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Invariant Currents and Dynamical Lelong Numbers

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ABSTRACT. Let f be a polynomial automorphism of $\mathbb{C}^k$ of degree λ, whose rational extension to $\mathbb{P}^k$ maps the hyperplane at infinity to a single point. Given any positive closed current S on $\mathbb{P}^k$ of bidegree (1,1), we show that the sequence $\lambda^{-n}(f^n)^*S$ converges in the sense of currents on $\mathbb{P}^k$ to a linear combination of the Green current T_+ of f and the current of integration along the hyperplane at infinity. We give an interpretation of the coefficients in terms of generalized Lelong numbers with respect to an invariant dynamical current for f^{-1}.

1. Introduction

Let $f = (P_1, \dots, P_k) : \mathbb{C}^k \to \mathbb{C}^k$ be a polynomial automorphism of first algebraic degree $\lambda = \max \deg P_j \geq 2$. We still denote by $f : \mathbb{P}^k \to \mathbb{P}^k$ the meromorphic extension of f to the complex projective space $\mathbb{P}^k = \mathbb{C}^k \cup (t = 0)$, where $(t = 0)$ denotes the hyperplane at infinity.

The mapping $f : \mathbb{P}^k \to \mathbb{P}^k$ is not well defined on the indeterminacy locus I^+, which is an algebraic subset of $(t = 0)$ of dimension $\leq k - 2$. Set $X^+ = f((t = 0) \setminus I^+)$. We assume throughout this article that X^+ is reduced to a point which does not belong to I^+. In particular, f is *weakly regular* (see [12]) hence, it is *algebraically stable*: the sequence $\lambda^{-n}(f^n)^*\omega$ converges in the weak sense of currents to a positive closed current T_+ of bidegree $(1, 1)$ such that $f^*T_+ = \lambda T_+$ (see [15]). Here ω denotes the Fubini-Study Kähler form on $\mathbb{P}^k$. Given S a positive closed current of bidegree (l, l) on $\mathbb{P}^k$, we set $\|S\| := \int_{\mathbb{P}^k} S \wedge \omega^{k-l}$.

We assume in the sequel that $\lambda > \lambda_2(f)$, the second dynamical degree of f. This allows us to construct an invariant positive closed current σ_- of bidimension $(1, 1)$ which we study in Section 2. We show (Theorem 2.2) that any quasiplurisubharmonic function is integrable with respect to the trace measure $\sigma_- \wedge \omega$. Using this we can define a generalized Lelong number $\nu(\cdot, \sigma_-)$ with respect to the dynamical weight σ_- (see Definition 2.3). The dynamical interest of these numbers lies in an invariance property (Proposition 3.1) which we establish when I^+ is an f^{-1}-attracting set. This last assumption has interesting dynamical consequences (see Theorem 2.13 in [12]).

Let S be a positive closed current of bidegree $(1, 1)$ and of unit mass in $\mathbb{P}^k$. Analyzing the behavior of the bounded sequence of currents $\lambda^{-n}(f^n)^*S$ is a natural problem since it is linked

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with ergodic properties of the invariant current T_+. This has been studied intensively in the past decade, starting with the work of Bedford–Smillie [2] and Fornæss–Sibony [8] on complex Hénon mappings (for further references see [15, 10]). In the context described above, our main result is the following.

Theorem 1.1. *Let f be a polynomial automorphism of $\mathbb{C}^k$ such that X^+ is a point not in I^+. Assume that $\lambda > \lambda_2(f)$ and that I^+ is an attracting set for f^{-1}. If S is a positive closed current on $\mathbb{P}^k$ of bidegree $(1,1)$ with $\|S\| = 1$, then*

$$\frac{1}{\lambda^n}\,(f^n)^* S \to c_S[t=0] + (1-c_S)T_+\,,$$

in the weak sense of currents on $\mathbb{P}^k$, where $c_S = \nu(S,\sigma_-) \in [0,1]$ is the generalized Lelong number of S with respect to the invariant weight σ_-. Moreover, $\nu(S,\sigma_-) > 0$ if and only if the Lelong number $\nu(S,X^+) > 0$.

It should be noted that this result is new even in the case when f is a complex Hénon mapping ($k=2$). In this case $\sigma_- = T_-$ is the Green current of f^{-1}, hence $\nu(S,\sigma_-)$ is a generalized Lelong number in the sense of Demailly [6]. For Hénon mappings, it was shown by Bedford and Smillie that $\lambda^{-n}(f^n)^*[\mathcal{C}] \to cT_+$ in $\mathbb{C}^2$, $c > 0$, for any algebraic curve $\mathcal{C} \subset \mathbb{C}^2$ (see Theorem 4.7 in [2]). Our result can be seen as a full generalization of this, in the sense that it yields global convergence on $\mathbb{P}^2$ (explaining what happens at infinity) and that it applies to any positive closed current S and in any dimension.

On our way to prove this theorem, we introduce an interesting invariant probability measure $\mu_f = T_+ \wedge \sigma_-$ (Section 2.3). We prove Theorem 1.1 in Section 3 and we check in Section 4 our hypotheses on the families of quadratic polynomial automorphisms of $\mathbb{C}^3$.

2. Invariant Lelong number

Let f be a polynomial automorphism of $\mathbb{C}^k$ which maps $(t=0)\setminus I^+$ to a point $X^+ \notin I^+$ and such that $\lambda > \lambda_2(f)$. Here $\lambda_2(f)$ denotes the second dynamical degree of f, $\lambda_2(f) = \lim[\delta_2(f^n)]^{1/n}$, where $\delta_2(f^n)$ is the second algebraic degree of f^n, i.e., the degree of $f^{-n}(L)$, L a generic linear subspace of codimension 2 (see [15]). Under these assumptions we can construct a positive closed current σ_- of bidegree $(k-1,k-1)$ and of unit mass such that $(f^{-1})^*\sigma_- = \lambda\,\sigma_-$ (see Theorem 3.1 in [12]).

2.1. Construction of σ_-

We recall the construction of σ_- since it is crucial for everything that follows. Let Θ be a smooth positive closed form of bidegree $(k-1,k-1)$ and of unit mass in $\mathbb{P}^k$ such that $\operatorname{Supp}\Theta \cap I^+ = \emptyset$. Then $\operatorname{Supp}(f^{-1})^*\Theta \cap (t=0) = X^+$, thus $(f^{-1})^*\Theta$ is smooth in $\mathbb{P}^k \setminus \{X^+\}$. Since $(f^{-1})^*\Theta$ has mass λ, there exists a current R of bidegree $(k-2,k-2)$ on $\mathbb{P}^k$, smooth in $\mathbb{P}^k \setminus \{X^+\}$, such that

$$\frac{1}{\lambda}\,(f^{-1})^*\Theta = \Theta + dd^c R\,.$$

For W_0 an arbitrarily small neighborhood of X^+ we may assume that $0 \le R \le C\omega^{k-2}$ in $\mathbb{P}^k \setminus W_0$, with a constant C depending on W_0. Then $0 \le (f^{-p})^* R \le C(f^{-p})^*\omega^{k-2}$ holds in $\mathbb{P}^k \setminus \overline{f^p(W_0)}$. We infer

$$\sigma_-^{(n)} := \frac{1}{\lambda^n}\,(f^{-n})^*\Theta = \Theta + dd^c R_n \longrightarrow \sigma_- := \Theta + dd^c R_\infty\,,$$

where $R_n = \sum_{j=0}^{n-1} \lambda^{-j}(f^{-j})^* R$ converges to R_∞ in the weak sense of currents: indeed, $\{R_n\}$ is an increasing sequence of positive currents in $\mathbb{P}^k \setminus W_0$ (because $R \geq 0$ in $\mathbb{P}^k \setminus W_0$ and we can assume $f(W_0) \subset W_0$) with bounded mass as $\lambda > \lambda_2(f)$. We will use over and over the following facts:

$$R_n \text{ is smooth in } \mathbb{C}^k \text{ and } R_\infty \geq 0 \text{ in } \mathbb{P}^k \setminus W_0 .$$

Remark 2.1. Let $K^- \subset \mathbb{C}^k$ be the set of points z with bounded backward orbit $\{f^{-n}(z)\}_{n>0}$. When I^+ is f^{-1}-attracting it was shown in [12] that the current σ_- is supported in the closure (in $\mathbb{P}^k$) of K^-, which intersects $(t = 0)$ only at the point X^+. This was used in particular, to show that σ_- has full mass 1 in $\mathbb{C}^k$. We will show here that σ_- has full mass 1 in $\mathbb{C}^k$ even when I^+ is not f^{-1}-attracting. This occurs for certain maps in the classes 4 and 5 from Theorem 4.1.

Let us recall that a function is quasiplurisubharmonic (qpsh) if it is locally given as the sum of a plurisubharmonic function and a smooth function.

Theorem 2.2. *Any quasiplurisubharmonic function is in $L^1(\sigma_- \wedge \omega)$. In particular, σ_- does not charge the hyperplane at infinity.*

Proof. Let φ be a qpsh function and let φ_ε be a smooth regularization of φ. Without loss of generality we can assume $\varphi, \varphi_\varepsilon \leq 0$ and $dd^c\varphi, dd^c\varphi_\varepsilon \geq -\omega$. Let β be a smooth positive closed form of bidegree $(1, 1)$ on $\mathbb{P}^k$ vanishing in W_0 such that $\omega = \beta + dd^c\chi$ with $\chi \geq 0$ on $\mathbb{P}^k$. By Stokes theorem, we have

$$\begin{aligned}
\int (-\varphi_\varepsilon)\sigma_- \wedge \omega &= \int (-\varphi_\varepsilon)\sigma_- \wedge \beta + \int (-\varphi_\varepsilon)\sigma_- \wedge dd^c\chi \\
&= \int (-\varphi_\varepsilon)\Theta \wedge \beta + \int dd^c(-\varphi_\varepsilon) \wedge R_\infty \wedge \beta + \int dd^c(-\varphi_\varepsilon) \wedge \chi\sigma_- \\
&\leq \int (-\varphi_\varepsilon)\Theta \wedge \beta + \int \omega \wedge R_\infty \wedge \beta + \int \omega \wedge \chi\sigma_- ,
\end{aligned}$$

since $R_\infty \wedge \beta \geq 0$, $\chi\sigma_- \geq 0$ and $-dd^c\varphi_\varepsilon \leq \omega$ in $\mathbb{P}^k$. Letting $\varepsilon \to 0$ we get

$$0 \leq \int (-\varphi)\sigma_- \wedge \omega \leq \int (-\varphi)\Theta \wedge \beta + \int \omega \wedge R_\infty \wedge \beta + \int \omega \wedge \chi\sigma_- < +\infty ,$$

since φ is integrable with respect to any smooth probability measure. In particular, when $\varphi = \log|t| - \log\|[z : t]\|$ is a potential of the current of integration along the hyperplane at infinity, this shows that the trace measure $\sigma_- \wedge \omega$ puts no mass on $(t = 0)$, hence σ_- has full mass in $\mathbb{C}^k$. $\square$

2.2. Dynamical Lelong number

Let S be a positive closed current of bidegree $(1, 1)$ and unit mass on $\mathbb{P}^k$, so $S = \omega + dd^c\varphi$ for some qpsh function φ. It follows from Theorem 2.2 that the probability measure $S \wedge \sigma_- := \omega \wedge \sigma_- + dd^c(\varphi\,\sigma_-)$ is well defined.

Definition 2.3. The generalized Lelong number of S with respect to the invariant current σ_- is $\nu(S, \sigma_-) := S \wedge \sigma_-(\{X^+\})$.

The following convergence result will help to compute generalized Lelong numbers.

Theorem 2.4. *Let S be a positive closed current of bidegree $(1,1)$ on $\mathbb{P}^k$. Then*

$$S \wedge \sigma_-^{(n)} \to S \wedge \sigma_- ,$$

in the weak sense of measures on $\mathbb{P}^k$.

Proof. We can assume S has mass 1, hence $S = \omega + dd^c\varphi$, where $\varphi \leq 0$ is qpsh. We are going to show that $\varphi\sigma_-^{(n)} \to \varphi\sigma_-$ in $\mathbb{P}^k \setminus X^+$.

Observe first that the currents $\varphi\sigma_-^{(n)}$ have uniformly bounded mass in $\mathbb{P}^k$: arguing as in the proof of Theorem 2.2, we get

$$0 \leq \int (-\varphi)\sigma_-^{(n)} \wedge \omega \leq \int (-\varphi)\Theta \wedge \beta + \int \omega \wedge R_n \wedge \beta + \int \omega \wedge \chi\sigma_-^{(n)} \leq C < +\infty$$

since R_n increases to R_∞ in $\mathbb{P}^k \setminus W_0$ and $\sigma_-^{(n)}$ has bounded total mass.

Let ν be a cluster point of $\{\varphi\sigma_-^{(n)}\}$. Let $\{\varphi_\varepsilon\}$ be a sequence of smooth qpsh functions decreasing pointwise to φ. Then $\varphi\sigma_-^{(n)} \leq \varphi_\varepsilon\sigma_-^{(n)}$, hence $\nu \leq \varphi_\varepsilon\sigma_-$. Letting $\varepsilon \to 0$ yields $\nu \leq \varphi\sigma_-$. To get equality, it suffices to show that the total mass of $(-\varphi)\sigma_-$ dominates that of $-\nu$. Recall that $\sigma_-^{(n)} = \Theta + dd^c R_n$, where $R_n = \sum_{j=0}^{n-1} \lambda^{-j}(f^{-j})^* R$, and R is smooth in $\mathbb{P}^k \setminus \{X^+\}$. Up to now, we have chosen $R \geq 0$ in $\mathbb{P}^k \setminus W_0$. Here it is actually more convenient to choose a negative potential. Set $T = R - C\omega^{k-2}$, where C is a positive constant so large that $T \leq 0$ in $\mathbb{P}^k \setminus W_0$. Then $\sigma_-^{(n)} = \Theta + dd^c T_n$, where $T_n = \sum_{j=0}^{n-1} \lambda^{-j}(f^{-j})^* T$ is a sequence of negative currents in $\mathbb{P}^k \setminus W_0$ decreasing to T_∞. Set

$$\hat{T}_n := \sum_{j \geq n} \frac{1}{\lambda^j} \left(f^{-j}\right)^* T \leq 0 \text{ in } \mathbb{P}^k \setminus W_0 ,$$

so that $\sigma_- - \sigma_-^{(n)} = dd^c \hat{T}_n$. Let β be a smooth closed form of bidegree $(1,1)$ on $\mathbb{P}^k$ vanishing in W_0 and strictly positive in $\mathbb{P}^k \setminus \overline{W_0}$. Using $-\hat{T}_n \wedge \beta \geq 0$ in $\mathbb{P}^k$, we get

$$\begin{aligned}
\int (-\varphi_\varepsilon)\sigma_- \wedge \beta &= \int (-\varphi_\varepsilon)\sigma_-^{(n)} \wedge \beta + \int (-\varphi_\varepsilon)\, dd^c \hat{T}_n \wedge \beta \\
&= \int (-\varphi_\varepsilon)\sigma_-^{(n)} \wedge \beta + \int dd^c\varphi_\varepsilon \wedge \left(-\hat{T}_n\right) \wedge \beta \\
&\geq \int (-\varphi_\varepsilon)\sigma_-^{(n)} \wedge \beta - \int \omega \wedge \left(-\hat{T}_n\right) \wedge \beta .
\end{aligned}$$

As $\varepsilon \to 0$

$$\int (-\varphi)\sigma_- \wedge \beta \geq \int (-\varphi)\sigma_-^{(n)} \wedge \beta + \int \omega \wedge \hat{T}_n \wedge \beta .$$

Now $\hat{T}_n \to 0$ as $n \to +\infty$, hence $\int (-\varphi)\sigma_- \wedge \beta \geq \int (-\nu) \wedge \beta$. This shows that $\nu = \varphi\sigma_-$ in $\mathbb{P}^k \setminus W_0$, hence in $\mathbb{P}^k \setminus X^+$ since W_0 is an arbitrarily small neighborhood of X^+.

It follows that $S \wedge \sigma_-^{(n)} \to S \wedge \sigma_-$ in $\mathbb{P}^k \setminus X^+$. Since these are all probability measures, we actually get $S \wedge \sigma_-^{(n)} \to S \wedge \sigma_-$ on $\mathbb{P}^k$. □

Example 2.5. If $\mu_n = \sigma_-^{(n)} \wedge [t = 0]$ then $\limsup \mu_n(\{X^+\}) \leq \nu([t = 0], \sigma_-) \leq 1$ by Theorem 2.4. Now $\mu_n(\{X^+\}) = 1$ because $\sigma_-^{(n)}$ clusters at infinity only at X^+. Therefore

$\nu([t=0],\sigma_-)=1$, i.e., $[t=0]\wedge\sigma_-$ is the Dirac mass at the point X^+. At the other end, observe that T_+ vanishes in a neighborhood of X^+ which is an attracting fixed point, so $\nu(T_+,\sigma_-)=0$.

Regular automorphisms were introduced by Sibony [15] and studied in [15, 12]. These are automorphisms such that $I^+\cap I^-=\emptyset$. In this case f^{-1} is algebraically stable, so there is a well defined invariant Green current T_- for f^{-1} (see [15]).

Proposition 2.6. *Assume f is a regular automorphism. Then $\sigma_-=T_-^{k-1}$, so $\nu(S,\sigma_-)$ is the Demailly number of S with respect to the weight T_-. In this case,*

$$\nu(S,\sigma_-)>0 \text{ if and only if } \nu\big(S,X^+\big)>0\,,$$

where $\nu(S,X^+)$ denotes the standard Lelong number at the point X^+.

Proof. When f is a regular automorphism as defined in [15], the inverse f^{-1} has first algebraic degree d_- such that $d_-^{k-1}=\lambda$ (recall that X^+ is a point), and $\lambda_2(f)=d_-^{k-2}<\lambda$. Note also that in this case $I^+=X^-$ is an f^{-1}-attracting set. We refer the reader to [15] for the construction of $T_-=\omega+dd^cg_-$, the Green current of bidegree $(1,1)$ for f^{-1}. It follows from the extension of the Bedford–Taylor theory of Monge–Ampère operators that T_-^{k-1} is well defined and equals $\lim\lambda^{-n}(f^{-n})^*(\omega^{k-1})$ (see [6, 15]). Thus, $T_-^{k-1}=\lim\lambda^{-n}(f^{-n})^*\Theta=\sigma_-$ since $\Theta=\omega^{k-1}+dd^c\alpha$, where α is a smooth form of bidegree $(k-2,k-2)$, hence $\|(f^{-n})^*(\alpha)\|=O(d_-^{n(k-2)})=o(\lambda^n)$. Note also that T_-^k is well defined and equals the Dirac mass at the point $X^+=I^-$. This is a situation where the Jensen type formulas of Demailly simplify and give a nice understanding of the generalized Lelong numbers $\nu(S,T_-^{k-1})$.

The potential g_- of T_- is obtained as $g_-=\sum_{n\ge0}d_-^{-n}\phi_-\circ f^{-n}$, where $d_-^{-1}(f^{-1})^*\omega=\omega+dd^c\phi_-$. Observe that g_- has positive Lelong number at $X^+=I^-$, hence $g_-(z)\le\gamma_1\log\operatorname{dist}(z,X^+)+C$.

We also have control from below, $\gamma_2\log\operatorname{dist}(z,X^+)-C\le g_-(z)$. This follows from a Lojasiewicz type inequality, since

$$\phi_-(z)=2^{-1}\log\Big[\big|Q_0(z)\big|^2+\ldots+\big|Q_k(z)\big|^2\Big]+\text{ smooth term near }X^+\,,$$

where Q_j are polynomials such that $\bigcap Q_j^{-1}(0)=X^+$. It follows from the Nullstellensatz that $|Q_0(z)|^2+\ldots+|Q_k(z)|^2\ge\operatorname{dist}(z,X^+)^\alpha$ near X^+ for some exponent $\alpha>0$. As X^+ is an attracting fixed point for f, we get $\operatorname{dist}(f(z),X^+)\le c\operatorname{dist}(z,X^+)$ for all $z\in\mathbb{C}^k$, hence $\operatorname{dist}(f^{-n}(z),X^+)\ge c^{-n}\operatorname{dist}(z,X^+)$. Therefore $g_-(z)\ge\gamma_2\log\operatorname{dist}(z,X^+)-C$ with $\gamma_2=2^{-1}\alpha d_-/(d_--1)$.

We conclude by the first comparison theorem of Demailly [6] that $\nu(S,\sigma_-)>0$ if and only if $\nu(S,X^+)>0$. □

Remark 2.7. For regular automorphisms T_-^k is the Dirac mass at the point $X^+=I^-$, thus $\nu(T_-,\sigma_-)=1$. It is an interesting question to characterize the closed positive currents $S\sim\omega$ such that $\nu(S,\sigma_-)=1$.

Theorem 2.8. *Let S be a positive closed current of bidegree $(1,1)$ on $\mathbb{P}^k$.*

1) *The sequence of currents $S\wedge R_n$ is well defined and convergent in $\mathbb{C}^k$. Set $S\wedge R_\infty:=\lim S\wedge R_n$ in $\mathbb{C}^k$. Then*

$$S\wedge\sigma_-=S\wedge\Theta+dd^c(S\wedge R_\infty)\text{ in }\mathbb{C}^k\,.$$

2) *Assume* $S_n \to S$, *where* S_n *are positive closed currents of bidegree* $(1,1)$ *on* $\mathbb{P}^k$. *Then* $S_n \wedge \sigma_- \longrightarrow S \wedge \sigma_-$ *in* $\mathbb{C}^k$. *Moreover, when* I^+ *is* f^{-1}*-attracting, then* $S_n \wedge \sigma_- \longrightarrow S \wedge \sigma_-$ *on* $\mathbb{P}^k$.

Corollary 2.9. *If* I^+ *is* f^{-1}*-attracting, the mapping* $S \mapsto \nu(S, \sigma_-)$ *is upper semicontinuous.*

Proof. Let $S_n \to S$. Then $S_n \wedge \sigma_- \to S \wedge \sigma_-$ on $\mathbb{P}^k$, so $\limsup S_n \wedge \sigma_-(\{X^+\}) \le S \wedge \sigma_-(\{X^+\})$. □

Lemma 2.10. *Let* S *be a positive closed current of bidegree* $(1,1)$ *on* $\mathbb{P}^k$ *and let* θ *be a positive closed current of bidimension* $(1,1)$ *which is smooth in an open subset* Ω *of* $\mathbb{P}^k$. *Then*

$$0 \le \int_\Omega S \wedge \theta \le \|S\| \cdot \|\theta\| \,,$$

where $\|S\| = \int_{\mathbb{P}^k} S \wedge \omega^{k-1}$ *and* $\|\theta\| = \int_{\mathbb{P}^k} \theta \wedge \omega$.

Proof. Since $\mathbb{P}^k$ is homogeneous (i.e., $\mathrm{Aut}(\mathbb{P}^k)$ acts transitively on $\mathbb{P}^k$), we can regularize S in the following sense: there exist smooth positive closed currents S_ε of bidegree $(1,1)$ on $\mathbb{P}^k$ such that $\|S_\varepsilon\| = \|S\|$ and $S_\varepsilon \to S$ on $\mathbb{P}^k$ (see [13]). Therefore $S_\varepsilon \wedge \theta \to S \wedge \theta$ in Ω, hence

$$0 \le \int_\Omega S \wedge \theta \le \liminf_{\varepsilon \to 0} \int_\Omega S_\varepsilon \wedge \theta \le \liminf_{\varepsilon \to 0} \int_{\mathbb{P}^k} S_\varepsilon \wedge \theta = \|S\| \cdot \|\theta\| \,. \qquad \square$$

Proof of Theorem 2.8. Let S be a positive closed current of bidegree $(1,1)$ on $\mathbb{P}^k$. Recall that $\sigma_- = \Theta + dd^c R_\infty$, where $R_\infty = R_n + \hat{R}_n = \lim R_n$, $R_n = \sum_{j=0}^{n-1} \lambda^{-j} (f^{-j})^* R$ being smooth in $\mathbb{C}^k$. Therefore $S \wedge R_n$ is a well-defined current of bidimension $(1,1)$ which is positive in $\mathbb{C}^k \setminus W_0$. We estimate its mass in $\mathbb{C}^k \setminus W_0$: if S_ε is a regularization of S as in the proof of Lemma 2.10, then

$$\begin{aligned} 0 &\le \int_{\mathbb{C}^k \setminus \overline{W_0}} S \wedge R_n \wedge \omega \le \liminf_{\varepsilon \to 0} \int_{\mathbb{C}^k \setminus \overline{W_0}} S_\varepsilon \wedge R_n \wedge \omega \\ &\le C \liminf_{\varepsilon \to 0} \sum_{j=0}^{n-1} \frac{1}{\lambda^j} \int_{\mathbb{P}^k} S_\varepsilon \wedge (f^{-j})^* \omega^{k-2} \wedge \omega \le C \|S\| \sum_{j \ge 0} \frac{\delta_2(f^j)}{\lambda^j} < +\infty \,, \end{aligned}$$

where $C > 0$ is a constant depending on the fixed neighborhood W_0. This shows that the increasing sequence $\{S \wedge R_n\}$ is convergent in $\mathbb{C}^k \setminus \overline{W}_0$. Observe that the sequence $\{R_n - R_p\}_{n \ge p}$ is positive and increasing in $\mathbb{C}^k \setminus \overline{f^p(W_0)}$. Thus, $S \wedge R_n$ converges in $\mathbb{C}^k \setminus \overline{f^p(W_0)}$, for all p, hence in $\mathbb{C}^k$, as $\overline{f^p(W_0)} \searrow X^+$. Set $S \wedge R_\infty := \lim S \wedge R_n$ in $\mathbb{C}^k$. Then

$$S \wedge \Theta + dd^c(S \wedge R_\infty) = \lim \left[S \wedge \Theta + dd^c\big(S \wedge R_n\big) \right] = \lim S \wedge \sigma_-^{(n)} = S \wedge \sigma_- \,,$$

by Theorem 2.4. This proves 1).

Let S_n, S now be positive closed currents of bidegree $(1,1)$ on $\mathbb{P}^k$ such that $S_n \to S$. Since R_N is smooth in $\mathbb{C}^k$, we get $S_n \wedge R_N \to S \wedge R_N$ for all fixed N. We want to show that $S_n \wedge R_\infty \to S \wedge R_\infty$. It is sufficient to get an estimate on $\|S_n \wedge \hat{R}_N\|_{\mathbb{C}^k \setminus \overline{f^N(W_0)}}$ which is uniform in n. This is the following

$$0 \le \int_{\mathbb{C}^k \setminus \overline{f^N(W_0)}} S_n \wedge \hat{R}_N \wedge \omega \le C \liminf_{\varepsilon \to 0} \sum_{j \ge N} \frac{1}{\lambda^j} \int_{\mathbb{P}^k} S_n^\varepsilon \wedge (f^{-j})^* \omega^{k-2} \wedge \omega \le C' \sum_{j \ge N} \frac{\delta_2(f^j)}{\lambda^j},$$

where S_n^ε is a regularization of S_n and the last inequality follows from Lemma 2.10 and the fact that the sequence of norms $\|S_n^\varepsilon\| = \|S_n\|$ is bounded. Therefore, $S_n \wedge R_\infty \to S \wedge R_\infty$ in $\mathbb{C}^k$, hence

$$S_n \wedge \sigma_- = S_n \wedge \Theta + dd^c(S_n \wedge R_\infty) \to S \wedge \sigma_- \text{ in } \mathbb{C}^k .$$

When I^+ is f^{-1}-attracting, the current σ_- clusters at infinity only at X^+. Since $S_n \wedge \sigma_-$ and $S \wedge \sigma_-$ are positive measures on $\mathbb{P}^k$ supported in Supp σ_- and $\|S_n \wedge \sigma_-\| = \|S_n\| \to \|S \wedge \sigma_-\|$, we infer in this case that $S_n \wedge \sigma_- \to S \wedge \sigma_-$ on $\mathbb{P}^k$. □

2.3. Invariant measure

In this section we introduce and study a dynamically interesting probability measure.

Definition 2.11. We write $T_+ = \omega + dd^c g_+$ and set

$$\mu_f = T_+ \wedge \sigma_- := \omega \wedge \sigma_- + dd^c\big(g_+\sigma_-\big) .$$

Note that this measure is well defined thanks to Theorem 2.2. It is clearly a probability measure since $\int_{\mathbb{P}^k} \omega \wedge \sigma_- = 1$.

We have $T_+ = 0$ in the basin of attraction of X^+. If I^+ is f^{-1}-attracting then the support of σ_- intersects $(t = 0)$ only at X^+ (see Remark 2.1). It follows that in this case μ_f has compact support in $\mathbb{C}^k$ and it is invariant, i.e., $f_*\mu_f = \mu_f$.

When f is a regular automorphism, we have $\sigma_- = T_-^{k-1}$, so $PSH(\mathbb{C}^k) \subset L^1(\mu_f)$, by the Chern–Levine–Nirenberg inequalities. More generally, when there exist partial Green functions for f^{-1}, one also gets $PSH(\mathbb{C}^k) \subset L^1(\mu_f)$ (see Section 4.2 in [12]). This requires however delicate estimates on the growth of f^{-1} near I^+. We now establish in the spirit of [11] the following integrability result.

Theorem 2.12. *If I^+ is f^{-1}-attracting and φ is a quasiplurisubharmonic function on $\mathbb{P}^k$, then $\varphi \in L^1(\mu_f)$.*

Proof. We can assume without loss of generality that $\varphi < 0$ and $dd^c\varphi \geq -\omega$. Let $\varphi_\varepsilon < 0$ be qpsh functions which decrease pointwise to φ such that $dd^c\varphi_\varepsilon \geq -\omega$. The current $T_+ = \omega + dd^c g_+$ has potential $g_+ < 0$ which is continuous in $\mathbb{P}^k \setminus I^+$. Since I^+ is an attracting set for f^{-1}, the current σ_- vanishes in a neighborhood V_0 of I^+. If $A = \|g_+\|_{L^\infty(\mathbb{P}^k \setminus V_0)}$ then $(g_+ + A)\sigma_- \geq 0$ on $\mathbb{P}^k$. We get

$$\begin{aligned}
\int (-\varphi_\varepsilon)\, d\mu_f &= \int (-\varphi_\varepsilon)\omega \wedge \sigma_- + \int (-\varphi_\varepsilon)\, dd^c((g_+ + A)\sigma_-) \\
&= \int (-\varphi_\varepsilon)\omega \wedge \sigma_- + \int dd^c(-\varphi_\varepsilon) \wedge ((g_+ + A)\sigma_-) \\
&\leq \int (-\varphi_\varepsilon)\omega \wedge \sigma_- + \int (g_+ + A)\omega \wedge \sigma_- \leq A + \int (-\varphi_\varepsilon)\omega \wedge \sigma_- .
\end{aligned}$$

The conclusion follows by letting $\varepsilon \to 0$ and using Theorem 2.2. □

Remark 2.13. If u is a plurisubharmonic (psh) function defined in a neighborhood of the support of μ_f, then $|u|^\alpha \in L^1(\mu_f)$ for every $\alpha \in (0, 1/k)$. Indeed, by Theorem 2.12 psh functions of logarithmic growth are integrable with respect to μ_f. The claim is straightforward using the following result of El Mir and Alexander–Taylor (see [1]): If $u \leq -1$ is psh in a ball $B(z_0, R) \subset \mathbb{C}^k$ and $r < R, 0 < \epsilon < 1/k$, then there exists a psh function v on $\mathbb{C}^k$ of logarithmic growth such that $v \leq -|u|^{1/k-\epsilon}$ on $B(z_0, r)$.

3. Equidistribution towards T_+

The purpose of this section is to prove Theorem 1.1 stated in the introduction.

Proof. The proof of the theorem is divided into four steps.

Step 1: Normalization of potentials. By Siu's theorem, we can write

$$\frac{1}{\lambda^n} (f^n)^* S = c_n[t=0] + (1-c_n)S_n \,, \tag{3.1}$$

where $c_n \in [0,1]$, and S_n are positive closed currents of bidegree $(1,1)$ and unit mass which do not charge $(t=0)$. Since $f^*[t=0] = \lambda[t=0]$, the sequence $\{c_n\}$ is increasing. Let c_S denote its limit. If $c_S = 1$ the convergence statement of the theorem is proved, so we assume hereafter that $c_S < 1$.

We write $S = \omega + dd^c v_0$, where the potential v_0 is uniquely determined up to additive constants. Using Theorem 2.12, we can normalize it so that $\int v_0 \, d\mu_f = 0$. Similarly, we fix potentials $S_n = \omega + dd^c v_n$, $T_+ = \omega + dd^c g_+$, $[t=0] = \omega + dd^c \varphi_\infty$ such that $\int v_n \, d\mu_f = \int g_+ \, d\mu_f = \int \varphi_\infty \, d\mu_f = 0$. If $\lambda^{-n}(f^n)^*\omega = \omega + dd^c g_+^{(n)}$, $\int g_+^{(n)} \, d\mu_f = 0$, then $g_+^{(n)} \to g_+$ in $L^1(\mathbb{P}^k)$ and $\lambda^{-n}(f^n)^*\omega \to T_+$. The desired convergence follows if we show that $\lambda^{-n} v_0 \circ f^n \to c_S(\varphi_\infty - g_+)$ in $L^1(\mathbb{P}^k)$.

Pulling back (3.1) (with $n = p$) by f^n yields

$$\begin{aligned}
\frac{1}{\lambda^{n+p}} (f^{n+p})^* S &= c_p[t=0] + (1-c_p)\frac{1}{\lambda^n} (f^n)^* S_p \\
&= c_p[t=0] + (1-c_p)\frac{1}{\lambda^n} (f^n)^*\omega + (1-c_p)\, dd^c\Big(\frac{1}{\lambda^n} v_p \circ f^n\Big) .
\end{aligned}$$

Using our normalization and the fact that μ_f is invariant, we infer

$$\frac{1}{\lambda^{n+p}} v_0 \circ f^{n+p} = c_p\big(\varphi_\infty - g_+^{(n)}\big) + \big(g_+^{(n)} - g_+^{(n+p)}\big) + (1-c_p)\frac{1}{\lambda^n} v_p \circ f^n \,. \tag{3.2}$$

Step 2: Control of the Lelong numbers. Since f^n is a biholomorphism in $\mathbb{C}^k$, it follows from (3.1) that for all $n \in \mathbb{N}$ and $z \in \mathbb{C}^k$,

$$\nu\big((1-c_n)S_n, z\big) = \frac{1}{\lambda^n}\, \nu\big((f^n)^* S, z\big) = \frac{1}{\lambda^n}\, \nu\big(S, f^n(z)\big) \leq \frac{1}{\lambda^n} \,,$$

hence, $\sup_{z\in\mathbb{C}^k} \nu(S_n, z) \leq (1-c_S)^{-1}\lambda^{-n} \to 0$.

Pulling back (3.1) by f we get

$$\frac{1}{\lambda} f^* S_n = \frac{c_{n+1} - c_n}{1 - c_n} [t=0] + \frac{1 - c_{n+1}}{1 - c_n} S_{n+1} \,. \tag{3.3}$$

Since S_{n+1} does not charge $(t=0)$, we have for a generic point $z \in (t=0)$

$$\nu\big(S_n, X^+\big) = \nu(S_n, f(z)) \leq \nu\big(f^* S_n, z\big) = \lambda \frac{c_{n+1} - c_n}{1 - c_n} \leq \lambda \frac{c_S - c_n}{1 - c_S} \,.$$

If $z \in (t=0) \setminus I^+$, it follows from [7] and [14] that there is an upper estimate $\nu(f^*S_n, z) \leq c_{f,z}\nu(S_n, f(z))$, where $z \mapsto c_{f,z}$ is locally upper bounded. Fix V_0 a small neighborhood of I^+ and set $C_{V_0} = \sup_{z\in(t=0)\setminus V_0} c_{f,z}$. Using (3.3) again, we get for all $z \in (t=0) \setminus V_0$,

$$\frac{1-c_{n+1}}{1-c_n}\,\nu(S_{n+1}, z) \leq \frac{1}{\lambda}\,\nu\big(f^*S_n, z\big) \leq \frac{C_{V_0}}{\lambda}\,\nu\big(S_n, X^+\big) \leq C_{V_0}\frac{c_S - c_n}{1-c_S}\,.$$

We conclude that $\sup_{z\in\mathbb{P}^k\setminus V_0} \nu(S_n, z) \to 0$ as $n \to +\infty$.

Step 3: Volume estimates. We have to prove that

$$w_n := \lambda^{-n} v_0 \circ f^n \to c_S(\varphi_\infty - g_+)\,.$$

Observe first that the sequence $\{w_n\}$ is relatively compact in $L^1(\mathbb{P}^k)$. Indeed,

$$\lambda^{-n}(f^n)^*S = \lambda^{-n}\big(f^n\big)^*\omega + dd^c(w_n) = \omega + dd^c\big(g_+^{(n)} + w_n\big)\,,$$

so $w_n + g_+^{(n)}$ are qpsh functions whose curvature is uniformly bounded from below by $-\omega$. Since $g_+^{(n)} \to g_+$ and $w_n \leq C\lambda^{-n}$, the sequence $\{w_n + g_+^{(n)}\}$ is uniformly upper bounded on $\mathbb{P}^k$. So either this sequence converges uniformly to $-\infty$, or it is relatively compact in $L^1(\mathbb{P}^k)$ (see Appendix in [10]). The former cannot happen since $\int (w_n + g_+^{(n)})\,d\mu_f = 0$. Thus, it suffices to show that w_n converges in measure to $c_S(\varphi_\infty - g_+)$. It follows from (3.2) that

$$\begin{aligned} &w_{n+p} - c_S\big(\varphi_\infty - g_+\big)\\ &\quad= (c_p - c_S)\Big(\varphi_\infty - g_+^{(n)}\Big) + c_S\Big(g_+ - g_+^{(n)}\Big) + \Big(g_+^{(n)} - g_+^{(n+p)}\Big) + (1-c_p)\lambda^{-n}v_p \circ f^n\,. \end{aligned}$$

Let $\varepsilon > 0$. Choose a small neighborhood V_0 of I^+ and fix p so large that

$$\sup_{z\in\mathbb{P}^k\setminus V_0} \nu(S_p, z) \leq \varepsilon^2 \ \text{ and } \ \big|c_p - c_S\big|\,\big\|\varphi_\infty - g_+^{(n)}\big\|_{L^1(\mathbb{P}^k)} < \varepsilon^2,\ \forall n \in \mathbb{N}\,.$$

By Chebyshev's inequality $\mathrm{Vol}(|(c_p - c_S)(\varphi_\infty - g_+^{(n)})| > \varepsilon/3) < 3\varepsilon$. Since $g_+^{(n)} \to g_+$ in $L^1(\mathbb{P}^k)$, we have for n large $\mathrm{Vol}(|c_S(g_+ - g_+^{(n)}) + (g_+^{(n)} - g_+^{(n+p)})| > \varepsilon/3) < \varepsilon$. Observe that

$$\begin{aligned} &\mathrm{Vol}\left(|w_{n+p} - c_S(\varphi_\infty - g_+)| > \varepsilon\right) \leq \mathrm{Vol}\left(\big|(c_p - c_S)\big(\varphi_\infty - g_+^{(n)}\big)\big| > \varepsilon/3\right)\\ &\quad + \mathrm{Vol}\left(\big|c_S\big(g_+ - g_+^{(n)}\big) + \big(g_+^{(n)} - g_+^{(n+p)}\big)\big| > \varepsilon/3\right) + \mathrm{Vol}\left(\big(1-c_p\big)\big|\lambda^{-n}v_p \circ f^n\big| > \varepsilon/3\right). \end{aligned}$$

Since v_p is bounded above on $\mathbb{P}^k$, it remains to show that

$$\mathrm{Vol}\left(\big|\lambda^{-n}v_p \circ f^n\big| > \varepsilon/3\right) = \mathrm{Vol}\left(\lambda^{-n}v_p \circ f^n < -\varepsilon/3\right) < C\varepsilon\,,$$

for all n sufficiently large.

Since I^+ is f^{-1}-attracting, there exist arbitrarily small neighborhoods V_0 of I^+ such that $f(\mathbb{P}^k \setminus V_0) \subset \mathbb{P}^k \setminus V_0$. Set

$$\Omega_n^\varepsilon := \Big\{z \in \mathbb{P}^k \setminus V_0 : \lambda^{-n}v_p \circ f^n(z) < -\varepsilon/3\Big\}\,.$$

We have $f^n(\Omega_n^\varepsilon) \subset \{z \in \mathbb{P}^k \setminus V_0 : v_p(z) < -\varepsilon\lambda^n/3\}$. It follows from [10] that there exists $C_1 > 0$ such that

$$\mathrm{Vol}\left(f^n\left(\Omega_n^\varepsilon\right)\right) \geq \exp\left(-\frac{C_1\lambda^n}{\mathrm{Vol}\left(\Omega_n^\varepsilon\right)}\right).$$

On the other hand, by Skoda's integrability theorem (see [14]) there exists $C_\varepsilon > 0$ such that

$$\begin{aligned}\mathrm{Vol}\left(\left\{z \in \mathbb{P}^k \setminus V_0 : v_p(z) < -\varepsilon\lambda^n/3\right\}\right) &\leq C_\varepsilon \exp\left(-\frac{\varepsilon\lambda^n}{3\sup_{z\in\mathbb{P}^k\setminus V_0} \nu(S_p, z)}\right)\\ &\leq C_\varepsilon \exp\left(-\frac{\lambda^n}{3\varepsilon}\right).\end{aligned}$$

Thus, $\mathrm{Vol}(\Omega_n^\varepsilon) \leq 4C_1\varepsilon$ for all $n > N(\varepsilon)$.

We conclude that $w_n \to c_S(\varphi_\infty - g_+)$ in measure on $\mathbb{P}^k \setminus V_0$. As V_0 was an arbitrarily small neighborhood of I^+, the convergence in measure holds on $\mathbb{P}^k$.

Step 4: Interpretation of c_S. We have shown that $\lambda^{-n}(f^n)^*S \to c_S[t = 0] + (1 - c_S)T_+$. It follows from [10] that $c_S > 0$ if and only if $\nu(S, X^+) > 0$. Assume now that I^+ is f^{-1}-attracting. We show below (Proposition 3.1) that $\nu((f^n)^*S, \sigma_-) = \lambda^n\nu(S, \sigma_-)$. It then follows from Example 2.5 that

$$\nu(S, \sigma_-) = \nu\left(\lambda^{-n}\left(f^n\right)^* S, \sigma_-\right) = c_n + \nu\left(\hat{S}_n, \sigma_-\right),$$

where $\hat{S}_n = (1 - c_n)S_n \to (1 - c_S)T_+$. Since $\nu(T_+, \sigma_-) = 0$, we infer from the upper semicontinuity property (Corollary 2.9) that $\nu(\hat{S}_n, \sigma_-) \to 0$, hence $c_S = \nu(S, \sigma_-)$. □

Proposition 3.1 (Transformation rule). $\nu(f^*S, \sigma_-) = \lambda\nu(S, \sigma_-)$.

Proof. Let S_j be a sequence of smooth closed positive currents of bidegree (1,1) with smooth potentials which decrease pointwise to a potential of S. Let W be a small neighborhood of X^+ so that $f(W) \subset\subset W$. Note that $f(W) = f(W \cap \mathbb{C}^k) \cup X^+$. Since f^*S_j is smooth in W and σ_- does not charge $(t = 0)$ (Theorem 2.2) we have

$$\int_W f^*S_j \wedge \sigma_- = \int_{W\cap\mathbb{C}^k} f^*S_j \wedge \sigma_- = \int_{f(W)\cap\mathbb{C}^k} S_j \wedge \left(f^{-1}\right)^*\sigma_- = \lambda\int_{f(W)} S_j \wedge \sigma_- .$$

By the monotone convergence theorem, one has $S_j \wedge \sigma_- \to S \wedge \sigma_-$ and $f^*S_j \wedge \sigma_- \to f^*S \wedge \sigma_-$. We infer $\int_W f^*S \wedge \sigma_- \leq \lambda \int_{\overline{W}} S \wedge \sigma_-$, hence $\nu(f^*S, \sigma_-) \leq \lambda\nu(S, \sigma_-)$.

For the opposite inequality, observe that the restriction of $f^{-1} : K^- \to K^-$ extends continuously at infinity by setting $f^{-1}(X^+) = X^+$. This shows f is an open mapping on $\overline{K^-}$, so there is a ball $B \subset W$ centered at X^+ such that $\overline{K^-} \cap B \subset f(W)$. Therefore $\int_W f^*S_j \wedge \sigma_- \geq \lambda \int_B S_j \wedge \sigma_-$, which yields

$$\int_{\overline{W}} f^*S \wedge \sigma_- \geq \lambda \int_B S \wedge \sigma_- \geq \lambda\nu(S, \sigma_-).$$

The desired inequality follows by shrinking $W \searrow X^+$. □

Remark 3.2. We showed in the proof of Theorem 1.1 that if $S = \omega + dd^c v_0$ then $\lambda^{-n}v_0 \circ f^n \to c_S(\phi_\infty - g_+)$ in $L^1(\mathbb{P}^k)$. Let $G^+(z, t)$, $(z, t) \in \mathbb{C}^{k+1}$, be the logarithmically homogeneous Green function of f. The function $h[z : t] = \log|t| - G^+(z, t)$ is well defined on $\mathbb{P}^k$ and $h = \phi_\infty - g_+ + c$ for some constant c. Since $h \circ f = \lambda h$ and $f_*\mu_f = \mu_f$ we have $\int h\, d\mu_f = 0$, so $\phi_\infty - g_+ = h$.

Remark 3.3. The convergence $\lambda^{-n}(f^n)^*S \to c_S[t = 0] + (1 - c_S)T_+$ holds without the hypotheses $\lambda > \lambda_2(f)$ and I^+ is f^{-1}-attracting. A proof can be given in the basin of X^+ by a similar argument, and on the complement of this basin one can conclude as in the proof of Theorem 2.7 in [10]. However, in this case we do not have an interpretation for c_S. As an example, our convergence theorem holds for the maps f and f^{-1}, where $f(x, y, z) = (P(y) + az, Q(y) + bx, y)$, $\deg(P) = \deg(Q) = 2$, $ab \neq 0$.

4. Quadratic polynomial automorphisms of $\mathbb{C}^3$

Let f be a quadratic polynomial automorphism of $\mathbb{C}^3$. Using the classification of Fornæss and Wu [9], we show that—up to conjugacy—f or f^2 (or f^{-1}) is weakly regular. Moreover, I^+ (resp. I^-) is f^{-1}-attracting (resp. f-attracting) except for certain mappings in the classes 4 or 5 below. Note that $\lambda_1(f^{-1}) = \lambda_2(f)$ since we are working in $\mathbb{C}^3$. Here $\lambda_1(f)$ is the first dynamical degree of f, $\lambda_1(f) = \lim[\delta_1(f^n)]^{1/n}$, where $\delta_1(f^n)$ is the first algebraic degree of f^n (see [15]).

Theorem 4.1. *Let f be a quadratic polynomial automorphism of $\mathbb{C}^3$ with $\lambda_1(f) \neq \lambda_1(f^{-1})$. Then one of the following holds:*

1) *f is conjugate to a regular automorphism with X^- reduced to a point. In this case $\lambda_1(f) = 2 < 4 = \lambda_1(f^{-1})$ and I^- is f-attracting.*

2) *f^2 or f^{-2} is conjugate to a mapping from 1).*

3) *f is conjugate to*

$$f(x, y, z) = \big(y[\alpha x + \beta y] + cx + dy + az, y^2 + x, y\big)$$

where $a\alpha \neq 0$. In this case f^{-1} is weakly regular with $X^- = [0 : 0 : 1 : 0]$, $\lambda_1(f^{-1}) = 3 > 2 = \lambda_1(f)$, and I^- is f-attracting.

4) *f or f^{-1} is conjugate to*

$$g(x, y, z) = \big(x^2 - xz + c + y, az, bx + c'\big)\,,$$

with $ab \neq 0$. In this case g is weakly regular with $X^+ = [1 : 0 : 0 : 0]$, $\lambda_1(g) = 2 > \lambda_1(g^{-1}) = (1 + \sqrt{5})/2$, and I^+ is g^{-1}-attracting if and only if $|b| < 1$.

5) *f is conjugate to*

$$f(x, y, z) = \big(x[y + \alpha x] + az + c, x^2 + dx + c' + by, x\big)$$

where $ab \neq 0$. In this case f^{-1} is weakly regular, $X^- = [0 : 0 : 1 : 0]$, $\lambda_1(f^{-1}) = 3 > 2 = \lambda_1(f)$, and I^- is f-attracting if and only if $|b| > 1$.

Proof. The quadratic polynomial automorphisms of $\mathbb{C}^3$ are classified into seven classes, up to affine conjugacy [9]. The growth of the degree of their forward iterates is studied in [3]. Two classes consist of affine and elementary automorphisms f, so $\lambda_1(f) = \lambda_1(f^{-1}) = 1$. We consider the remaining five classes $H_1, \ldots, H_5$ [9].

The classes H_1 and H_2. By considering the degrees of forward and backward iterates of the maps H in these classes, it is easy to see that $\lambda_1(H) = \lambda_1(H^{-1}) \in \{1, 2\}$.

The class H_3. This class contains maps H of the form

$$H(x, y, z) = \big(P(x, z) + a'y,\, Q(x) + z,\, x\big)\,,\ \max\{\deg(P), \deg(Q)\} = 2,\ a' \neq 0\,.$$

We let $h = F \circ H \circ F^{-1}$, where $F(x, y, z) = (x, y - Q(z), z)$. Then

$$h(x, y, z) = \big(\alpha x^2 + \alpha' xz + \alpha'' z^2 + c_1 x + c_2 z + c_3 + a'y, z, x\big)\,. \tag{4.1}$$

The inverse map is

$$h^{-1}(x,y,z) = \left(z, \frac{1}{a'}\left(x - \alpha z^2 - \alpha' yz - \alpha'' y^2 - c_1 z - c_2 y - c_3\right), y\right).$$

Using the change of variables $(x,y,z) \to (y,x,z)$ we see that h^{-1} is conjugated to h, and the role of the coefficients α, α'' interchanges. We have the following cases:

Case A. $\alpha \neq 0 \neq \alpha''$. Then $\deg(h^n) = \deg(h^{-n}) = 2^n$, so $\lambda_1(h) = \lambda_1(h^{-1}) = 2$.

Case B. $\alpha \neq 0$, $\alpha'' = 0$, $\alpha' \neq 0$. Then as before $\deg(h^n) = 2^n$ and $\lambda_1(h) = 2$. The degrees of the backward iterates $d_n = \deg(h^{-n})$ are given by Fibonacci's numbers, $d_{n+2} = d_{n+1} + d_n$. So $\lambda_1(h^{-1}) = (1+\sqrt{5})/2$. Using the change of variables

$$F(x,y,z) = \left(\alpha x + v, \alpha a' y + s, -\alpha' z + r\right), \ v = c_2\alpha/\alpha', \ r = 2v - c_1, \ s = -\alpha a' r/\alpha',$$

we see that $F \circ h \circ F^{-1} = g$, the map from 4). We have $I^+(g) = \{t = x = 0\} \cup \{t = x - z = 0\}$ and $g(\{t=0\} \setminus I^+) = X^+ = [1:0:0:0]$. If $c = c' = 0$ and $a = b^2$ the line $\tau(\zeta) = (\zeta, b\zeta, \zeta)$ is g-invariant and $g(\tau(\zeta)) = \tau(b\zeta)$. So in this case I^+ is not g^{-1}-attracting if $|b| \geq 1$. We show in Lemma 4.2 following this proof that I^+ is always g^{-1}-attracting if $|b| < 1$.

Case C. $\alpha \neq 0$, $\alpha'' = \alpha' = 0$. Then h^2 is regular, $\lambda_1(h^2) = 4$, $\lambda_1(h^{-2}) = 2$, and $X^+ = [1:0:0:0]$.

Case D. $\alpha'' \neq 0$, $\alpha = 0$, $\alpha' \neq 0$. This is similar to Case B, with the roles of h and h^{-1} interchanged, $\lambda_1(h) = (1+\sqrt{5})/2$ and $\lambda_1(h^{-1}) = 2$.

Case E. $\alpha'' \neq 0$, $\alpha = \alpha' = 0$. As in Case C, h^2 is regular, $\lambda_1(h^2) = 2$, $\lambda_1(h^{-2}) = 4$, and $X^- = [0:1:0:0]$. The fact that I^- is attracting for f holds for any regular automorphism f.

Case F. $\alpha = \alpha'' = 0$, $\alpha' \neq 0$. As in Cases B and D, $\lambda_1(h) = \lambda_1(h^{-1}) = (1+\sqrt{5})/2$.

Case G. $\alpha = \alpha'' = \alpha' = 0$. Then h is linear, $\lambda_1(h) = \lambda_1(h^{-1}) = 1$.

The class H_4. The maps H in this class have the form

$$\begin{aligned} H(x,y,z) &= (P(x,y) + az, Q(y) + x, y), \ \max\{\deg(P), \deg(Q)\} = 2, \ a \neq 0, \\ H^{-1}(x,y,z) &= \left(y - Q(z), z, \frac{x}{a} + \widetilde{P}(y,z)\right), \ \widetilde{P}(y,z) = -\frac{1}{a} P(y - Q(z), z). \end{aligned}$$

We write $P(x,y) = c_1x^2 + c_2xy + c_3y^2 + l.d.t.$, $Q(y) = c_4y^2 + l.d.t.$.

Case A. $c_4 \neq 0 \neq c_1$. H is regular, $\lambda_1(H) = 2$, $\lambda_1(H^{-1}) = 4$, $X^- = [0:0:1:0]$.

Case B. $c_4 \neq 0$, $c_1 = 0$, $c_2 \neq 0$. Then H is conjugated to the map f of 3), $\lambda_1(f) = 2$, $\lambda_1(f^{-1}) = 3$, f^{-1} is weakly regular, $X^- = [0:0:1:0]$, I^- is f-attracting (see [4]).

Case C. $c_4 \neq 0$, $c_1 = c_2 = 0$. By [5] p. 446, either H^2 is regular, $\lambda_1(H^2) = 4$, $\lambda_1(H^{-2}) = 2$, $X^+ = [c_3 : c_4 : 0 : 0]$, or we have $\deg(H^{\pm n}) = 2^n$.

Case D. $c_4 = 0$. If $F(x,y,z) = (x + Q(y), z, y)$, $F \circ H \circ F^{-1}$ is the map from (4.1).

The class H_5. The maps H in this class have form

$$\begin{aligned} H(x,y,z) &= (P(x,y) + az, Q(x) + by, x), \ \max\{\deg(P), \deg(Q)\} = 2, \ a \neq 0 \neq b, \\ H^{-1}(x,y,z) &= \left(z, \frac{y - Q(z)}{b}, \frac{x}{a} + \widetilde{P}(y,z)\right), \ \widetilde{P}(y,z) = -\frac{1}{a} P\left(z, \frac{y - Q(z)}{b}\right). \end{aligned}$$

Let $P(x,y) = c_1x^2 + c_2xy + c_3y^2 + d_1x + d_2y + d_3$, $Q(x) = c_4x^2 + e_1x + e_2$.

Case A. $c_4 \neq 0 \neq c_3$. H is regular, $\lambda_1(H) = 2$, $\lambda_1(H^{-1}) = 4$, $X^- = [0:0:1:0]$.

Case B. $c_4 \neq 0$, $c_3 = 0$, $c_2 \neq 0$. Then $\deg(H^n) = 2^n$ and $\deg(H^{-n}) = 3^n$. If

$$F(x, y, z) = (px + q, c_2 y + r, pz + q),\ p^2 = c_2 c_4,\ q = pd_2/c_2,\ r = d_1 - 2qc_1/p\ ,$$

then $F \circ H \circ F^{-1}$ is the map f from 5), $I^- = \{t = z = 0\}$, $f^{-1}(\{t = 0\} \setminus I^-) = X^- = [0:0:1:0]$. If $|b| > 1$ it is shown in [12] that I^- is f-attracting. If $|b| \leq 1$ and if f fixes the origin, then $f(0, y, 0) = (0, by, 0)$, so I^- is not f-attracting.

Case C. $c_4 \neq 0$, $c_3 = c_2 = 0$. The inverse map is

$$H^{-1}(x, y, z) = \left(z, \frac{y - c_4 z^2 - e_1 z - e_2}{b}, \frac{x}{a} + \frac{\gamma z^2}{a} - \frac{d_2 y}{ab} + L(z)\right),$$

where $\gamma = (d_2 c_4/b) - c_1$ and $\deg(L) \leq 1$. If $c_1 \neq 0 \neq \gamma$ then $\lambda_1(H) = \lambda_1(H^{-1}) = 2$. If $c_1 \neq 0$ and $\gamma = 0$ then $d_2 \neq 0$ and H^2 is regular, $\lambda_1(H^2) = 4$, $\lambda_1(H^{-2}) = 2$. If $c_1 = 0$ and $d_2 \neq 0$ then H^2 is regular, $\lambda_1(H^2) = 2$, $\lambda_1(H^{-2}) = 4$. If $c_1 = d_2 = 0$ then the degrees of all iterates are bounded by 2.

Case D. $c_4 = e_1 = 0$. If $c_1 \neq 0$ then $\lambda_1(H) = \lambda_1(H^{-1}) = 2$. If $c_1 = 0$ then $\deg(H^{\pm n}) \leq n + 1$, so $\lambda_1(H) = \lambda_1(H^{-1}) = 1$.

Case E. $c_4 = 0$, $e_1 \neq 0$. We have that $F \circ H \circ F^{-1}$ is the map h from (4.1), where

$$F(x, y, z) = \left(e_1 x + by + e_2 + \frac{e_2}{b}, -\frac{e_1 z}{b} + \frac{y}{b}, y + \frac{e_2}{b}\right). \qquad \square$$

Lemma 4.2. *If $g(x, y, z) = (x^2 - xz + c + y, az, bx + c')$ is the map from Theorem 4.1, case 4), and $|b| < 1$, then I^+ is g^{-1}-attracting.*

Proof. The inverse of g has the form

$$g^{-1}(x, y, z) = (x_1, y_1, z_1) = \left(\frac{z}{b} + c'', \frac{z}{b}\left(\frac{y}{a} - \frac{z}{b}\right) + L(y, z) + x, \frac{y}{a}\right),$$

where $c'' \in \mathbb{C}$ and $\deg(L) \leq 1$. Recall that $I^+ = \{t = x = 0\} \cup \{t = x - z = 0\}$. We let $\alpha = |b|/(4|a|)$ and define for $R > 1$

$$\begin{aligned} V_R &= \left\{(x, y, z) \in \mathbb{C}^3 : \max\{2\alpha|y|, |z|\} > \max\left\{2R, R^{1/3}|x|\right\}\right\}, \\ W_R &= \left\{(x, y, z) \in \mathbb{C}^3 : \max\{\alpha|y|, |x|\} > \max\left\{R, R^{1/3}|x - z|\right\}\right\}. \end{aligned}$$

Since $|b| < 1$ we can find $\varepsilon > 0$ such that $|b| < (1 - 2\varepsilon)/(1 + \varepsilon)$. The lemma follows if we show that for all R sufficiently large we have

$$g^{-1}(V_R) \subset V_{2R} \cup W_{2R}\ ,\ g^{-1}(W_R) \subset V_{2R} \cup W_{(1+\varepsilon)R}\ . \tag{4.2}$$

We denote in the sequel by C_g all constants which depend only on the coefficients of g. For the first inclusion of (4.2), let $(x, y, z) \in V_R$. We have two cases:

Case A. $2\alpha|y| \geq |z|$, so $|y| > R/\alpha$, $|y| > R^{1/3}|x|/(2\alpha)$. We show that in this case $g^{-1}(x, y, z) \in V_{2R}$. If $|y|/|a| > 4R^{1/3}|z|/|b|$ then

$$2R^{1/3}|x_1| \leq 2R^{1/3}\frac{|z|}{|b|} + 2\left|c''\right| R^{1/3} < |z_1|\ ,\ |z_1| > \frac{R}{\alpha|a|} > 4R\ .$$

If $|y|/|a| \le 4R^{1/3}|z|/|b|$, using $|z|/|b| \le 2\alpha|y|/|b| = |y|/(2|a|)$, we get

$$|y_1| \ge \frac{|z|}{|b|}\left(\frac{|y|}{|a|} - \frac{|z|}{|b|}\right) - |x| - |L(y,z)| \ge \frac{C_g|y|^2}{R^{1/3}} > \max\left\{4R, 2R^{1/3}|x_1|\right\}.$$

Case B. $2\alpha|y| < |z|$, so $|z| > 2R$, $|z| > R^{1/3}|x|$. If $|x_1| > 2R^{1/3}|x_1 - z_1|$ then $g^{-1}(x,y,z) \in W_{2R}$, since $|x_1| \ge |z|/|b| - |c''| > 2R$. If $|x_1| \le 2R^{1/3}|x_1 - z_1|$ then $|z/b - y/a| \ge C_g|z|/R^{1/3}$, so $|y_1| > C_g|z|^2/R^{1/3}$ and $g^{-1}(x,y,z) \in V_{2R}$.

To prove the second inclusion of (4.2), let $(x,y,z) \in W_R$ and consider two cases:

Case A. $\alpha|y| \ge |x|$, so $|y| > R/\alpha$, $|y| > R^{1/3}|x - z|/\alpha$. If $|z_1| > 2R^{1/3}|x_1|$ then $g^{-1}(x,y,z) \in V_{2R}$, since also $|z_1| = |y|/|a| > 4R$. If $|z_1| \le 2R^{1/3}|x_1|$ then

$$\frac{|z|}{|b|} \ge \frac{|y|}{2|a|R^{1/3}} - \left|c''\right| \ge \frac{|y|}{3|a|R^{1/3}}, \quad |z| \le |z - x| + |x| \le \frac{\alpha|y|}{R^{1/3}} + \alpha|y| < 2\alpha|y|.$$

It follows that $g^{-1}(x,y,z) \in V_{2R}$, since

$$|y_1| \ge \frac{|y|}{3|a|R^{1/3}}\left(\frac{|y|}{|a|} - \frac{2\alpha|y|}{|b|}\right) - C_g|y| > \frac{C_g|y|^2}{R^{1/3}}.$$

Case B. $\alpha|y| < |x|$, so $|x| > R$, $|x| > R^{1/3}|x - z|$. There exists a large constant M depending only on g, such that if $|z/b - y/a| \ge M$ then $g^{-1}(x,y,z) \in W_{2R}$. Indeed, if R is large we have $||z| - |x|| < |x|/100$, so

$$\alpha|y_1| > \frac{|x|}{5|a|}\left|\frac{y}{a} - \frac{z}{b}\right| - C_g|x| \ge \frac{|x|}{6|a|}\left|\frac{y}{a} - \frac{z}{b}\right|,$$

provided that $M = M_g$ is sufficiently large. Therefore

$$\alpha|y_1| > \frac{RM}{6|a|} \ge 2R, \quad (2R)^{1/3}|x_1 - z_1| \le 2R^{1/3}\left|\frac{y}{a} - \frac{z}{b}\right| < \alpha|y_1|,$$

so $g^{-1}(x,y,z) \in W_{2R}$. Finally, we assume that $|z/b - y/a| < M$. For R large we have $||z| - |x|| < \varepsilon|x|$, so $|x_1| \ge |z|/|b| - |c''| > (1 - 2\varepsilon)|x|/|b| > (1+\varepsilon)|x|$. Since $|x| > R$ and $|x_1 - z_1| \le M + |c''|$, we conclude that in this case $g^{-1}(x,y,z) \in W_{(1+\varepsilon)R}$. □

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