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A Priori L^∞-Estimates for Degenerate Complex Monge–Ampère Equations

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We study families of complex Monge–Ampère equations, focusing on the case where the cohomology classes degenerate to a nonbig class. We establish uniform a priori L^∞-estimates for the normalized solutions, generalizing the recent work of S. Kołodziej and G. Tian. This has interesting consequences in the study of the Kähler–Ricci flow.

1 Introduction

Let $\pi : X \longrightarrow Y$ be a nondegenerate holomorphic mapping between compact Kähler manifolds such that $n := \dim_{\mathbb{C}} X \geq m := \dim_{\mathbb{C}} Y$. Let ω_X and ω_Y be Kähler forms on X and Y, respectively. Let $F : X \longrightarrow \mathbb{R}^+$ be a real function such that $F \in L^p(X)$ for some $p > 1$.

Set $\omega_t := \pi^*(\omega_Y) + t\omega_X$, $t > 0$. We consider the following family of complex Monge–Ampère equations

$$\begin{cases} (\omega_t + dd^c\varphi_t)^n = c_t t^{n-m} F \omega_X^n \\ \max_X \varphi_t = 0 \end{cases} \qquad (\star)_t$$

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where φ_t is ω_t-plurisubharmonic on X and $c_t > 0$ is a constant given by

$$c_t t^{n-m} \int_X F\omega_X^n = \int_X \omega_t^n.$$

It follows from the seminal work of Yau [16] and Kołodziej [9], that Equation $(\star)_t$ admits a unique continuous solution (see also [2] for uniqueness). (Observe that for $t \in]0,1]$, ω_t is a Kähler form.)

Our aim here is to understand what happens when $t \to 0^+$, motivated by recent geometrical developments [11–13]. When $n = m$, the cohomology class ω_0 is big and semiample and this problem has been addressed by several authors recently (see [4, 6, 14, 15]).

We focus here on the case $m < n$. This situation is motivated by the study of the Kähler–Ricci flow on manifolds X of intermediate Kodaira dimension $1 \leq \mathrm{kod}(X) \leq n-1$. When $n = 2$ this has been studied by Song and Tian [12].

In a very recent and interesting paper [11], Kołodziej and Tian were able to show, under a technical geometric assumption on the fibration π, that the solutions (φ_t) are uniformly bounded on X when $t \searrow 0^+$.

The purpose of this note is to (re)prove this result without any technical assumption and with a different method: we actually follow the strategy introduced by Kołodziej in [9, 10] and further developed in [3, 6].

Theorem A. There exists a uniform constant $M = M(p, \pi, \|F\|_p) > 0$ such that the solutions to the Monge–Ampère equations $(\star)_t$ satisfy

$$\|\varphi_t\|_{L^\infty(X)} \leq M, \ \forall t \in]0,1].$$

An independent proof of Theorem A has been given in [5]. □

In [11, Theorem 2], an important application of Theorem A to convergence of the expanding Kähler–Ricci flow is given.

Let X be a projective manifold and m a positive integer such that mK_X is base point free. Let Y be the normalization of the image of the morphism $|mK_X|$. The morphism $\pi : X \longrightarrow Y$ is then a privileged model of the Iitaka fibration. Then, it is known [4, 14] that the expanding Kähler–Ricci flow

$$\begin{cases} \frac{d}{dt}\omega(t,\cdot) = -\mathrm{Ric}\omega(t,\cdot) - \omega(t,\cdot) \\ \omega(0,\cdot) = \omega_0 \end{cases} \qquad (\star\star)$$

has a global solution on X, for any fixed Kähler form ω_0. It can be expressed as

$$\omega(t,\cdot) = e^{-t}\omega_0 + (1-e^{-t})\pi^*\omega_Y + dd^c\varphi(t,\cdot),$$

where $\varphi(t,\cdot)$ is a smooth family of functions on X.

Assume furthermore that Y is smooth and π has no multiple fiber. Then [11, Theorem 2] asserts, under a technical condition, $\varphi(t,\cdot)$ converges in L^1-topology to $\pi^*\varphi_\infty$ as $t \to +\infty$, where

(1) φ_∞ is a bounded ω_Y-plurisubharmonic function on Y;
(2) φ_∞ satisfies a degenerate Monge–Ampère equation (see [12]).

The proof given in [11] is sketchy and only uses the above technical condition to establish a special case of Theorem A. Hence Theorem A can be used to remove it. The details—and much more—have indeed been given subsequently in the very interesting preprint [13].

1.1 **Preliminary remarks**

1.1.1 *Uniform control of c_t*

Observe that $\omega_0^k = 0$ for $m < k \le n$, hence for all $t \in]0,1]$,

$$\omega_t^n = \sum_{k=1}^{m} \binom{n}{k} t^{n-k} {\omega_0}^k \wedge {\omega_X}^{n-k}.$$

Note that $]0,1] \ni t \longmapsto t^{m-n}\omega_t^n$ is increasing (hence decreases as $t \searrow 0^+$) and satisfies for $t \in]0,1]$

$$\binom{n}{m} \qquad \frac{\omega_0^m \wedge \omega_X^{n-m}}{\int_X \omega_0^m \wedge \omega_X^{n-m}} \le \frac{\omega_t^n}{t^{n-m}\int_X \omega_0^m \wedge \omega_X^{n-m}} \le \frac{\omega_1^n}{\int_X \omega_0^m \wedge \omega_X^{n-m}}. \tag{1}$$

In particular, $t \longmapsto c_t$ is increasing in $t \in]0,1]$ and

$$0 < \binom{n}{m} \frac{\int_X \omega_0^m \wedge \omega_X^{n-m}}{\int_X F\omega_X^n} =: c_0 \le c_t \le c_1.$$

1.1.2 *Uniform control of densities*

Let J_π denote the (modulus square) of the Jacobian of the mapping π, defined through

$$\omega_0^m \wedge \omega_X^{n-m} = J_\pi \omega_X^n.$$

Let us rewrite Equation $(\star)_t$ as follows:

$$(\omega_t + dd^c \varphi_t)^n = f_t \omega_t^n,$$

where for $t \in]0, 1]$

$$0 \leq f_t := c_t t^{n-m} F \frac{\omega_X^n}{\omega_t^n} \leq c_1 \frac{F}{J_\pi}.$$

Observe that

$$\int_X f_t \omega_t^n = c_t t^{n-m} \int_X F \omega_t^n = \int_X \omega_t^n =: \mathrm{Vol}_{\omega_t}(X),$$

hence (f_t) is uniformly bounded in $L^1(\omega_t / V_t)$, $V_t := \mathrm{Vol}_{\omega_t}(X)$. We actually need a slightly stronger information.

Lemma 1.1. There exists $p' > 1$ and a constant $C = C(\pi, \|F\|_{L^p(X)}) > 0$ such that for all $t \in]0, 1]$

$$\int_X f_t^{p'} \omega_t^n \leq C \,\mathrm{Vol}_{\omega_t}(X).$$

□

Proof of the Lemma. Set $V_t := \mathrm{Vol}_{\omega_t} = \int_X \omega_t^n$ and observe that

$$0 \leq f_t \frac{\omega_t^n}{V_t} \leq c_1 F \frac{\omega_X^n}{\int_X \omega_0^m \wedge \omega_X^{n-m}} = C_2 F \omega_X^n,$$

where $C_2 := c_1 \int_X J_\pi \omega_X^n$.

This shows that the densities f_t are uniformly in L^1 with respect to the normalized volume forms ω_t^n / V_t.

Since J_π is locally given as the square of the modulus of a holomorphic function that does not vanish identically, there exists $\alpha \in]0, 1[$ such that $J_\pi^{-\alpha} \in L^1(X)$. Fix $\beta \in]0, \alpha[$

satisfying the condition $\beta/p+\beta/\alpha=1$. It follows from Hölder's inequality that

$$\int_X f_t^\beta \omega_X^n \leq \left(\int_X F^p \omega_X^n\right)^{\beta/p} \left(\int_X J_\pi^{-\alpha} \omega_X^n\right)^{\beta/\alpha}.$$

Setting $\varepsilon := \beta/q$ and using Hölder's inequality again, we obtain

$$\int_X f_t^{1+\varepsilon} \frac{\omega_t^n}{V_t} \leq C_2 \int_X f_t^\varepsilon F \omega_X^n.$$

Now applying again Hölder inequality, we get

$$\int_X f_t^{1+\varepsilon} \frac{\omega_t^n}{V_t} \leq C_2 \left(\int_X f_t^\beta \omega_X\right)^{1/q} \|F\|_{L^p(X)}.$$

Therefore denoting by $p' := 1+\varepsilon$, we have the following uniform estimate:

$$\int_X f_t^{p'} \frac{\omega_t^n}{V_t} \leq C(\pi, \|F\|_{L^p(X)}), \forall t \in]0,1],$$

where

$$C(\pi, \|F\|_{L^p(X)}) := C_2 \left(\int_X J_\pi^{-\alpha} \omega_X^n\right)^{\beta/\alpha q} \|F\|_{L^p(X)}^{1+\beta/q}.$$

■

1.2 Uniform domination by capacity

We now show that the measure $\mu_t := f_t \omega_t^n / \mathrm{Vol}_{\omega_t}$ are uniformly strongly dominated by the normalized capacity $\mathrm{Cap}_{\omega_t} / \mathrm{Vol}_{\omega_t}(X)$. It actually follows from a careful reading of the no-parameter proof given in [3, 6].

Lemma 1.2. There exists a constant $C_0 = C_0(\pi, \|F\|_{L^p(\omega_X^n)}) > 0$ such that for any compact set $K \subset X$ and $t \in]0,1]$,

$$\mu_t(K) \leq C_0^n \left(\frac{\mathrm{Cap}_{\omega_t}(K)}{\mathrm{Vol}_{\omega_t}(X)}\right)^2.$$

□

Proof. Fix a compact set $K \subset X$. Set $V_t := \mathrm{Vol}_{\omega_t}(X)$. Hölder's inequality yields

$$\mu_t(K) \leq \left(\int_X f_t^{p'} \frac{\omega_t^n}{V_t} \right)^{1/p'} \left(\int_K \frac{\omega_t^n}{V_t} \right)^{1/q'} .$$

It remains to dominate uniformly the normalized volume forms ω_t^n / V_t by the normalized capacities $\mathrm{Cap}_{\omega_t} / V_t$. Fix $\sigma > 0$ and observe that for any $t \in]0, 1]$,

$$\int_K \frac{\omega_t^n}{V_t} \leq \int_X e^{-\sigma(V_{K,\omega_t} - \max_X V_{K,\omega_t})} \frac{\omega_t^n}{V_t} T_{\omega_t}(K)^{\sigma},$$

where

$$V_{K,\omega_t} := \sup\{\psi \in PSH(X, \omega_t); \psi \leq 0, \text{ on } K\}$$

is the ω_t-extremal function of K and $T_{\omega_t}(K) := \exp(-\sup_X V_{K,\omega_t})$ is the associated ω_t-capacity of K (see [7] for their properties).

Observe that $\omega_t^n / V_t \leq c_1 \omega_1^n$ and $\omega_t \leq \omega_1$, hence the family of functions $V_{K,\omega_t} - \max_X V_{K,\omega_t}$ is a normalized family of ω_1-psh functions. Thus there exists $\sigma > 0$, which depends only on (X, ω_1) and a constant $B = B(\sigma, X, \omega_1)$ such that ([17])

$$\int_X e^{-\sigma(V_{K,\omega_t} - \max_X V_{K,\omega_t})} \frac{\omega_t^n}{V_t} \leq B, \forall t \in]0, 1].$$

The Alexander–Taylor comparison theorem (see Theorem 7.1 in [7]) now yields for a constant $C_3 = C_3(\pi, \|F\|_{L^p(X)})$

$$\mu_t(K) \leq C_3 \exp\left[-\sigma \left(\frac{V_t}{\mathrm{Cap}_{\omega_t}(K)} \right)^{1/n} \right], \quad \forall t \in]0, 1].$$

We infer that there is a constant $C_4 = C_4(\pi, \|F\|_{L^p(X)})$ such that

$$\mu_t(K) \leq C_4 \left(\frac{\mathrm{Cap}_{\omega_t}(K)}{V_t} \right)^2, \quad \forall t \in]0, 1]. \tag{2}$$

■

1.3 Uniform normalization

The comparison principle (see [10]) yields for any $s > 0$ and $\tau \in [0, 1]$

$$\tau^n \frac{\mathrm{Cap}_{\omega_t}(\{\varphi_t \le -s - \tau\})}{V_t} \le \int_{\{\varphi_t \le -s\}} \frac{(\omega_t + dd^c\varphi_t)^n}{V_t}.$$

It is now an exercise to derive from this inequality an a priori L^∞-estimate,

$$\|\varphi_t\|_{L^\infty(X)} \le C_5 + s_0(\omega_t),$$

where $s_0(\omega_t)$ (see [3, 6]) is the smallest number $s > 0$ satisfying the condition $e^n C_0^n \mathrm{Cap}_{\omega_t}(\{\psi \le -s\})/V_t \le 1$ for all $\psi \in PSH(X, \omega_t)$ such that $\sup_X \psi = 0$. Recall from ([7], Proposition 3.6) that

$$\frac{\mathrm{Cap}_{\omega_t}(\{\psi \le -s - \tau\})}{V_t} \le \frac{1}{s}\left(\int_X (-\psi)\frac{\omega_t^n}{V_t} + n\right).$$

Since $\frac{\omega_t^n}{V_t} \le C_1 \omega_1^n$, it follows that

$$\frac{\mathrm{Cap}_{\omega_t}(\{\psi \le -s - \tau\})}{V_t} \le \frac{1}{s}\left(C_1 \int_X (-\psi)\omega_1^n + n\right).$$

Since ψ is ω_1-psh and normalized, we know that there is a constant $A = A(X, \omega_1) > 0$ such that $C_1 \int_X (-\psi)\omega_1^n \le A$ for any such ψ. Therefore $s_0(\omega_t) \le s_0 := e^n C_0^n (A + n)$ for any $t \in]0, 1]$. Finally, we obtain the required uniform estimate for all $t \in]0, 1]$.

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