k) \subset (\varphi > -j)$, therefore

$$j \geq k \longrightarrow 1_{\{\varphi > -j\}}[\omega + dd^c \varphi_j]^n \geq 1_{\{\varphi > -k\}}[\omega + dd^c \varphi_k]^n,$$

in the weak sense of Borel measures. □

Since the μ_j's all have total mass bounded from above by 1 (the total mass of the measure $MA(\varphi_j)$), we can consider

$$\mu_\varphi := \lim_{j \to +\infty} \mu_j,$$

which is a positive Borel measure on X, with total mass ≤ 1.

Definition 2.2.2. *We set*

$$\mathcal{E}(X,\omega) := \{\varphi \in PSH(X,\omega) \mid \mu_\varphi(X) = 1\}.$$

For $\varphi \in \mathcal{E}(X,\omega)$, we set $MA(\varphi) := \mu_\varphi$.

Of course every bounded ω-psh function belongs to $\mathcal{E}(X,\omega)$ with $\mu_\varphi = MA(\varphi)$, since in this case $\{\varphi > -j\} = X$ for j large enough, hence

$$\mu_\varphi \equiv \mu_j = MA(\varphi_j) = MA(\varphi).$$

The class $\mathcal{E}(X,\omega)$ also contains many ω-psh functions which are unbounded. When X is a compact Riemann surface ($n = \dim_{\mathbb{C}} X = 1$), the set $\mathcal{E}(X,\omega)$ is the set of ω-sh functions whose Laplacian does not charge polar sets.

Remark 2.2.3. *The functions which belong to the class $\mathcal{E}(X,\omega)$, although usually unbounded, have relatively mild singularities. In particular they have zero Lelong numbers.*

Let us stress that the convergence of μ_j towards μ_φ holds in the (strong) sense of Borel measures, i.e. for all Borel sets B,

$$\mu_\varphi(B) := \lim_{j\to+\infty} \mu_j(B).$$

In particular when $B = P$ is a pluripolar set, we obtain $\mu_j(P) = 0$, hence

$$\mu_\varphi(P) = 0 \text{ for all pluripolar sets } P.$$

Conversely, one can show [GZ07, BEGZ10] that a probability measure μ equals μ_φ for some $\varphi \in \mathcal{E}(X,\omega)$ whenever μ does not charge pluripolar sets (one then says that μ is non-pluripolar).

We shall use without proving it the following important fact:

Theorem 2.2.4. *The complex Monge-Ampère operator $\varphi \mapsto MA(\varphi)$ is well defined on the class $\mathcal{E}(X,\omega)$.*

2.2.2 The class $\mathcal{E}^1(X,\omega)$

The following class is going to play a central role:

Definition 2.2.5. *We let $\mathcal{E}^1(X,\omega)$ denote the class of those functions $\varphi \in \mathcal{E}(X,\omega)$ such that $\varphi \in L^1(MA(\varphi))$.*

Set

$$E(\varphi) = \frac{1}{(n+1)V} \sum_{j=0}^{n} \int_X \varphi(\omega + dd^c\varphi)^j \wedge \omega^{n-j}.$$

One can show, by using the next proposition that

$$\varphi \in \mathcal{E}^1(X,\omega) \longleftrightarrow E(\varphi) > -\infty,$$

where $E(\varphi)$ can be defined first for bounded functions and then setting

$$E(\varphi) := \inf\{E(\psi) \,|\, \psi \in PSH(X,\omega) \cap L^\infty(X) \text{ and } \psi \geq \varphi\}.$$

Proposition 2.2.6. *The functional E is a primitive of the complex Monge-Ampère operator, namely whenever $\varphi + tv$ belongs to $\mathcal{E}^1(X,\omega)$,*

$$\frac{dE(\varphi + tv)}{dt}_{|t=0} = \int_X vMA(\varphi).$$

The functional E is concave increasing, satisfies $E(\varphi+c) = E(\varphi)+c$ for all $c \in \mathbb{R}$, $\varphi \in \mathcal{E}^1(X,\omega)$, and the cocycle condition

$$E(\varphi) - E(\psi) = \frac{1}{(n+1)V} \sum_{j=0}^{n} \int_X (\varphi - \psi)(\omega + dd^c\varphi)^j \wedge (\omega + dd^c\psi)^{n-j}.$$

for all $\varphi, \psi \in \mathcal{E}^1(X,\omega)$.

Proof. By using the continuity property of the Monge-Ampère operators, it suffices to treat the case of bounded functions. We first justify that $E' = MA$. Setting $\varphi_t = \varphi + tv$, we compute

$$\begin{aligned}(n+1)V\frac{dE(\varphi_t)}{dt} &= \sum_{j=0}^{n} \int_X v\,(\omega + dd^c\varphi_t)^j \wedge \omega^{n-j} \\ &+ \sum_{j=0}^{n} j \int_X \varphi_t\, dd^c v \wedge (\omega + dd^c\varphi_t)^{j-1} \wedge \omega^{n-j}.\end{aligned}$$

Integrating by part in the last integral yields

$$\int_X \varphi_t\, dd^c v \wedge (\omega + dd^c\varphi_t)^{j-1} \wedge \omega^{n-j} = \int_X v\,(\omega_{\varphi_t} - \omega) \wedge (\omega + dd^c\varphi_t)^{j-1} \wedge \omega^{n-j+1}.$$

Rearranging the second sum yields a telescopic series and we end up with

$$(n+1)V\frac{dE(\varphi_t)}{dt} = (n+1) \int_X v\,(\omega + dd^c\varphi_t)^n,$$

as claimed.

If we formally replace ω by ω_ψ and φ by $\varphi - \psi$ in all these computations (this amounts to changing the representative of the cohomology class $\{\omega\}$), we end up similarly with

$$g'(t) = \frac{dG(\varphi + tv)}{dt}_{|t=0} = \int_X v \frac{(\omega_\psi + dd^c[\varphi - \psi])^n}{V} = \int_X vMA(\varphi),$$

where

$$G(\varphi) := \frac{1}{(n+1)V} \sum_{j=0}^{n} \int_X (\varphi - \psi)(\omega + dd^c\varphi)^j \wedge (\omega + dd^c\psi)^{n-j}.$$

For $v = \psi - \varphi$ and $e(t) = E(\varphi + tv)$, we therefore obtain $e' \equiv g'$ and $e(0) = E(\varphi), e(1) = E(\psi)$, while $g(0) = G(\varphi)$ and $g(1) = G(\psi) = 0$. This yields the cocycle condition $G(\varphi) = E(\varphi) - E(\psi)$.

The invariance under translations $E(\varphi + c) = E(\varphi) + c$ is immediate, it remains to show that E is concave (it is increasing as $E' = MA$ is a non-negative measure). Set again, for $v = \psi - \varphi$,

$$e(t) := E(\varphi + tv) = E(t\psi + (1-t)\varphi).$$

We have already shown that

$$e'(t) = \int_X vMA(t\psi + (1-t)\varphi) = V^{-1} \int_X v\,(t\omega_\psi + (1-t)\omega_\varphi)^n.$$

Taking derivative again yields

$$e''(t) = \frac{n}{V} \int_X v dd^c v \wedge (t\omega_\psi + (1-t)\omega_\varphi)^{n-1} \leq 0,$$

as follows from Stokes theorem. □

Proposition 2.2.7. *For all $C > 0$ the subsets*

$$\mathcal{E}^1_C(X,\omega) := \{\varphi \in \mathcal{E}^1(X,\omega) \,|\, E(\varphi) \geq -C \text{ and } \varphi \leq 0\}$$

are compact and convex.

Proof. The convexity property easily follows from the concavity of E. We let the reader check that E is upper semi-continuous, hence $\mathcal{E}^1_C(X,\omega)$ is a closed subset of the set

$$F_C := \{\varphi \in PSH(X,\omega) \,|\, -C \leq \sup_X \varphi \leq 0\}$$

which is easily seen to be compact. □

2.3 Solutions in $\mathcal{E}^1(X,\omega)$

2.3.1 Euler-Lagrange equations

Given $\lambda \in \mathbb{R}$, we consider the functional $\mathcal{F}_\lambda : \mathcal{E}^1(X,\omega) \to \mathbb{R}$ defined by

$$\mathcal{F}_\lambda(\varphi) = E(\varphi) + \frac{1}{\lambda} \log \left[\int_X e^{-\lambda\varphi} d\mu \right].$$

Note that $E(\varphi)$ is well defined for all functions $\varphi \in \mathcal{E}^1(X,\omega)$. The second term is easily seen to be well defined when $\lambda \leq 0$ (since ω-psh functions are bounded from above on X), while its finiteness follows from Skoda's integrability theorem, Hölder's inequality and the fact that functions in $\mathcal{E}^1(X,\omega)$ have zero Lelong numbers when $\lambda > 0$.

Our first basic observation is that $(MA)_\lambda$ is the Euler-Lagrange equation of the functional $\mathcal{F}_\lambda$:

Lemma 2.3.1.

$$\frac{d\mathcal{F}_\lambda(\varphi + tv)}{dt}_{|t=0} = \int_X vMA(\varphi) - \int_X v \frac{e^{-\lambda\varphi} d\mu}{\int_X e^{-\lambda\varphi} d\mu}.$$

Therefore if φ is a ω-plurisubharmonic critical point of $\mathcal{F}_\lambda$, then $\varphi + c$ is a solution of $(MA)_\lambda$ for an appropriate choice of $c \in \mathbb{R}$.

It is understood here that $\frac{e^{-\lambda\varphi}\mu}{\int_X e^{-\lambda\varphi} d\mu} = \mu$, when $\lambda = 0$ (there is no need to change φ in $\varphi + c$ in this case).

Proof. We have already observed that

$$\frac{dE(\varphi + tv)}{dt}_{|t=0} = \int_X vMA(\varphi)$$

while the second term comes from the derivative of the log term.

If φ is a ω-plurisubharmonic critical point of $\mathcal{F}_\lambda$, the above derivative vanishes for all test functions $v \in \mathcal{C}^\infty(X,\mathbb{R})$, therefore

$$MA(\varphi) = \frac{e^{-\lambda\varphi}\mu}{\int_X e^{-\lambda\varphi} d\mu}.$$

When $\lambda = 0$, this is thus a solution to $(MA)_0$. When $\lambda \neq 0$, we note that

$$\tilde{\varphi} := \varphi + \frac{1}{\lambda} \log \left[\int_X e^{-\lambda\varphi} d\mu \right]$$

is a solution to $(MA)_\lambda$. □

Lemma 2.3.2. *The functional $\mathcal{F}_\lambda : \mathcal{E}^1(X,\omega) \to \mathbb{R}$ is invariant by translations and it is upper semi-continuous on each compact subset $\mathcal{E}^1_C(X,\omega)$.*

Proof. The invariance by translations is straightforward. Recall that $\varphi \mapsto E(\varphi)$ is upper semi-continuous (the class $\mathcal{E}^1(X,\omega)$ is endowed with the induced L^1-topology). We check the semi-continuity property separately for each case $\lambda > 0, \lambda = 0, \lambda > 0$. We leave the cases $\lambda \leq 0$ to the reader and treat the most difficult one:

Case $\lambda > 0$. We can assume without loss of generality that $\lambda = +1$ (up to rescaling ω) We need to check that

$$\varphi \in \mathcal{E}^1_C(X,\omega) \mapsto \int_X e^{-\varphi} d\mu \in \mathbb{R}$$

is upper semi-continuous. Here $\mu = f\omega^n$ is absolutely continuous with respect to Lebesgue measure, with density $f \in L^p(\omega^n)$ for some $p > 1$. It follows from Fatou's lemma that $\varphi \in \mathcal{E}^1(X,\omega) \mapsto \int_X e^{-\varphi} d\mu \in \mathbb{R}$ is lower semi-continuous. To establish an upper-semi continuity property we restrict ourselves to the compact convex subset $\mathcal{E}^1_C(X,\omega)$: let φ_j be a sequence in $\mathcal{E}^1_C(X,\omega)$ converging to φ; since all these functions have zero Lelong number, it follows from Skoda's uniform integrability theorem and Hölder inequality that

$$\int_X e^{-2(\varphi_j+\varphi)} d\mu = \int_X e^{-2(\varphi_j+\varphi)} f dV \leq \|f\|_{L^p} \left(\int_X e^{-2q(\varphi_j+\varphi)} dV \right)^{1/q} \leq C'.$$

Now

$$\left| \int_X e^{-\varphi_j} d\mu - \int_X e^{-\varphi} d\mu \right| \leq \int_X |\varphi_j - \varphi| e^{-(\varphi_j+\varphi)} d\mu \leq C' ||\varphi_j - \varphi||_{L^2(\mu)},$$

as follows from Cauchy-Schwarz inequality and the elementary inequality

$$\left| e^a - e^b \right| \leq |a-b| \, e^{a+b}, \text{ for all } a, b \geq 0.$$

The conclusion follows since (φ_j) converges to φ in $L^2(\mu)$. □

2.3.2 Properness of the functional

Definition 2.3.3. *We say that the functional $\mathcal{F}_\lambda$ is proper if whenever $\varphi_j \in \mathcal{E}^1(X,\omega)$ are such that $E(\varphi_j) \to -\infty$ and $\int_X \varphi_j \omega^n = 0$, then $\mathcal{F}_\lambda(\varphi_j) \to -\infty$.*

In many articles, this condition is expressed by saying that $\mathcal{F}_\lambda(\varphi_j) \to -\infty$ whenever $J(\varphi_j) \to -\infty$, where $J(\varphi) = E(\varphi) - \frac{1}{V} \int_X \varphi \omega^n$ is the translation invariant functional associated to E. Note that there exists $C_0 > 0$ such that for all $\varphi \in PSH(X,\omega)$,

$$\int_X \varphi\, \omega^n \leq \sup_X \varphi \leq \int_X \varphi \omega^n + C_0,$$

thus one can equivalently normalize φ by asking for $\sup_X \varphi = 0$.

An important issue is to determine whether $\mathcal{F}_\lambda$ is proper. This turns out to be always the case when $\lambda \leq 0$. This is not necessarily so when $\lambda > 0$ and indeed one can not always solve $(MA)_\lambda$ when $\lambda > 0$.

Theorem 2.3.4.
(i) $\mathcal{F}_\lambda(\varphi) \leq E(\varphi) - \sup_X \varphi + C_{\mu,\lambda}$ *is always proper when* $\lambda < 0$*;*
(ii) $\mathcal{F}_0$ *is proper: there exists* $C > 0$ *such that for all* $\varphi \in \mathcal{E}^1(X,\omega)$

$$\mathcal{F}_0(\varphi) \leq E(\varphi) - \sup_X \varphi + C \left| E(\varphi) - \sup_X \varphi \right|^{1/2}.$$

(iii) There exists $\alpha(\{\omega\}) > 0$ *such that for all* $0 < \lambda < \alpha(\{\omega\})$*, there exists* $C_\lambda > 0$ *such that for all* $\varphi \in \mathcal{E}^1(X,\omega)$*,*

$$\mathcal{F}_\lambda(\varphi) \leq E(\varphi) - \sup_X \varphi + C_\lambda,$$

hence $\mathcal{F}_\lambda$ *is proper for* $0 < \lambda < \alpha(\{\omega\})$*.*

Proof. The first point is straightforward: it suffices to check, setting $\tau = -\lambda > 0$, that there exists $\delta > 0$ such that

$$\int_X e^{\tau(\varphi - \sup_X \varphi)} d\mu \geq \delta > 0$$

for all $\varphi \in PSH(X,\omega)$. This easily follows from the fact that the set of normalized $\tau\omega$-psh functions is compact.

We assume now that $\lambda = 0$. It follows from Hölder's inequality that $\mu = f\omega_X^n$ verifies $\mathcal{E}^1(X,\omega) \subset L^1(\mu)$. We let the reader check that in this case, for each $C > 0$ there exists $M_C > 0$ such that

$$\forall \psi \in \mathcal{E}^1_C(X,\omega), \quad \int_X \psi d\mu \geq -M_C.$$

Fix $\varphi \in \mathcal{E}^1(X,\omega)$ such that $\sup_X \varphi = 0$ and $E(\varphi) \geq -1$. We set $\varepsilon := |E(\varphi)|^{-1/2} \in]0,1]$ and consider

$$\psi := \varepsilon\varphi = \frac{\varphi}{|E(\varphi)|^{1/2}} \in PSH(X,\omega).$$

Observe that

$$E(\psi) = \frac{\varepsilon}{(n+1)V} \sum_{j=0}^{n} \int_X \varphi\, \omega_\psi^j \wedge \omega^{n-j}.$$

We let the reader check that there exists $N \in \mathbb{N}$ such that for all $j \geq 1$,

$$\omega_\psi^j \wedge \omega^{n-j} \leq \omega^j + N\varepsilon \sum_{l=1}^{n} \omega_\varphi^l \wedge \omega^{n-l}.$$

We infer, since $\varphi, \psi \leq 0$,

$$E(\psi) \geq \frac{1}{V}\varphi\omega^n + \varepsilon^2 N' E(\varphi) \geq -C.$$

Since $I(\psi) := \int_X \psi d\mu$ is uniformly bounded on $\mathcal{E}^1_C(X,\omega)$, this yields

$$\int_X \varphi\, d\mu = |E(\varphi)|^{1/2} \int_X \psi\, d\mu \geq -C' |E(\varphi)|^{1/2},$$

hence the proof of (ii) is complete.

We finally treat the case $\lambda > 0$. The upper bound (iii) is a consequence of the uniform upper bound

$$\int_X e^{-\lambda(\varphi - \sup_X \varphi)} f \omega^n \leq C$$

which follows from Hölder inequality and Skoda's uniform integrability theorem (see Theorem 2.1.1): for all $\varphi \in PSH(X,\omega)$,

$$\int_X e^{-q\lambda(\varphi - \sup_X \varphi)} \omega^n \leq C(q\lambda)$$

since the set of normalized ω-psh functions $\psi = \varphi - \sup_X \varphi$ is compact, hence these integrals are uniformly bounded from above as soon as λ is so small that the Lelong numbers of the functions $q\lambda\psi$ are smaller than 2. □

2.3.3 Maximizers are critical points

Theorem 2.3.5. *If $\mathcal{F}_\lambda$ is proper, then there exists $\varphi \in \mathcal{E}^1(X,\omega)$ which solves $(MA)_\lambda$ and such that*

$$\mathcal{F}_\lambda(\varphi) = \sup_{\mathcal{E}^1(X,\omega)} \mathcal{F}_\lambda.$$

Proof. Since $\mathcal{F}_\lambda$ is invariant by translations and proper, we can find $C > 0$ so large that

$$\sup_{\mathcal{E}^1(X,\omega)} \mathcal{F}_\lambda = \sup_{\mathcal{E}^1_C(X,\omega)} \mathcal{F}_\lambda,$$

where we recall that

$$\mathcal{E}^1_C(X,\omega) := \{\varphi \in \mathcal{E}^1(X,\omega) \,|\, E(\varphi) \geq -C \text{ and } \varphi \leq 0\}$$

is a compact convex subset of $\mathcal{E}^1(X,\omega)$. Now $\mathcal{F}_\lambda$ is upper semi-continuous on $\mathcal{E}^1_C(X,\omega)$, thus we can find $\varphi \in \mathcal{E}^1_C(X,\omega) \subset \mathcal{E}^1(X,\omega)$ which maximizes the functional $\mathcal{F}_\lambda$ on $\mathcal{E}^1(X,\omega)$.

If φ where known to be smooth and strictly ω-psh, we could consider $\varphi + tv$ for $t > 0$ small and v an arbitrary smooth function and conclude that φ is a critical point of $\mathcal{F}_\lambda$, hence $\varphi + c$ solves $(MA)_\lambda$ (by Lemma 2.3.1).

This turns out to be the case when μ is a smooth volume form, however this requires a lot of extra work and our measures μ are here allowed to vanish, hence we cannot expect φ to be strictly ω-psh.

We therefore use the following detour. Fix $v \in \mathcal{C}^0(X,\mathbb{R})$ an arbitrary continuous function and consider

$$g(t) := E \circ P(\varphi + tv) + \frac{1}{\lambda} \log \left[\int_X e^{-\lambda(\varphi + tv)} d\mu \right],$$

where, given h an upper semi-continuous function on X, we let

$$P(h)(x) := \sup\{\psi(x) \in \mathbb{R} \,|\, \psi \in PSH(X,\omega) \text{ and } \psi \leq h\}$$

denote the ω-psh envelope of h (see section 2.1.3). We show in Theorem 2.3.6 below that the function g is differentiable at the origin, with

$$g'(0) = \langle MA(\varphi), v\rangle - \left\langle \frac{e^{-\lambda\varphi}\mu}{\int_X e^{-\lambda\varphi} d\mu}, v \right\rangle.$$

Since $\varphi \in \mathcal{E}^1(X,\omega)$ and tv is bounded below, we see that $P(\varphi + tv) \in \mathcal{E}^1(X,\omega)$. Now $P(\varphi + tv) \leq \varphi + tv$ hence $E \circ P(\varphi + tv) \leq E(\varphi + tv)$ and

$$g(t) \leq \mathcal{F}_\lambda \circ P(\varphi + tv) \leq g(0),$$

as φ is a maximizer. We infer $g'(0) = 0$, which finishes the proof. □

2.3.4 The projection theorem

The goal of this section is to prove the following result of Berman-Boucksom [BeBo10], on which relies the variational approach:

Theorem 2.3.6. *Fix* $\varphi \in \mathcal{E}^1(X,\omega)$ *and* $v \in \mathcal{C}^0(X,\mathbb{R})$. *Then* $t \mapsto E \circ P(\varphi+tv)$ *is differentiable at zero, with*

$$\frac{dE \circ P(\varphi+tv)}{dt}_{|t=0} = \int_X vMA(\varphi).$$

Proof. We first let the reader convince him/herself that it suffices to treat the case when v is smooth. We shall use this assumption by the end of the proof. We first claim that the quantities

$$\frac{E \circ P(\varphi+tv) - E(\varphi)}{t} \quad \text{and} \quad \int_X \frac{P(\varphi+tv)-\varphi}{t} MA(\varphi)$$

have the same asymptotic as $t \to 0$. It follows on the one hand from the concavity of E together with $E' = MA$ that

$$E \circ P(\varphi+tv) \leq E(\varphi) + E'(\varphi) \cdot [P(\varphi+tv) - \varphi]$$

hence

$$E \circ P(\varphi+tv) - E(\varphi) \leq \int_X [P(\varphi+tv)-\varphi] \; MA(\varphi).$$

On the other hand the projection P is concave hence

$$P(\varphi+tsv) = P((1-t)\varphi + t(\varphi+sv)) \geq (1-t)P(\varphi) + tP(\varphi+sv)$$

yields, by using the monotonicity of E,

$$\begin{aligned} E \circ P(\varphi+tsv) &\geq E\left(\varphi + t[P(\varphi+sv)-\varphi]\right) \\ &= E(\varphi) + t\int_X [P(\varphi+sv)-\varphi] MA(\varphi) + o(t), \end{aligned}$$

where the last equality corresponds to the differentiability of E at φ. Thus

$$\frac{E \circ P(\varphi+tsv) - E(\varphi)}{st} \geq \int_X \frac{P(\varphi+sv)-\varphi}{s} MA(\varphi) + o(1)$$

where $s > 0$ is arbitrary and $t \to 0$. This proves our first claim.

We now want to prove that

$$\int_X \frac{P(\varphi+tv)-\varphi}{t} MA(\varphi) = \int_X v\, MA(\varphi) + o(1),$$

which is equivalent to showing that $\int_X f_t MA(\varphi) \to 0$ as $t \to 0$, where

$$f_t = \frac{P(\varphi + tv) - \varphi - tv}{t}.$$

Observe that $f_t \leq 0$ since by definition $P(\varphi + tv) \leq \varphi + tv$. Note also that the projection P is uniformly 1-Lipschitz,

$$\sup_X |P(\varphi_1) - P(\varphi_2)| \leq \sup_X |\varphi_1 - \varphi_2|$$

hence $\sup_X |P(\varphi + tv) - P(\varphi)| \leq t \sup_X |v|$ and there exists $C > 0$ such that $\inf_X f_t \geq -C$. It therefore suffices to show that

$$MA(\varphi)(\{f_t < 0\}) = MA(\varphi)(\{P(\varphi + tv) < \varphi + tv\}) \overset{t \to 0}{\longrightarrow} 0.$$

This follows from the comparison principle as we now explain. Since v is smooth, we can find $\varepsilon > 0$ such that v is $\varepsilon\omega$-psh. To simplify we assume $\varepsilon = 1$ in the sequel. Set $\Omega_t := \{P(\varphi + tv) < \varphi + tv\}$ and use the comparison principle to deduce that

$$\int_{\Omega_t} MA(\varphi) \leq V^{-1} \int_{\Omega_t} (\omega_\varphi + t\omega_v)^n \leq V^{-1} \int_{\Omega_t} (\omega_{P(\varphi+tv)} + t\omega)^n.$$

Since $MA(P(\varphi + tv))$ vanishes on the set $\Omega_t = \{P(\varphi + tv) < \varphi + tv\}$ (by Proposition 2.1.4), we can develop the last term to obtain

$$\int_{\Omega_t} MA(\varphi) \leq (1 + t)^n - 1 = O(t),$$

and the proof is complete. □

2.4 Kähler-Einstein currents

2.4.1 Uniqueness

Theorem 2.4.1. *The solutions to* $(MA)_\lambda$ *is unique (respectively unique up to an additive constant) when* $\lambda < 0$ *(resp.* $\lambda = 0$*).*

Proof. The proof is easy when $\lambda < 0$ and follows from the comparison principle: indeed if ψ is another solution of $(MA)_\lambda$, then

$$MA(\psi) = e^{\tau\psi}\mu = e^{\tau(\psi - \varphi)} MA(\varphi)$$

setting $\tau = -\lambda > 0$, and the comparison principle shows that

$$\int_{(\varphi<\psi)} MA(\psi) = \int_{(\varphi<\psi)} e^{\tau(\psi-\varphi)} MA(\psi) \leq \int_{(\varphi<\psi)} MA(\varphi),$$

hence $\mu(\varphi < \psi) = 0$. We infer $\psi \leq \varphi$ almost everywhere with respect to Lebesgue measure, hence everywhere by quasi-plurisubharmonicity. By symmetry we conclude that $\varphi \equiv \psi$.

Assume now $\lambda = 0$ and $\varphi, \psi \in \mathcal{E}^1(X, \omega)$ verify $MA(\varphi) = MA(\psi) = \mu$. Observe that $0 = MA(\varphi) - MA(\psi) = dd^c(\varphi - \psi) \wedge S$, where

$$S = \sum_{j=0}^{n-1} \omega_\varphi^j \wedge \omega_\psi^{n-1-j} \geq 0.$$

Integrating against $(\psi - \varphi)$ and using Stokes theorem yields

$$0 = \int_X d(\varphi - \psi) \wedge d^c(\varphi - \psi) \wedge S \geq 0.$$

This shows that $(\varphi - \psi)$ is constant when $\omega_\varphi, \omega_\psi$ are Kähler forms (this argument is due to Calabi). Tricky integration by parts allow to extend this argument to the case of general functions in $\mathcal{E}^1(X, \omega)$ (see [Blo03, Kol05, GZ07, Din09, BEGZ10] for more details). □

The uniqueness problem is very delicate and largely open when $\lambda > 0$. When $\{\omega\} = c_1(X)$ is the First Chern class of X which is thus weak Fano, it follows from the work of Bando-Mabuchi [BM87] and Berndtsson [Bern11] that any two solutions to $(MA)_\lambda$ –when they exist– are connected by the time-one map of a holomorphic flow. In particular we note for later use the following:

Theorem 2.4.2. *When $\{\omega\} = c_1(X)$ and $H^0(X, TX) = \{0\}$, there exists at most one solution to $(MA)_1$.*

2.4.2 The variational characterization of K-E currents

We now specialise to the case where X has mild singularities and the cohomology class $\lambda\{\omega\} = c_1(X)$ is proportional to the first Chern class of X, so that solutions of $(MA)_\lambda$ correspond to Kähler-Einstein currents of finite energy (cf Lecture 1).

The following result summarizes previous results and will be used as such in Lecture 3:

Theorem 2.4.3. *Let X be a $\mathbb{Q}$-Gorenstein variety with only klt singularities. Let $\alpha \in H^{1,1}(X, \mathbb{R})$ be a Kähler class and $\lambda \in \mathbb{R}$ such that $c_1(X) = \lambda\alpha$.*

We assume that the functional $\mathcal{F}_\lambda$ is proper so that there exists a unique Kähler-Einstein current $T_{KE} \in \alpha$ with finite energy. Fix $T \in \mathcal{E}^1(\alpha)$ a positive current with finite energy. The following are equivalent:

1. *T maximizes the functional $\mathcal{F}_\lambda$;*

2. *$T = T_{KE}$ is the unique Kähler-Einstein current.*

By convention we write here $T \in \mathcal{E}^1(\alpha)$ if $T = \omega + dd^c\varphi$ with $\varphi \in \mathcal{E}^1(X, \omega)$. This does not depend on the choice of representatives. Note that there can be no non zero holomorphic vector field if $\mathcal{F}_\lambda$ is proper.

It was first realized by Ding-Tian [DT92] that, when X is a Kähler-Einstein Fano manifold with no holomorphic vector field, the Kähler-Einstein metric is the unique Kähler metric maximizing $\mathcal{F}_1$. This result being extended to the class of finite energy currents allows to use the soft compacity criteria available in these Sobolev-like spaces, we have in particular the following useful result:

Corollary 2.4.4. *Under the same assumptions as above, assume that $T_j \in \mathcal{E}^1_C(\alpha)$ is a sequence of positive currents with uniformly bounded energies. If $\mathcal{F}_\lambda(T_j)$ increases towards $\sup_{\mathcal{E}^1} \mathcal{F}_\lambda$, then T_j weakly converges towards T_{KE}.*

An interesting application will be given in Lecture 3, when we will study the long term behavior of the normalized Kähler-Ricci flow on Fano varieties.

2.4.3 Regularity

Since we are dealing from the beginning with probability measures $\mu = f\omega_X^n$ whose density is in L^p, $p > 1$, it follows from the work of Kolodziej [Kol98] and its extensions to non Kähler classes [EGZ09, BEGZ10] that the solution $\varphi \in \mathcal{E}^1(X, \omega)$ to

$$MA(\varphi) = e^{\lambda\varphi}\mu$$

is actually *bounded*.

When μ is moreover a smooth volume form, it was proved by Yau [Yau78] that φ is smooth, when $\lambda \leq 0$ and ω is a Kähler form. Yau's proof was extended to the case of Fano manifolds (see [Tian]).

The regularity theory has been extended to the case of varieties by various authors (see [Tsu88, TZha06, ST07, Pau08, EGZ09]). An elegant result of Szekelyhidi-Tosatti [SzTo09] insures that, for many Monge-Ampère type equations and when the reference cohomology class is Kähler, any bounded solution is automatically smooth. This result can be extended to some cases where the class is non Kähler (see [ST09, BBEGZ11]), providing smooth solutions on the ample locus of the reference class (equivalently on the regular part of a variety, when the original problem comes from a singular setting).

A major **open problem** is then to understand the asymptotic regularity/behavior at the boundary of the ample locus/near singular points. We refer the reader to [EGZ11] for a first result in this direction.

Lecture 3

Convergence of the singular Kähler-Ricci flow

The Ricci flow, first introduced by Hamilton [Ham82, Ham88] three decades ago, is the equation

$$\frac{\partial g_t}{\partial t} = -Ric(g_t),$$

evolving a Riemannian metric by its Ricci curvature. When the initial metric g_0 is Kähler, so is g_t for all $t > 0$ and the flow is thus called the Kähler-Ricci flow.

The Kähler-Ricci flow was used by H.D.Cao [Cao85] to give a new proof of the existence of Kähler-Einstein metrics of non positive curvature. The idea was pushed further by Tsuji [Tsu88], Tian-Zhang [TZha06] and Song-Tian [ST07, ST08, ST09] who studied extensively the case of manifolds with non-negative Kodaira dimension. We refer the interested reader to the survey by [SW] for a neat treatment of these cases.

The situation is far less understood on manifolds of Kodaira dimension $-\infty$, and the Kähler-Einstein problem is still largely open on Fano manifolds, despite several works and progress by many authors.

The goal of this third lecture is to explain how the (normalized) Kähler-Ricci flow can be used to detect the unique Kähler-Einstein current, on a Kähler-Einstein Fano variety with no holomorphic vector field, following [BBEGZ11]. This result was first obtained on smooth Fano manifolds in [TZ07] (see also [PSS07]), by using deep estimates of Perelman. We provide an alternative approach, by using the variational characterization explained in Lecture 2.

3.1 Perelman's theorem

Let X be a Fano manifold. It follows from the work of Bando-Mabuchi [BM87] and Ding-Tian [DT92, Tian97] that X admits a unique Kähler-Einstein metric ω_{KE} if and only if the Mabuchi functional is proper.

It is natural to wonder whether the Kähler-Ricci flow can detect it, hence serve as a useful tool in studying the Kähler-Einstein problem on Fano manifolds. Perelman established in 2003 deep estimates (see [SeT08]), which allowed Tian and Zhu [TZ07] to prove the following:

Theorem 3.1.1. *Let X be a Fano manifold whose Mabuchi functional is proper. Fix $\omega_0 \in c_1(X)$ an arbitrary Kähler form. Then the normalized Kähler-Ricci flow*

$$\begin{cases} \frac{\partial \omega_t}{\partial t} = -Ric(\omega_t) + \omega_t \\ \omega_{t|t=0} = \omega_0 \end{cases}$$

$\mathcal{C}^\infty$-converges, as $t \to +\infty$, to the unique Kähler-Einstein metric ω_{KE}.

In other words, the normalized Kähler-Ricci flow detects the (unique) Kähler-Einstein metric if it exists.

This result has been generalized by Tian and Zhu [TZ07] to the case of Kähler-Ricci soliton. Other generalizations by Phong and his collaborators can be found in [PS06, PSS07, PSSW08, PS10]. All proofs rely on deep estimates due to Perelman [SeT08].

It is natural to try and understand similar problems on Fano *varieties*, i.e. allowing for mild singularities like those arising in the Minimal Model Program. The goal of this third lecture is to explain the proof of the following result [BBEGZ11]:

Theorem 3.1.2. *Let X be a $\mathbb{Q}$-Fano variety whose Mabuchi functional is proper. Fix $\omega_0 \in c_1(X)$ an arbitrary Kähler form. Then the normalized Kähler-Ricci flow*

$$\begin{cases} \frac{\partial \omega_t}{\partial t} = -Ric(\omega_t) + \omega_t \\ \omega_{t|t=0} = \omega_0 \end{cases}$$

weakly converges, as $t \to +\infty$, to the unique Kähler-Einstein current T_{KE}.

Recall that a $\mathbb{Q}$-Fano variety means here a normal variety whose anticanonical bundle $-K_X$ is $\mathbb{Q}$-Cartier and ample, and such that X has only log terminal singularities. The existence and uniqueness of such Kähler-Einstein currents was addressed in Lecture 2.

Since Perelman's estimates are not available in this singular context, we actually need to produce an alternative proof of Theorem 3.1.1. This is certainly of independent interest and we will actually focus on the smooth case to simplify the exposition. The drawback is of course that the convergence only holds in a weak (energy) sense.

The *idea of the proof* is to show that $(\omega_t)_{t>0}$ are relatively compact in the finite energy class $\mathcal{E}^1(X,\omega)$ and form a maximizing sequence for the Ding functional $\mathcal{F}$. Any cluster point will thus be Kähler-Einstein current (by the variational characterization from Lecture 2), but the properness assumption insures that there is no non-trivial holomorphic vector field, hence a unique Kähler-Einstein current by Bando-Mabuchi-Berndtsson' uniqueness result.

Here is a *sketch of the proof*:

(i) we first observe that the Mabuchi $\mathcal{M}$ and the Ding $\mathcal{F}$ functionals are non decreasing along the normalized Kähler-Ricci flow;

(ii) the properness assumption insures then that the ω_t's have uniformly bounded energies, hence belong to a compact sublevel set of finite energy;

(iii) if we can show that $\mathcal{F}(\omega_t)$ increases towards the absolute maximum of $\mathcal{F}$, every cluster point of (ω_t) will maximize $\mathcal{F}$ hence coincide with the unique Kähler-Einstein current with finite energy, thus $\omega_t \to T_{KE}$;

(iv) the last step consists in carefully selecting a subsequence $t_j \to +\infty$ along which one can insure that $\mathcal{F}(\omega_{t_j}) \nearrow \sup \mathcal{F}$: this is done by choosing t_j so that the time derivative $\frac{\partial \omega_{t_j}}{\partial t} \to 0$ and by noting that the ω_t's have uniformly bounded entropies.

Comparison with Perelman-Tian-Zhu's result. Although convergence results in the smooth category (see [TZ07, PSS07]) are not formulated in a variational manner, one can try and prove them following the scheme we've just presented. The difficult task is then Step 2 (show relative compactness of the ω_t's in $\mathcal{C}^\infty$) which requires the use of Perelman deep estimates and further extra work, while Step 4 (the delicate part here) is then a formality.

The *plan of the lecture* is as follows. We actually work at the level of potentials, so our first task is to explain how the normalized Kähler-Ricci flow can be reduced to a scalar parabolic equation on potentials. We then establish each step of the sketch in the (simpler) smooth setting, and we briefly address the singular context in the last section.

3.2 Reduction to a scalar parabolic equation

The Ricci flow is the parabolic evolution equation

$$(KRF) \qquad \frac{\partial \omega_t}{\partial t} = -\mathrm{Ric}(\omega_t) \quad \text{with initial data } \omega_0.$$

When ω_0 is a Kähler form, so is ω_t, hence it is called the Kähler-Ricci flow.

3.2.1 Long time existence

The short time existence is guaranteed by standard parabolic theory: in the Kähler context, this translates into a parabolic scalar equation as we explain below.

It is more convenient to analyze the long time existence by considering the normalized Kähler-Ricci flow. For a Fano manifold this is

$$(NKRF) \qquad \frac{\partial \omega_t}{\partial t} = -\mathrm{Ric}(\omega_t) + \omega_t.$$

One passes from (KRF) to (NKRF) by changing $\omega(t)$ in $e^t\omega(1-e^{-t})$. At the level of cohomology classes,

$$\frac{d\{\omega_t\}}{dt} = -c_1(X) + \{\omega_t\} \in H^{1,1}(X,\mathbb{R})$$

therefore $\{\omega_t\} \equiv c_1(X)$ is constant if we start from $\omega_0 \in c_1(X)$. This justifies the name (NKRF) since in this case

$$\mathrm{vol}_{\omega_t}(X) = \mathrm{vol}_{\omega_0}(X) = c_1(X)^n$$

is constant. Note that the volume blows up exponentially fast if $\{\omega_0\} > c_1(X)$. The following result is due to H.D.Cao [Cao85]:

Theorem 3.2.1. *Let X be a Fano manifold and pick a Kähler form $\omega_0 \in c_1(X)$. Then the normalized Kähler-Ricci flow exists for all times $t > 0$.*

The main issue is then whether (ω_t) converges as $t \to +\infty$. Hopefully $\partial\omega_t/\partial t \to 0$ and $\omega_t \to \omega_{KE}$ such that $\mathrm{Ric}(\omega_{KE}) = \omega_{KE}$.

3.2.2 Normalization of potentials

Let $\omega = \omega_0 \in c_1(X)$ denote the initial data. Since ω_t is cohomologous to ω, we can find $\varphi_t \in PSH(X,\omega)$ a smooth function such that $\omega_t = \omega + dd^c\varphi_t$. The function φ_t is defined up to a time dependent additive constant. Then

$$\frac{d\omega_t}{dt} = dd^c\dot{\varphi}_t = -\mathrm{Ric}(\omega_t) + \omega + dd^c\varphi_t,$$

where $\dot{\varphi}_t := \partial\varphi_t/\partial t$. Let $h \in \mathcal{C}^\infty(X,\mathbb{R})$ be the unique function such that

$$\mathrm{Ric}(\omega) = \omega - dd^c h, \text{ normalized so that } \int_X e^{-h}\omega^n = V.$$

We also consider $h_t \in \mathcal{C}^\infty(X, \mathbb{R})$ the unique function such that

$$\mathrm{Ric}(\omega_t) = \omega_t - dd^c h_t, \text{ normalized so that } \int_X e^{-h_t} \omega_t^n = V.$$

The following deep estimates of Perelman are crucial in analyzing the smooth convergence of the normalized flow:

Theorem 3.2.2 (Perelman 03)**.** *The Ricci deviation h_t is uniformly bounded along the flow, as well as its gradient and Laplacian,*

$$\sup_{t>0} ||h_t|| + ||\nabla h_t|| + ||\Delta h_t|| < +\infty.$$

The gradient and Laplacian are here computed with respect to the (evolving) metric $\omega_t := \omega + dd^c \varphi_t$. We refer the reader to [SeT08] for a proof and to [TZ07, PSS07, PS10] for applications in various contexts. Our goal in the sequel is actually to avoid using these estimates which are not available in the singular setting.

Observe that $\mathrm{Ric}(\omega_t) = \omega - dd^c h - dd^c \log(\omega_t^n / \omega^n)$, hence

$$dd^c \left\{ \log\left(\frac{\omega_t^n}{\omega^n}\right) + h + \varphi_t - \dot{\varphi}_t \right\} = 0,$$

therefore

$$(\omega + dd^c \varphi_t)^n = e^{\dot{\varphi}_t - \varphi_t - h - \beta(t)} \omega^n,$$

for some normalizing constant $\beta(t)$.

Note also that $dd^c \dot{\varphi}_t = -\mathrm{Ric}(\omega_t) + \omega_t = dd^c h_t$ hence

$$\dot{\varphi}_t(x) = h_t(x) + \alpha(t)$$

for some time dependent constant $\alpha(t)$. Our plan is to show the convergence of the metrics $\omega_t = \omega + dd^c \varphi_t$ by studying the properties of the potentials φ_t, so we should be very careful in the way we normalize the latter.

Observe that $\omega_0 = \omega + dd^c \varphi_0 = \omega$, hence $\varphi_0(x) \equiv c_0 \in \mathbb{R}$ is a constant which may play an important role, depending on the choice of normalization.

There are at least two "natural" choices of normalization. One can impose $\beta \equiv 0$, by changing $\varphi_t(x)$ in $\varphi_t(x) + B(t)$ with $B' - B = -\beta$, as proposed by Chen-Tian in [CT02]. This yields

$$(\omega + dd^c \varphi_t)^n = e^{\dot{\varphi}_t - \varphi_t - h} \omega^n.$$

One can then further change $\varphi_t(x)$ in $\varphi_t(x) + \kappa e^t$ without affecting the previous flow equation. It is then crucial to choose c_0 suitably, since it clearly affects (by an exponential term) the long term behavior of $\varphi_t(x)$.

One may on the other hand prefer to impose $\alpha \equiv 0$, as we choose to do in the sequel, following [BBEGZ11]. This normalization is probably the one Perelman had in mind[1], as it then yields $\dot\varphi_t \equiv h_t$ and allows to apply directly Perelman's deep estimates. We let the reader verify that this is equivalent to choosing

$$\beta(t) = \log\left[\frac{1}{V}\int_X e^{-\varphi_t - h_0}\omega^n\right]$$

so that

$$\dot\varphi_t = \log\left[\frac{MA(\varphi_t)}{\mu(\varphi_t)}\right],$$

where

$$MA(\varphi_t) := \frac{1}{V}(\omega + dd^c\varphi_t)^n \text{ and } \mu(\varphi_t) := \frac{e^{-\varphi_t - h_0}\omega^n}{\int_X e^{-\varphi_t - h_0}\omega^n}$$

are both probability measures.

Note that the equation is now invariant under $\varphi_t(x) \mapsto \varphi_t(x) - c_0$, so that we can assume without loss of generality that $c_0 = 0$.

3.3 Monotonicity of functionals along the flow

3.3.1 Ding functional

Recall that Ding's functional is defined by

$$\mathcal{F}(\varphi) := E(\varphi) + \log\left[\frac{1}{V}\int e^{-\varphi - h}\omega^n\right],$$

where

$$E(\varphi) = \frac{1}{(n+1)V}\sum_{j=0}^{n}\int_X \varphi\,\omega_\varphi^j \wedge \omega^{n-j}$$

is the primitive of the Monge-Ampère operator normalized so that $E(0) = 0$.

Lemma 3.3.1. *The functional $\mathcal{F}$ is non-decreasing along the normalized Kähler-Ricci flow. More precisely,*

$$\frac{d\mathcal{F}(\varphi_t)}{dt} = H_{MA(\varphi_t)}(\mu_t) + H_{\mu_t}(MA(\varphi_t)) \geq 0.$$

[1] In his 2003 seminar talk at MIT, Perelman apparently focused on his key estimates and did not say much about the remaining details.

Here $H_\mu(\nu)$ denotes the relative entropy of the probability measure ν with respect to the probability measure μ. It is defined by

$$H_\mu(\nu) = \int_X \log\left(\frac{\nu}{\mu}\right) d\nu$$

if ν is absolutely continuous with respect to μ, and $H_\mu(\nu) = +\infty$ otherwise. It follows from the concavity of the logarithm that

$$H_\mu(\nu) = -\int_X \log\left(\frac{\mu}{\nu}\right) d\nu \geq -\log(\mu(X)) = 0,$$

with strict inequality unless $\nu = \mu$.

Proof. Recall that $\mathcal{F}(\varphi) = E(\varphi) + \log\left[\int_X e^{-\varphi-h_0}\,\omega^n/V\right]$, where E is a primitive of the complex Monge-Ampère operator. We thus obtain along the NKRF,

$$\frac{dE(\varphi_t)}{dt} = \int_X \dot{\varphi}_t MA(\varphi_t) = \int_X \log\left(\frac{MA(\varphi_t)}{\mu_t}\right) MA(\varphi_t) = H_{\mu_t}(MA(\varphi_t)),$$

while

$$\frac{d\log\left[\int_X e^{-\varphi_t-h_0}\,\omega^n\right]}{dt} = -\int_X \dot{\varphi}_t d\mu_t = H_{MA(\varphi_t)}(\mu_t).$$

This proves the lemma. □

Since the relative entropy $H_\mu(\nu)$ is positive unless $\mu \equiv \nu$, the normalized Kähler-Ricci flow is increasing unless we have reached a fixed point, i.e. a Kähler-Einstein metric.

Pinsker's inequality (see [Villani, Remark 22.12]) gives an explicit lower bound for $H_\mu(\nu)$ in terms of the total variation of the signed measure $\mu - \nu$: for all Borel subsets $A \subset X$,

$$\begin{aligned} H_\mu(\nu) &\geq \nu(A)\log\left[\frac{\nu(A)}{\mu(A)}\right] + [1-\nu(A)]\log\left[\frac{1-\nu(A)}{1-\mu(A)}\right] \\ &\geq 2\|\nu(A) - \mu(A)\|^2, \end{aligned}$$

where the first inequality follows from the concavity of the logarithm, while the second is the elementary inequality, for all $0 < a, b < 1$,

$$a\log\left(\frac{a}{b}\right) + (1-a)\log\left(\frac{1-a}{1-b}\right) \geq 2(a-b)^2.$$

We infer $H_\mu(\nu) \geq 2\|\mu - \nu\|^2$, hence:

Corollary 3.3.2.

$$\frac{d\mathcal{F}(\varphi_t)}{dt} \geq 4\|MA(\varphi_t) - \mu_t\|^2.$$

3.3.2 Mabuchi functional

Recall that the scalar curvature of a Kähler form ω is the trace of the Ricci curvature,

$$\operatorname{Scal}(\omega) := n\frac{\operatorname{Ric}(\omega)\wedge\omega^{n-1}}{\omega^n}.$$

Its mean value is denoted by

$$\overline{\operatorname{Scal}(\omega)} := V^{-1}\int_X \operatorname{Scal}(\omega)\omega^n = n\frac{c_1(X)\cdot\{\omega\}^{n-1}}{\{\omega\}^n}.$$

The *Mabuchi energy*[2] is defined by its derivative: if $\omega_t = \omega + dd^c\psi_t$ is any path of Kähler forms within the cohomology class $\{\omega\}$, then

$$\frac{d\mathcal{M}(\psi_t)}{dt} := V^{-1}\int_X \dot{\psi}_t\left[\operatorname{Scal}(\omega_t) - \overline{\operatorname{Scal}(\omega_t)}\right]\omega_t^n.$$

We normalize $\mathcal{M}$ so that $\mathcal{M}(0) = 0$. As we work here with $\omega \in c_1(X)$, we obtain $\overline{\operatorname{Scal}(\omega_t)} = n$. Since

$$\operatorname{Ric}(\omega_t) = \omega_t - dd^c h_t,$$

we observe that

$$\operatorname{Scal}(\omega_t) - \overline{\operatorname{Scal}(\omega_t)} = -\Delta_{\omega_t} h_t := -n\frac{dd^c h_t\wedge\omega_t^{n-1}}{\omega_t^n}.$$

Now $dd^c\dot{\varphi}_t = dd^c h_t$ thus along the normalized Kähler-Ricci flow,

$$\frac{d\mathcal{M}(\varphi_t)}{dt} = -\frac{1}{V}\int_X \dot{\varphi}_t\Delta_{\omega_t}(\dot{\varphi}_t)\omega_t^n = +\frac{n}{V}\int_X d\dot{\varphi}_t\wedge d^c\dot{\varphi}_t\wedge\omega_t^{n-1} \geq 0.$$

We have thus proved the following important property:

Lemma 3.3.3. *The Mabuchi energy is non-decreasing along the normalized Kähler-Ricci flow. More precisely,*

$$\frac{d\mathcal{M}(\varphi_t)}{dt} = \frac{n}{V}\int_X d\dot{\varphi}_t\wedge d^c\dot{\varphi}_t\wedge\omega_t^{n-1} \geq 0.$$

When the Mabuchi functional is bounded from above, the previous computation yields in particular

$$\int_0^{+\infty} \|\nabla_t\dot{\varphi}_t\|^2_{L^2(X)}dt < +\infty.$$

We refer the reader to the discussion before Lemma 1 in [PSS07] for more details on how this condition can be used for a suitable choice of c_0.

[2] The Mabuchi energy is often denoted by K or ν in the literature; our sign convention is the opposite of the traditional one, so we call it here $\mathcal{M}$ to avoid any confusion.

3.3.3 Comparison of functionals

Lemma 3.3.4. *Along the normalized Kähler-Ricci flow, one has*

$$\frac{1}{V}\int_X \dot{\varphi}_t \omega_t^n = \frac{1}{V}\int_X h_0\,\omega^n + \mathcal{F}(\varphi_t) - \mathcal{M}(\varphi_t).$$

Proof. Recall that

$$\dot{\varphi}_t = \log(\omega_t^n/\omega^n) + \varphi_t + h_0 + \beta(t), \text{ with } \beta(t) = \log\left[\frac{1}{V}\int_X e^{-\varphi_t - h_0}\omega^n\right].$$

We let $a(t) = \int_X \dot{\varphi}_t MA(\varphi_t)$ denote the left hand side and compute

$$a'(t) = \int_X \ddot{\varphi}_t MA(\varphi_t) - \frac{d\mathcal{M}(\varphi_t)}{dt},$$

where

$$\ddot{\varphi}_t = \Delta_{\omega_t}\dot{\varphi}_t + \dot{\varphi}_t + \beta'(t).$$

Therefore

$$a'(t) = a(t) + \beta'(t) - \frac{d\mathcal{M}(\varphi_t)}{dt} = \frac{d}{dt}\left\{\mathcal{F}(\varphi_t) - \mathcal{M}(\varphi_t)\right\},$$

noting that $a(t) = \frac{dE(\varphi_t)}{dt}$.

The conclusion follows since $a(0) = \int_X h\,\frac{\omega^n}{V}$ while $\mathcal{F}(0) = \mathcal{M}(0) = 0$. $\square$

We now show that the Mabuchi energy and the $\mathcal{F}$ functional are bounded from above simultaneously. This seems to have been noticed only recently (see [Li08, CLW09]).

Proposition 3.3.5. *Let X be a Fano manifold. The Mabuchi functional $\mathcal{M}$ is bounded from above if and only if the $\mathcal{F}$ functional is so. Moreover*

$$\sup \mathcal{M} = \sup \mathcal{F} + \int_X h_0\,\frac{\omega^n}{V}.$$

Proof. We have noticed in previous lemma that

$$\mathcal{M}(\varphi_t) + \frac{1}{V}\int_X \dot{\varphi}_t\omega_t^n = \mathcal{F}(\varphi_t) + \frac{1}{V}\int_X h_0\,\omega^n.$$

It follows from Perelman's estimates that $\dot{\varphi}_t$ is uniformly bounded along the flow. Thus $\mathcal{M}(\varphi_t)$ is bounded if and only if $\mathcal{F}(\varphi_t)$ is so. We assume such is the case. The error term $a(t) = \frac{1}{V}\int_X \dot{\varphi}_t\omega_t^n$ is non-negative, with

$$0 \le a(t) = \frac{dE(\varphi_t)}{dt}.$$

Since $\mathcal{F}(\varphi_t) = E(\varphi_t)+\beta(t)$ is bounded from above and $t \mapsto \beta(t)$ is increasing, the energies $t \mapsto E(\varphi_t)$ are bounded from above as well. Thus $\int^{+\infty} a(t)dt < +\infty$, hence there exists $t_j \to +\infty$ such that $a(t_j) \to 0$. We infer

$$\sup_{t>0} \mathcal{M}(\varphi_t) = \sup_{t>0} \mathcal{F}(\varphi_t) + \int_X h_0 \frac{\omega^n}{V}.$$

It is actually possible to avoid Perelman's estimates and obtain the same result. We have already noted that the "error term" $a(t) = \frac{1}{V}\int_X \dot{\varphi}_t \omega_t^n$ is non-negative which yields an inequality. To obtain the reverse inequality, we fix $\varphi \in \mathcal{E}^1(X, \omega)$ and let $\psi \in \mathcal{E}^1(X, \omega)$ be a solution to

$$MA(\psi) = \mu(\varphi).$$

The existence of ψ is guaranteed by [GZ07, BEGZ10], the curvature ω_ψ may be called the Ricci inverse of ω_φ, following [Kel09, Rub08]. We let the reader check that

$$\mathcal{M}(\psi) \geq \mathcal{F}(\varphi) + \frac{1}{V}\int_X h_0\, \omega^n,$$

which yields the desired bound. □

3.4 Bounding the entropies

3.4.1 Finite energy classes

Since $E' = MA \geq 0$ is a non negative measure, the mapping $\varphi \mapsto E(\varphi)$ is non decreasing. This allows to extend the definition of E to non smooth ω-plurisubharmonic functions, by setting

$$E(\varphi) := \inf\{E(\psi) \,|\, \psi \in PSH(X, \omega) \cap \mathcal{C}^\infty(X, \mathbb{R}) \text{ with } \psi \geq \varphi\}.$$

Recall here that $PSH(X, \omega)$ denotes the set of ω-psh functions, i.e. those functions $\varphi : X \to \mathbb{R} \cup \{-\infty\}$ which are locally given as the sum of a plurisubharmonic and a smooth function, and such that $\omega + dd^c\varphi \geq 0$ in the sense of currents. The set $PSH(X, \omega)$ is naturally endowed with the L^1-topology, as well as with any of the L^p-topologies, $p \geq 1$, which are all equivalent on it.

The set $PSH(X, \omega)$ is closed for these topologies, and the subsets

$$\{\varphi \in PSH(X, \omega) \,|\, -C \leq \sup_X \varphi \leq +C'\}$$

are compact for all $C, C' \geq 0$ (this is Hartog's celebrated lemma [GZ05]).

Definition 3.4.1. *We set*

$$\mathcal{E}^1(X,\omega) := \{\varphi \in PSH(X,\omega) \,|\, E(\varphi) > -\infty\}$$

and for all $C \geq 0$,

$$\mathcal{E}^1_C(X,\omega) := \{\varphi \in PSH(X,\omega) \,|\, E(\varphi) \geq -C \text{ and } \varphi \leq +C\}.$$

These classes have been discussed in Lecture 2, so we only list here the most important facts:

1. the complex Monge-Ampère operator $MA(\varphi) := V^{-1}(\omega + dd^c\varphi)^n$ is well-defined on the class $\mathcal{E}^1(X,\omega)$;

2. for all $C \geq 0$, the sets $\mathcal{E}^1_C(X,\omega)$ are compact;

3. the operator MA is not continuous for the L^1-topology, however if $\varphi_j, \varphi \in \mathcal{E}^1(X,\omega)$ are s.t. $\varphi_j \to \varphi$ in L^1 and $\int_X |\varphi_j - \varphi| MA(\varphi_j) \to 0$, then $MA(\varphi_j) \to MA(\varphi)$ and moreover $E(\varphi_j) \to E(\varphi)$;

4. if $\mathcal{F}$ is proper, then there exists a unique $\varphi \in \mathcal{E}^1(X,\omega)$ such that $MA(\varphi) = \mu(\varphi)$, it is the unique maximizer of $\mathcal{F}$.

Recall here that $\mathcal{F}$ is *proper* means, setting $\tilde{\varphi}_j := \varphi_j - \sup_X \varphi_j$, that $\mathcal{F}(\tilde{\varphi}_j) \to -\infty$ whenever $E(\tilde{\varphi}_j) \to -\infty$. We can now summarize the information we've obtained so far:

- the functional $\mathcal{F}$ is invariant by translations and non decreasing along the normalized Kähler-Ricci flow, hence $(\mathcal{F}(\tilde{\varphi}_t))_{t>0}$ is bounded;

- the functional $\mathcal{F}$ is assumed to be proper so $(E(\tilde{\varphi}_t))_{t>0}$ is bounded as well hence the functions $\tilde{\varphi}_t$ belong to a compact subset $\mathcal{E}^1_C(X,\omega)$;

- it suffices to prove that there is only one cluster point (the Kähler-Einstein potential), and this will be the case if we can find one subsequence $\tilde{\varphi}_{t_j}$ such that $\mathcal{F}(\tilde{\varphi}_{t_j})$ increases, as $t_j \to +\infty$, towards the absolute maximum of $\mathcal{F}$: this is our next goal.

3.4.2 Maximizing subsequences

The final step consists in making a careful choice of a subsequence $(\tilde{\varphi}_{t_j})$ which is "maximizing", i.e. such that $\mathcal{F}(\tilde{\varphi}_{t_j}) \nearrow \sup \mathcal{F}$. In the sequel we denote by $\phi_j := \tilde{\varphi}_{t_j}$ such a subsequence, to simplify notations.

Since $\mathcal{F}$ is bounded from above, it follows from Corollary 3.3.2 that one can select a subsequence such that

$$MA(\phi_j) - \mu(\phi_j) \to 0.$$

Extracting again if necessary, we can further assume that

$$\phi_j \to \phi \quad \text{and} \quad \mu(\phi_j) \to \mu(\phi).$$

If we could show that $MA(\phi_j) \to MA(\phi)$, we would conclude that

$$MA(\phi) = \mu(\phi)$$

hence ϕ is a Kähler-Einstein potential: it would then follow from [BBGZ09] (see Lecture 2) that ϕ maximizes the functional $\mathcal{F}$, hence

$$\lim \nearrow \mathcal{F}(\tilde{\varphi}_t) = \lim \nearrow \mathcal{F}(\phi_j) = \mathcal{F}(\phi) = \sup \mathcal{F}.$$

This would thus show that $(\tilde{\varphi}_t)$ is a maximizing family, hence $\tilde{\varphi}_t$ converges, as $t \to +\infty$, towards the unique (normalized) Kähler-Einstein potential.

It thus only remains to show that $MA(\phi_j) \to MA(\phi)$. Recall that

$$MA(\phi_j) = F_j \mu,$$

where $\mu = e^{-h}\omega^n/V$ and

$$F_t = e^{\dot{\varphi}_t} \frac{e^{-\tilde{\varphi}_t}}{\int_X e^{-\tilde{\varphi}_t}\, d\mu}.$$

If we knew that the densities F_t are uniformly in L^p for some $p > 1$, it would follow from Hölder inequality that

$$\int_X |\phi_j - \phi| MA(\phi_j) = \int_X |\phi_j - \phi| F_j d\mu \leq C ||\phi_j - \phi||_{L^q(\mu)} \to 0.$$

The ϕ_j's would thus converge "in energy" towards ϕ and we would be done.

We don't have such strong control on the densities, we can however get a uniform control on their *entropies*:

Lemma 3.4.2. *We have*

$$0 \leq \mathcal{M}(\tilde{\varphi}_t) = E(\tilde{\varphi}_t) - \int_X F_t \log F_t d\mu - \int_X \tilde{\varphi}_t MA(\tilde{\varphi}_t) + \int_X h \frac{\omega^n}{V}.$$

In particular there exists $C > 0$ such that for all $t > 0$,

$$0 \leq \int_X F_t \log F_t d\mu \leq C.$$

Proof. This is a straightforward consequence of Lemma 3.3.4. Observe indeed that

$$\int F_t \log F_t d\mu = \int \dot{\varphi}_t MA(\tilde{\varphi}_t) - \int \tilde{\varphi}_t MA(\tilde{\varphi}_t) - \log\left[\int e^{-\tilde{\varphi}_t} d\mu\right]$$

while

$$\mathcal{F}(\tilde{\varphi}_t) = E(\tilde{\varphi}_t) + \log\left[\int e^{-\tilde{\varphi}_t} d\mu\right],$$

thus

$$\begin{aligned} E(\tilde{\varphi}_t) - \int F_t \log F_t d\mu - \int \tilde{\varphi}_t MA(\tilde{\varphi}_t) & \\ = \mathcal{F}(\tilde{\varphi}_t) - \int \dot{\varphi}_t MA(\tilde{\varphi}_t) = \mathcal{M}(\tilde{\varphi}_t) - \int_X h \frac{\omega^n}{V}, \end{aligned}$$

as follows from Lemma 3.3.4. □

We can therefore replace Hölder's inequality by Hölder-Young inequality: we let the reader check that $\chi(t) = (t+1)\log(t+1) - t$ defines a convex increasing weight on $\mathbb{R}^+$, whose Legendre transform is

$$\chi^*(s) := \sup\{st - \chi(t) \,|\, t \geq 0\} = e^s - s - 1.$$

Young's additive inequality states that for all $s, t \geq 0$,

$$st \leq \chi(t) + \chi^*(s),$$

while Hölder-Young multiplicative inequality yields

$$\int_X |\phi_j - \phi| MA(\phi_j) = \int_X |\phi_j - \phi| F_j d\mu \leq C' \|\phi_j - \phi\|_{L^{\chi^*}(\mu)},$$

where

$$\|\phi_j - \phi\|_{L^{\chi^*}(\mu)} := \inf\left\{\alpha > 0 \,\Big|\, \int_X \chi^*\left(\frac{|\phi_j - \phi|}{\alpha}\right) d\mu \leq 1\right\}$$

denotes the Luxemburg norm of the functions $\phi_j - \phi$, which converges to zero as $j \to +\infty$. This completes the proof of the theorem.

3.5 Concluding remarks

3.5.1 Weak Kähler-Ricci flow

J.Song and G.Tian have proposed in [ST09] to study the interplay between the Minimal Model Program (MMP for short) and the long run properties of the (normalized) Kähler-Ricci flow.

An optimistic goal, in view of the successful use of the Ricci flow towards proving the Poincaré conjecture, would be to establish that the Kähler-Ricci flow produces in finitely many steps a minimal model of the initial variety, and can then be run in infinite time to produce a canonical (e.g. Kähler-Einstein) metric. This requires to define and study the Kähler-Ricci flow on varieties, starting from a non smooth initial datum.

This program has been studied intensively by Tian and his co-authors on manifolds of non-negative Kodaira dimension (see notably [TZha06, ST07, ST08, ST09]). The remaining cases of manifolds X with $kod(X) = -\infty$ is largely open (see [SW] for some results and references) and most works have focused on the case of Fano manifolds. The work [BBEGZ11] is the first step towards such a study in a mildly singular context.

It would be too technical to give here the precise definitions and statements that allow to make sense of the (weak) Kähler-Ricci flow in this singular setting, we rather refer the interested reader to [ST09, BBEGZ11].

3.5.2 Open problems

There is a huge number of open problems related to the study of the long term behavior of the (normalized) Kähler-Ricci flow. We only indicate two, in relation with the result we've just exposed.

Problem 1. *Let X be a $\mathbb{Q}$-Fano Kähler-Einstein Fano manifold with no holomorphic vector field, such that $\mathcal{M}$ is proper and there exists a unique Kähler-Einstein current which is known to be smooth on X_{reg}.*

We know by Theorem 3.1.2 that the normalized Kähler-Ricci flow weakly converges to the unique Kähler-Einstein current. Does the convergence hold in the $\mathcal{C}^\infty$-sense on X_{reg} ?

This certainly requires to extend (at least) some of Perelman's estimates to this singular context, an interesting but delicate task.

Problem 2. *Let X be a Kähler-Einstein Fano manifold with non trivial holomorphic vector fields. What is the long term behavior of the normalized Kähler-Ricci flow starting from an arbitrary Kähler form in $c_1(X)$?*

*When $X = \mathbb{CP}^1$ is the Riemann sphere, it follows from the work of Hamilton [**?**] and Chow [Chow91] that it converges -after renormalization- towards a Kähler-Einstein metric. Even the case $X = \mathbb{CP}^n$, $n \geq 2$, of the complex projective space is largely open in higher dimension. The interested reader will find in [SW] some results and references, when the initial metric has enough symmetries.*

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