ON THE EXTENSION OF QUASIPLURISUBHARMONIC FUNCTIONS

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ABSTRACT. Let (V, ω) be a compact Kähler manifold such that V admits a cover by Zariski-open Stein sets with the property that ω has a strictly plurisubharmonic exhaustive potential on each element of the cover. If $X \subset V$ is an analytic subvariety, we prove that any $\omega|_X$-plurisubharmonic function on X extends to a ω-plurisubharmonic function on V.

This result generalizes a previous result of ours on the extension of singular metrics of ample line bundles. It allows one to show that any transcendental Kähler class in the real Neron-Severi group $NS_{\mathbb{R}}(V)$ has this extension property.

1. INTRODUCTION

Let (V, ω) be a compact Kähler manifold of dimension n. We recall that a function $\psi : V \to [-\infty, +\infty)$ is called quasiplurisubharmonic (qpsh) if it is locally the sum of a plurisubharmonic (psh) function and a smooth one. If ψ is qpsh then $\psi \in L^1(V, \omega^n)$ and ψ is called ω-plurisubharmonic (ω-psh) if $\omega + dd^c\psi \geq 0$ in the sense of currents on V. Here $d = \partial + \bar{\partial}$, $d^c = \frac{1}{2\pi i}(\partial - \bar{\partial})$. We denote by $PSH(V, \omega)$ the class of ω-psh functions on V and refer the reader to [GZ05, GZ17] for their basic properties.

Let now $X \subset V$ be a (closed) analytic subvariety. An upper semicontinuous function $\varphi : X \to [-\infty, +\infty)$ is called $\omega|_X$-psh if $\varphi \not\equiv -\infty$ on X and if there exist an open cover $\{U_\alpha\}_{\alpha \in A}$ of X and psh functions $\varphi_\alpha, \rho_\alpha$ defined on U_α, where ρ_α is smooth and $dd^c\rho_\alpha = \omega$, such that $\rho_\alpha + \varphi = \varphi_\alpha$ holds on $X \cap U_\alpha$, for every $\alpha \in A$. Note that we allow φ to be identically $-\infty$ on some (but not all) irreducible components of X.

The function φ is called *strictly* $\omega|_X$-psh if φ is not identically $-\infty$ on any irreducible component of X and if it is $(1 - \varepsilon)\omega|_X$-psh for some small $\varepsilon > 0$. The current $\omega|_X + dd^c\varphi$ is then called a Kähler current on X. We denote by $PSH(X, \omega|_X)$, resp. $PSH^+(X, \omega|_X)$, the class of $\omega|_X$-psh, resp. strictly $\omega|_X$-psh functions on X. For a detailed discussion of psh functions on complex spaces, we refer the reader to [FN80] and [D85, section 1].

By restriction, ω-psh functions on V yield $\omega|_X$-psh functions on X. An interesting problem is whether every $\omega|_X$-psh function φ on X arises in this way:

Problem. Does one have $(PSH(V, \omega))|_X = PSH(X, \omega|_X)$?

In [CGZ13, Theorem B], this was shown to be the case when ω is a Hodge form, i.e. ω is a representative of the Chern class $c_1(L)$ of a positive holomorphic line bundle L

over V (this requires the manifold V to be projective). The proof relies on an extension result with growth control of psh functions on analytic subvarieties of Stein manifolds [CGZ13, Theorem A]. This in turn used methods of Coltoiu based on Runge domains [Co91, Proposition 2], and of Sadullaev [Sa82] (see also [BL03, Theorem 3.2]).

Besides being a natural one, this problem has found important applications in Kähler geometry, allowing one to prove the continuity of Kähler-Einstein potentials (see e.g. [EGZ09, EGZ17, Dar17, GGZ20]).

A related and slightly simpler question is:

Problem+. Does one have $(PSH^+(V, \omega))|_X = PSH^+(X, \omega|_X)$?

When X is a complex submanifold of V, it is not difficult to show that any smooth strictly $\omega|_X$-psh function on X extends to a smooth strictly ω-psh function on V (see e.g. [CGZ13, Proposition 2.1] and [Sch98]).

In [CT14] it was shown that the answer to Problem+ is again positive for functions with analytic singularities. The proof uses resolution of singularities and a gluing argument similar to that of Richberg [R68].

A further result in this direction was recently obtained in [WZ20, NWZ21]. Assuming that there exists a holomorphic retraction $U \to X$ in a neighborhood $U \subset V$, it is proved in [NWZ21] that any strictly $\omega|_X$-psh on X extends to a strictly ω-psh function on V. These results all assume that X is smooth.

When X is singular, the equality $(PSH^+(V, \omega))|_X = PSH^+(X, \omega|_X)$ holds if the class $\{\omega\}$ belongs to the real Neron-Severi group $NS_{\mathbb{R}}(V)$. This follows from [CGZ13, Theorem B] through a simple density argument. Recall that $NS_{\mathbb{R}}(V) = NS(V) \otimes \mathbb{R}$, where

$$NS(V) = H^{1,1}(V, \mathbb{R}) \cap H^2(V, \mathbb{Z})/torsion.$$

Thus Hodge classes are rational points of $NS_{\mathbb{R}}(V)$.

The main goal of this note is to show the following stronger result:

Theorem 1.1. *Let (V, ω) be a compact Kähler manifold and let X be an analytic subvariety of V. If $\{\omega\} \in NS_{\mathbb{R}}(V)$ then $(PSH(V, \omega))|_X = PSH(X, \omega|_X)$.*

Note that a compact Kähler manifold (V, ω) such that $\{\omega\} \in NS_{\mathbb{R}}(V)$ must be projective (see Lemma 3.1).

We actually show that the methods developed in [CGZ13] can be adapted to prove that this extension property holds for a more general class of compact Kähler manifolds (V, ω) than the one of polarized projective manifolds considered in [CGZ13, Theorem B]:

Definition 1.2. We say that (V, ω) admits an *adapted Zariski-open Stein cover* if
  - V can be covered by finitely many open Stein sets V_j, $1 \le j \le N$, such that $H_j := V \setminus V_j$ is an analytic subvariety of V, and
  - there exist smooth exhaustion functions $\rho_j \ge 0$ on each V_j such that $\omega = dd^c \rho_j$.

In this case we also say that (V, ω) verifies condition (ZOS).

Note that the sets V_j are connected, as we assume V is connected and since H_j are analytic subvarieties. This notion has the following basic properties:

Proposition 1.3. *Let (V, ω) be a compact Kähler manifold. We have the following:*
(i) Condition (ZOS) only depends on the cohomology class of ω.
(ii) The set of Kähler classes $\{\omega\}$ such that (V, ω) satisfies condition (ZOS) is a convex cone.
(iii) If $\{\omega\}$ is a Hodge class then (V, ω) satisfies condition (ZOS).

It follows from the above observations that condition (ZOS) is satisfied for any Kähler class in the real Neron-Severi group $NS_{\mathbb{R}}(V)$ (see Lemma 3.1). We show moreover that these conditions are stable under restriction and products, as well as by pull-back by holomorphic coverings.

The main technical result of the note is the following extension theorem in this setting, which is is a generalization of [CGZ13, Theorem B and Theorem 2.2]:

Theorem 1.4. *Let (V, ω) be a compact Kähler manifold of dimension n which verifies condition (ZOS) and let X be an analytic subvariety of V. If $\varphi \in PSH(X, \omega|_X)$ then, given any constant $a > 0$, there exists $\psi \in PSH(V, \omega)$ such that $\psi|_X = \varphi$ and $\max_V \psi < \max_X \varphi + a$.*

Note that in Theorem 1.4 we do not assume that V is projective. Since qpsh functions on a compact Kähler manifold can be regularized (see [D92, BK07]), one has the following immediate corollary:

Corollary 1.5. *Let (V, ω) be a compact Kähler manifold which verifies condition (ZOS) and let X be an analytic subvariety of V. If $\varphi \in PSH(X, \omega|_X)$ then there exists a sequence of smooth functions $\varphi_j \in PSH(X, \omega|_X) \cap \mathcal{C}^{\infty}(X)$ which decrease pointwise to φ.*

This smoothing property plays a key role in establishing global continuity of singular Kähler-Einstein potentials (see [EGZ09, Theorem 2.1]). Corollary 1.5 thus provides an alternative proof of [GGZ20, Theorem 3.9]. Note that the regularizing techniques of Demailly [D92] and Blocki-Kolodziej [BK07] break down when X is singular.

Theorem 1.4 is proved in Section 2. Theorem 1.1 follows from Theorem 1.4 and Proposition 1.3 (see Section 3.1 (v) and Lemma 3.1). In Section 3 we prove Proposition 1.3 and we collect some examples of compact Kähler manifolds which admit an adapted Zariski-open Stein cover. Following [Mat13], we also show that the reference cohomological class has to be Kähler for the extension property to hold (a similar observation has been made by Meng-Wang [MW21]).

2. PROOF OF THE THEOREM 1.4

Let X be an analytic subvariety of a compact Kähler manifold (V, ω) of dimension n, which verifies condition (ZOS). Note that the function $-\rho_j \le 0$ is an ω-psh function on V_j. Since $V \setminus V_j = H_j$ is an analytic set, $-\rho_j$ extends to an ω-psh function θ_j on V, such that $\theta_j \le 0$ on V. Since ρ_j is an exhaustion of V_j, we see that $\theta_j = -\infty$ on H_j. Let

$$(2.1) \qquad \theta := \max\{\theta_j : 1 \le j \le N\}.$$

It follows that $\theta \leq 0$ is a continuous ω-psh function on V, and we set

$$(2.2) \qquad\qquad m := -\min_{z \in V} \theta(z) > 0.$$

The following lemma gives special subextensions of $\omega|_X$-psh functions on X.

Lemma 2.1. *Let $\varepsilon \geq 0$ and u be a continuous $(1+\varepsilon)\omega$-psh function on V such that $u(z) \leq 0$ for all $z \in V$. If $c > 1$ and φ is an $\omega|_X$-psh function on X such that $\varphi < u$, then there exists a $c\omega$-psh function ψ on V such that*

$$\frac{1}{c} \psi(z) \leq \frac{1}{1+\varepsilon} u(z), \ \forall z \in V,$$

and

$$\psi(z) = \varphi(z) + (c-1)\theta(z) + (c-1)\min_{\zeta \in V} u(\zeta), \ \forall z \in X.$$

Proof. Let $X_j = X \cap V_j$. We may assume that $X_j \neq \emptyset$ for $1 \leq j \leq N'$, and $X_j = \emptyset$ for $N' < j \leq N$, where N' is an integer with $1 \leq N' \leq N$. Set

$$M = -\min_{\zeta \in V} u(\zeta) \geq 0.$$

Let $1 \leq j \leq N'$. The function $\varphi + \rho_j + M$ is psh on X_j and since $u \leq 0$ we have that

$$\varphi + \rho_j + M < u + \rho_j + M \leq \frac{1}{1+\varepsilon} u + \rho_j + M \ \text{ on } X_j.$$

Fix c', $1 < c' < c$. Note that the function $(1+\varepsilon)^{-1}u + \rho_j + M \geq 0$ is a continuous psh exhaustion function of V_j. It follows from [CGZ13, Theorem A] that there exists a psh function $\widetilde{\psi}_j$ on V_j such that

$$\widetilde{\psi}_j < \frac{c'}{1+\varepsilon} u + c'\rho_j + c'M \ \text{ on } V_j, \ \ \widetilde{\psi}_j = \varphi + \rho_j + M \ \text{ on } X_j.$$

The function

$$\psi_j := \widetilde{\psi}_j - c\rho_j - cM$$

is $c\omega$-psh on V_j and, since $c' < c$, it verifies

$$\psi_j < \frac{c}{1+\varepsilon} u + (c' - c)\rho_j + (c' - c)\Big(\frac{1}{1+\varepsilon} u + M\Big) \leq \frac{c}{1+\varepsilon} u \leq 0.$$

Since $\rho_j(z) \to +\infty$ as $V_j \ni z \to H_j$ and since u is continuous on V, we conclude that ψ_j extends to a $c\omega$-psh function ψ_j on V such that

$$\psi_j \leq \frac{c}{1+\varepsilon} u \ \text{ on } V, \ \ \psi_j = -\infty \ \text{ on } H_j, \ \ \psi_j = \varphi - (c-1)\rho_j - (c-1)M \ \text{ on } X_j.$$

Hence $\psi_j = \varphi + (c-1)\theta_j - (c-1)M$ on X, as both sides are equal to $-\infty$ on $X \cap H_j$.

We define $\psi := \max_{1 \leq j \leq N'} \psi_j$. Then ψ is $c\omega$-psh on V and

$$\psi \leq \frac{c}{1+\varepsilon} u \ \text{ on } V, \ \ \psi = \varphi + (c-1)\max_{1 \leq j \leq N'} \theta_j - (c-1)M \ \text{ on } X.$$

If $j > N'$ then $X \subset H_j$, so $\theta_j = -\infty$ on X. Therefore

$$\theta = \max_{1 \leq j \leq N} \theta_j = \max_{1 \leq j \leq N'} \theta_j, \ \ \psi = \varphi + (c-1)\theta - (c-1)M,$$

hold on X.   □

Proof of Theorem 1.4. Using Lemma 2.1, we repeat the proof of [CGZ13, Theorem 2.2]. Assuming that $\max_X \varphi = -a$, where $a > 0$, we have to find a negative ω-psh function ψ on V with $\psi = \varphi$ on X. Let X' be the union of the irreducible components Y of X such that $\varphi|_Y \not\equiv -\infty$. We construct by induction on $j \geq 1$ a sequence of numbers $\varepsilon_j \searrow 0$ and a sequence of negative smooth $(1 + \varepsilon_j)\omega$-psh functions ψ_j on V such that, for all $j \geq 2$,

$$\frac{\psi_j}{1 + \varepsilon_j} < \frac{\psi_{j-1}}{1 + \varepsilon_{j-1}} \ \text{ on } V \ , \ \ \psi_{j-1} > \varphi \text{ on } X \ , \ \ \int_{X'} (\psi_j - \varphi) < \frac{1}{j} \ , \ \ \int_W \psi_j < -j \ ,$$

for every irreducible component W of X where $\varphi|_W \equiv -\infty$. Here the integrals are with respect to the area measure on each irreducible component X_j of X, i.e.

$$\int_X f := \sum_{X_j} \int_{X_j} f \, \omega^{\dim X_j} \ , \ \ |X_j| := \int_{X_j} \omega^{\dim X_j}.$$

Let $\varepsilon_1 = 1$, $\psi_1 = 0$, and assume that $\varepsilon_{j-1}, \psi_{j-1}$, where $j \geq 2$, are constructed with the above properties. Since $\varphi < \psi_{j-1}|_X$ and the latter is continuous on the compact set X, we can find $\delta > 0$ such that $\varphi < \psi_{j-1} - \delta$ on X.

Let $c > 1$. Lemma 2.1 implies that there exists a $c\omega$-psh function ψ_c such that

$$\frac{\psi_c}{c} \leq \frac{\psi_{j-1} - \delta}{1 + \varepsilon_{j-1}} \ \text{ on } V \ , \ \ \psi_c = \varphi + (c-1)\theta - (c-1)M_{j-1} \ \text{ on } X,$$

where $M_{j-1} = \delta - \min_{\zeta \in V} \psi_{j-1}(\zeta) \geq 0$. By [BK07] and [D92], there exists a sequence of smooth $c\omega$-psh functions $\{\eta_k\}$ decreasing pointwise to ψ_c on V. Let $\psi_c' := \eta_k$, where k is chosen sufficiently large such that ψ_c' is a negative smooth $c\omega$-psh function on V and it verifies the following:

$$\frac{\psi_c'}{c} < \frac{\psi_{j-1} - \frac{\delta}{2}}{1 + \varepsilon_{j-1}} \ \text{ on } V,$$

$$\psi_c' > \varphi + (c-1)\theta - (c-1)M_{j-1} \geq \varphi - (c-1)(m + M_{j-1}) \ \text{ on } X,$$

$$\int_{X'} (\psi_c' - \varphi) \leq \int_{X'} (\psi_c' - \varphi - (c-1)\theta + (c-1)M_{j-1}) < c - 1,$$

$$\int_W \psi_c' < -j - (c-1)(m + M_{j-1})|W|,$$

where W is as above. The last two requirements are possible thanks to the dominated, respectively monotone, convergence theorems.

Let $\psi_c'' := \psi_c' + (c-1)(m + M_{j-1})$. Then on V we have that

$$\frac{\psi_c''}{c} < \frac{\psi_{j-1} - \frac{\delta}{2}}{1 + \varepsilon_{j-1}} + \frac{(c-1)(m + M_{j-1})}{c} < \frac{\psi_{j-1}}{1 + \varepsilon_{j-1}} - \frac{\delta}{4} + (c-1)(m + M_{j-1}).$$

Moreover, $\psi_c'' > \varphi$ on X and

$$\int_{X'} (\psi_c'' - \varphi) = \int_{X'} (\psi_c' - \varphi) + (c-1)(m + M_{j-1})|X'| < (c-1)(1 + m|X'| + M_{j-1}|X'|),$$

$$\int_W \psi_c'' = \int_W \psi_c' + (c-1)(m + M_{j-1})|W| < -j.$$

If $\varepsilon_j > 0$ is such that $\varepsilon_j < \varepsilon_{j-1}/2$, $\varepsilon_j(m + M_{j-1}) < \delta/4$, $\varepsilon_j(1 + m|X'| + M_{j-1}|X'|) < j^{-1}$, and $\psi_j := \psi_{1+\varepsilon_j}''$, then ε_j, ψ_j have the desired properties.

We conclude that $\varphi_j = (1 + \varepsilon_j)^{-1}\psi_j$ is a decreasing sequence of negative smooth ω-psh functions on V such that $\varphi_j > (1 + \varepsilon_j)^{-1}\varphi > \varphi$ on X. Hence $\psi = \lim_{j\to\infty} \varphi_j$ is a negative ω-psh function on V and $\psi \geq \varphi$ on X. Note that

$$\int_{X'} (\varphi_j - \varphi) = \frac{1}{1 + \varepsilon_j} \int_{X'} (\psi_j - \varphi) - \frac{\varepsilon_j}{1 + \varepsilon_j} \int_{X'} \varphi \to 0 \,, \quad \int_W \varphi_j = \frac{1}{1 + \varepsilon_j} \int_W \psi_j \to -\infty \,,$$

as $j \to +\infty$, hence $\psi = \varphi$ on X. $\qquad\square$

3. Zariski open Stein covers

We study in this section the existence of Zariski open covers on various types of Kähler manifolds. This requires the existence of closed subvarieties of positive dimension, so it is natural to explore in particular the case of projective varieties (many Kähler manifolds -e.g. generic tori or very generic K3 surfaces- do not admit any closed subvariety of positive dimension).

3.1. **Basic properties.** We establish here the proof of Proposition 1.3, and some of its consequences.

(i) Let (V, ω) be a compact Kähler manifold that satisfies condition (ZOS), and let ω' be another Kähler form cohomologous to ω. By the $\partial\bar{\partial}$-Lemma, $\omega' = \omega + dd^c\eta$ for a smooth function $\eta : V \to \mathbb{R}$. Let $\{V_j\}_j$ be a Zariski-open Stein cover of V such that $\omega = dd^c\rho_j$ on V_j, where ρ_j is a smooth strictly psh exhaustion function on V_j. Then $\omega' = dd^c(\rho_j + \eta)$ and $\rho_j + \eta$ is a smooth strictly psh exhaustion function on V_j, since η is globally bounded on V. Thus (V, ω') also satisfies condition (ZOS).

(ii) Let ω_1, ω_2 be Kähler forms on V such that $(V, \omega_1), (V, \omega_2)$ verify condition (ZOS). If $\omega = a_1\omega_1 + a_2\omega_2$, where $a_1, a_2 > 0$, then (V, ω) verifies condition (ZOS). Indeed, assume that $V = \bigcup_{j=1}^{N_1} V_j = \bigcup_{k=1}^{N_2} W_k$, where V_j, W_k are open Stein sets such that $H_j = V \setminus V_j$, $L_k = V \setminus W_k$ are analytic subvarieties of V. Moreover, $\omega_1 = dd^c\rho_j$ on V_j, $\omega_2 = dd^c\eta_k$ on W_k, where ρ_j, η_k are smooth strictly psh exhaustion functions on V_j, respectively on W_k. Then $V \setminus (V_j \cap W_k) = H_j \cup L_k$ is a proper analytic subvariety of V. Moreover, the function $\tau_{jk} := a_1\rho_j + a_2\eta_k$ is a smooth strictly psh exhaustion function on $V_j \cap W_k$, hence $V_j \cap W_k$ is Stein. It follows that $\{V_j \cap W_k\}_{j,k}$ is an adapted Zariski-open Stein cover for (V, ω).

(iii) If (V, ω) is a compact Kähler manifold which verifies condition (ZOS) and if W is a (closed) complex submanifold of V then it is easy to see that $(W, \omega|_W)$ verifies condition (ZOS), by intersecting the open sets V_j with W.

(iv) If V is a projective manifold and ω is a Hodge form on V then (V, ω) verifies condition (ZOS). Indeed, replacing ω by $k\omega$, we can assume that V is an algebraic submanifold of the complex projective space $\mathbb{P}^N$, and $\omega = \omega_{FS}|_V$ is the Fubini-Study Kähler form. Using the canonical covering by affine charts $\mathbb{C}^N \hookrightarrow \mathbb{P}^N$, and the Fubini-Study potentials $\rho(z) = \log\sqrt{1 + \|z\|^2}$, $z \in \mathbb{C}^N$, we have that $(\mathbb{P}^N, \omega_{FS})$ verifies condition (ZOS), hence so does (V, ω) by the previous restriction property.

(v) If V is a projective manifold and $\{\omega\} = \sum_{\ell=1}^m a_\ell\{\omega_\ell\}$, where $a_\ell > 0$ and $\{\omega_\ell\}$ are Hodge classes on V, then (V, ω) verifies condition (ZOS), as follows from the previous observations. It is easy to see that any Kähler class $\{\omega\}$ in the real Neron-Severi group $NS_{\mathbb{R}}(V)$ can be expressed as a convex combination of Hodge classes. For the convenience of the reader we include a proof in Lemma 3.1 below. Hence (V, ω) satisfies condition (ZOS) if the Kähler class $\{\omega\} \in NS_{\mathbb{R}}(V)$.

Lemma 3.1. *Let (V, ω) be a compact Kähler manifold such that $\{\omega\} \in NS_{\mathbb{R}}(V)$. Then V is projective and $\{\omega\} = \sum_{\ell=1}^m a_\ell\vartheta_\ell$, where $a_\ell > 0$ and ϑ_ℓ are Hodge classes on V.*

Proof. Let $\mathcal{K}_V \subset H^{1,1}(V, \mathbb{R})$ denote the set of Kähler classes on V. Then $\mathcal{K}_V$ is a convex cone and it is open. Let $c_1(L_j)$, $L_j \in \mathrm{Pic}(V)$, $1 \leq j \leq k$, be a basis of $NS_{\mathbb{R}}(V)$, so

$$\{\omega\} = \sum_{j=1}^k c_j c_1(L_j), \text{ where } c_j \in \mathbb{R}.$$

Since $\mathcal{K}_V$ is open it follows that there exists $\varepsilon > 0$ such that the class

$$\{\omega'\} = \sum_{j=1}^k r_j c_1(L_j) \in \mathcal{K}_V, \text{ if } r_j \in \mathbb{Q}, |r_j - c_j| < \varepsilon.$$

Note that $\{\omega'\} = N^{-1}\vartheta$ where $N \in \mathbb{N}$ and ϑ is a Hodge class on V, hence V is projective. Since $\mathcal{K}_V \cap NS_{\mathbb{R}}(V) \neq \emptyset$ is open in $NS_{\mathbb{R}}(V)$, we have that $NS_{\mathbb{R}}(V)$ has a basis of Kähler classes, and hence a basis of Hodge classes, by the preceding discussion. So we can write

$$\{\omega\} = \sum_{j=1}^k c_j\vartheta_j, \text{ where } \vartheta_j \text{ are Hodge classes and } c_j \in \mathbb{R}.$$

Choosing $r_j \in \mathbb{Q}$, $r_j < c_j$ sufficiently close to c_j, we infer by above that

$$\{\omega\} = \sum_{j=1}^k r_j\vartheta_j + \sum_{j=1}^k a_j\vartheta_j = N^{-1}\vartheta + \sum_{j=1}^k a_j\vartheta_j,$$

where $a_j = c_j - r_j > 0$, $N \in \mathbb{N}$ and ϑ is a Hodge class on V. $\qquad\square$

(vi) Let $(V^1, \omega_1), (V^2, \omega_2)$ be compact Kähler manifolds which verify condition (ZOS). Then $(V^1 \times V^2, \omega)$ verifies condition (ZOS), where $\omega = \pi_1^\star\omega_1 + \pi_2^\star\omega_2$ and

$$\pi_j : V^1 \times V^2 \to V^j$$

are the canonical projections. Indeed, if $V^1 = \bigcup_k V_k^1$ and $V^2 = \bigcup_\ell V_\ell^2$ are adapted Zariski-open Stein covers for (V^1, ω_1) and (V^2, ω_2), then it is easy to see that $\{V_k^1 \times V_\ell^2\}_{k,\ell}$ is an adapted Zariski-open Stein cover for $(V^1 \times V^2, \omega)$.

(vii) Let V, W be compact Kähler manifolds and $f : W \to V$ be a (unbranched) holomorphic covering map. Assume that ω is a Kähler form on V such that (V, ω) verifies condition (ZOS). Then $(W, f^\star \omega)$ verifies condition (ZOS). Indeed, since W is compact f is a finite map. If $U \subset V$ is an open Stein set such that $H = V \setminus U$ is an analytic subvariety of V and if ρ is a smooth strictly psh exhaustion function on U, then $\rho \circ f$ is a smooth strictly psh exhaustion function on $f^{-1}(U)$. Moreover $f^{-1}(H)$ is a proper analytic subvariety of W.

In the case of a *branched* holomorphic cover $f : W \to V$, and with the above notation, it still holds that $f^{-1}(U)$ is a Zariski-open Stein set (see e.g. [For17, p. 49]). However in this case the cohomology class $\{f^\star \omega\}$ is merely semi-positive. As noted by Matsumura [Mat13], strict positivity is required to expect the extension property to hold true (see Proposition 3.4).

3.2. **The real Neron-Severi group.** A Fano manifold is a projective n-dimensional manifold V whose anticanonical bundle $-K_V$ is ample. It follows from the Kodaira-Nakano vanishing theorem that $H^{0,q}(V, \mathbb{C}) = 0$ for $q > 0$, so Hodge symmetry and Hodge decomposition theorem ensure that

$$H^2(V, \mathbb{R}) = H^{1,1}(V, \mathbb{R}),$$

hence any Kähler class belongs to the real Neron-Severi group. Note also that in this case the map $c_1 : H^1(V, \mathscr{O}_V^\star) \cong \operatorname{Pic}(V) \to H^2(V, \mathbb{Z})$ is an isomorphism. Thus condition (ZOS) is satisfied and the extension property holds for *any* Kähler class in this case.

Since $h^{0,2}(V) = \dim H^{0,2}(V, \mathbb{C})$ is a birational invariant (see e.g. [H77]), the extension property holds in particular on any *rational* manifold V (i.e. birationally equivalent to $\mathbb{P}^n$), and hence on any toric manifold.

The Picard number $\rho(V)$ (rank of the Neron-Severi group) is always bounded from above by the Hodge number $h^{1,1}(V)$. Besides Fano manifolds, there are many manifolds with maximal Picard number $\rho(V) = h^{1,1}(V)$, especially when $\dim_{\mathbb{C}} V \geq 3$. We refer the interested reader to [Be14] for a list of examples and a classification in complex dimension 2. The extension property holds true for any Kähler class on such manifolds.

At the other extreme the Picard number is one on a generic abelian variety, as well as on any generic projective $K3$ surface, so most Kähler classes do not belong to the real Neron-Severi group and we do not know if the extension property holds for these.

3.3. **Projective toric manifolds.** Let V be a compact toric manifold of dimension n. Then V is a compactification of the complex torus $(\mathbb{C}^\star)^n$ such that the canonical action by multiplication of $(\mathbb{C}^\star)^n$ on itself extends to a holomorphic action of $(\mathbb{C}^\star)^n$ on V. As noted in Section 3.2 the extension property is valid for any Kähler class on V, since V is rational.

A toric Kähler form on V is a Kähler form ω which is $(S^1)^n$-invariant. We give here an alternative proof that (V, ω) satisfies condition (ZOS). For this, we will need the following lemma:

Lemma 3.2. *Let $f : [0, +\infty)^n \to \mathbb{R}$ be such that $\rho(z_1, \ldots, z_n) := f(|z_1|, \ldots, |z_n|)$ is a smooth strictly psh function on $\mathbb{C}^n$. Then ρ is an exhaustion function for $\mathbb{C}^n$.*

Proof. It follows from the maximum principle applied to ρ on polydiscs that for each $1 \leq j \leq n$ and for fixed $x_k \geq 0$, $k \neq j$, the function $x_j \to f(x_1, \ldots, x_j, \ldots, x_n)$ is increasing. For $x_j > 0$ and $x_k \geq 0$, $k \neq j$, we obtain using Green's formula that

$$\frac{\partial}{\partial x_j} f(x_1, \ldots, x_n) = \frac{1}{2\pi} \frac{\partial}{\partial x_j} \int_0^{2\pi} \rho(x_1, \ldots, x_j e^{it}, \ldots, x_n) \, dt$$

(3.1)

$$= \frac{2}{\pi x_j} \int_{\{|z_j| \leq x_j\}} \frac{\partial^2 \rho}{\partial z_j \partial \overline{z}_j} (x_1, \ldots, z_j, \ldots, x_n) \, d\lambda(z_j),$$

where λ is the Lebesgue measure on $\mathbb{C}$. Let

$$m := \min \left\{ \frac{\partial^2 \rho}{\partial z_j \partial \overline{z}_j} (0, \ldots, z_j, \ldots, 0) : |z_j| \leq 1, \ 1 \leq j \leq n \right\}.$$

We have $m > 0$ since ρ is strictly psh. Equation (3.1) implies that

$$\frac{\partial}{\partial x_j} f(0, \ldots, x_j, \ldots, 0) \geq \frac{2m}{x_j}$$

holds for $x_j \geq 1$. Since f is increasing in each variable we infer that

$$f(x_1, \ldots, x_j, \ldots, x_n) \geq f(0, \ldots, x_j, \ldots, 0) \geq 2m \log x_j + f(0),$$

for every $1 \leq j \leq n$, $x_j \geq 1$, and $x_k \geq 0$ for $k \neq j$. Hence

$$\rho(z) \geq 2m \log^+ \max\{|z_1|, \ldots, |z_n|\} + \rho(0), \ \forall z \in \mathbb{C}^n.$$

$\square$

Proposition 3.3. *Let V be a projective toric manifold of dimension n equipped with a toric Kähler form ω. Then (V, ω) verifies condition (ZOS).*

Proof. Let $T := (\mathbb{C}^\star)^n$ be the complex torus in V, and let $p_1, \ldots, p_N$ be the toric points of V. By [ALZ16, Proposition 4.4] we have that $V = \bigcup_{j=1}^N V_j$, where the sets $V_j \ni p_j$ are open, $T \subset V_j$, $V \setminus V_j$ are analytic subvarieties of V, and the following properties hold: For $1 \leq j \leq N$, there exists a biholomorphic map $\Phi_j : V_j \to \mathbb{C}^n$ such that $\Phi_j(p_j) = 0$, $\Phi_j(T) = (\mathbb{C}^\star)^n \subset \mathbb{C}^n$, V_j is invariant under the action of T and the action of T on V_j coincides via Φ_j with the canonical action by multiplication of $(\mathbb{C}^\star)^n$ on $\mathbb{C}^n$.

Set $\omega_j = (\Phi_j^{-1})^\star(\omega|_{V_j})$. Then ω_j is a Kähler form on $\mathbb{C}^n$, invariant under the canonical action of $(S^1)^n$ on $\mathbb{C}^n$. We have $\omega_j = dd^c v_j$ for some smooth strictly psh function v_j on $\mathbb{C}^n$. It follows that

$$w_j(z_1, \ldots, z_n) := \frac{1}{(2\pi)^n} \int_{[0.2\pi]^n} v_j(e^{i\theta_1} z_1, \ldots, e^{i\theta_n} z_n) \, d\theta_1 \ldots d\theta_n$$

is a smooth, polyradial, strictly psh function such that $dd^c w_j = \omega_j$. Lemma 3.2 shows that w_j is an exhaustion function for $\mathbb{C}^n$. Thus $\rho_j := w_j \circ \Phi_j$ is a smooth strictly psh exhaustion function on V_j with $dd^c \rho_j = \omega$. $\square$

For a brief description of the construction of n-dimensional projective toric manifolds V, and of very ample toric line bundles on V, from fans of cones in $\mathbb{R}^n$ we refer to [ALZ16, Section 4] (see also [Fu93]).

By Lemma 3.2 (and its proof) it follows that every smooth strictly psh function ρ on $\mathbb{C}^n$ which is polyradial is an exhaustion, and in fact it grows at least like $c \log \|z\|$ for some constant $c > 0$. We conclude this section by noting that the hypothesis of Lemma 3.2 that ρ is polyradial is necessary. Namely, we give below an example of a smooth strictly psh function on $\mathbb{C}^n$ which is not an exhaustion, and an example of a smooth strictly psh exhaustion function on $\mathbb{C}^n$ which does not grow at least logarithmically (i.e. it is not bounded below by $c \log \|z\|$, for any constant $c > 0$).

Example 1. The function $u(z) = \sum_{j=1}^{\infty} \frac{1}{2^{j+1}} \log(3\|z - j\|)$ is psh on $\mathbb{C}^n$ and $\mathscr{C}^{\infty}$-smooth on $\mathbb{C}^n \setminus \{1, 2, \ldots\}$. Moreover $u(z) \leq \frac{1}{2} \log^+ \|z\| + C$ on $\mathbb{C}^n$ for some constant C, and $u(z) \geq 0$ on $D := \mathbb{C}^n \setminus \left(\bigcup_{j \geq 1} B_j \right)$, where B_j is the open ball in $\mathbb{C}^n$ centered at j and of radius $\frac{1}{3}$. Let $\{a_j\}_{j \geq 1}$ be a sequence such that $a_j < 0$. By a regularized maximum construction on each B_j we obtain a $\mathscr{C}^{\infty}$-smooth psh function v on $\mathbb{C}^n$ which has the following properties:

(i) $v = u$ on D and $v(z) \leq \frac{1}{2} \log^+ \|z\| + C$ on $\mathbb{C}^n$ for some constant C.

(ii) $v \geq a_j$ on B_j and $v(j) = a_j$.

Let $\rho(z) = \frac{1}{4} \log(1 + \|z\|^2) + v(z)$. Then ρ is a smooth strictly psh function $\mathbb{C}^n$ and by (i),

$$\rho(z) = \frac{1}{4} \log(1 + \|z\|^2) + u(z) \geq \frac{1}{4} \log(1 + \|z\|^2) \text{ on } D.$$

Moreover ρ is in the Lelong class, i.e. $\rho(z) \leq \log^+ \|z\| + C$ on $\mathbb{C}^n$ for some constant C. Choosing $a_j = -\frac{1}{4} \log(1 + j^2)$ we have $\rho(j) = 0$ so ρ is not an exhaustion. If we take $a_j = \log \log(j + 1) - \frac{1}{4} \log(1 + j^2)$ we obtain a smooth strictly psh exhaustion function ρ such that $\rho(j) = \log \log(j + 1)$.

3.4. **Positivity vs. extension property.** We finally study a necessary condition for a cohomology class α to have the extension property. Recall that a class $\alpha \in H^{1,1}_{psef}(V, \mathbb{R})$ is pseudo-effective if it is the deRham cohomology class of a positive closed current T of bidegree $(1, 1)$. Fixing $\theta \in \alpha$ a smooth closed $(1, 1)$-form, it follows from the $\partial \bar{\partial}$-lemma that $T = \theta + i \partial \bar{\partial} \varphi$, for some qpsh function $\varphi \in PSH(V, \theta)$. It is natural to wonder when the class α has the extension property, i.e. $(PSH(V, \theta))|_X = PSH(X, \theta|_X)$ holds for any analytic subset $X \subset V$. Following Matsumura [Mat13] we observe that α should actually be a Kähler class:

Proposition 3.4. *Let V be a projective manifold of dimension n. If a pseudo-effective class $\alpha \in H^{1,1}_{psef}(V, \mathbb{R})$, $\alpha \neq 0$, has the extension property, then α is a Kähler class.*

This result is due to Matsumura [Mat13] when $\alpha = c_1(L)$ is the first Chern class of some holomorphic line bundle L on V. His proof can be adapted straightforwardly, replacing the algebraic Nakai-Moishezon criterion by the transcendental version due to Demailly-Paun [DP04]. A similar observation has been made by Meng-Wang in [MW21].

Proof. We fix a Kähler form ω_V on V. Let $\theta \in \alpha$ be a smooth closed $(1, 1)$-form and let $\varphi_0 \in PSH(V, \theta)$.

Step 1: The class α is nef. Following Demailly we consider

$$W_\theta(x) := (\sup\{\varphi(x) : \varphi \in PSH(V, \theta) \text{ with } \varphi \leq 0\})^* .$$

This is a θ-psh function with minimal singularities on V.

We observe first that the restriction of α to any irreducible curve $\mathcal{C} \subset V$ is Kähler, provided that $\mathcal{C} = D_1 \cap \cdots \cap D_{n-1}$ is a complete intersection of ample divisors (this is in analogy to [Mat13, Lemma 2.1]). Indeed, let h_j be a Hermitian metric on the line bundle $\mathscr{O}_V(D_j)$ such that $\omega_j := c_1(\mathscr{O}_V(D_j), h_j)$ is a Kähler form. By the Lelong-Poincaré formula we have $[D_j] = \omega_j + dd^c \log |S_j|_{h_j}$, where $[D_j]$ is the current of integration on D_j and S_j is the canonical section of $\mathscr{O}_V(D_j)$. Fix $\varepsilon > 0$ such that $\omega_j \geq \varepsilon \omega_V$ for $1 \leq j \leq n-1$. Since $\mathcal{C}$ is a complete intersection it follows from [D93] that $[C] = [D_1] \wedge \ldots \wedge [D_{n-1}]$. Therefore

$$\int_{\mathcal{C}} \theta = \int_V \theta \wedge [\mathcal{C}] = \int_V (\theta + dd^c \varphi_0) \wedge \omega_1 \wedge \ldots \wedge \omega_{n-1} \geq \varepsilon^{n-1} \int_V (\theta + dd^c \varphi_0) \wedge \omega_V^{n-1} > 0,$$

as $\alpha \neq 0$. Hence $\alpha|_{\mathcal{C}}$ is Kähler.

Next, a Bertini type theorem [Zha09, Theorem 1.3] ensures that for any $p, q \in V$ there exists a smooth complete intersection curve $\mathcal{C}$ as above with $p, q \in \mathcal{C}$ (see [Mat13, Theorem 2.2]). This is where we use the projectivity assumption on the underlying manifold V.

Fix now $x \in V$ and $\mathcal{C}$ a curve as above such that $x \in \mathcal{C}$. Since $\alpha|_{\mathcal{C}}$ is Kähler, one can find $\phi \in PSH(\mathcal{C}, \theta|_{\mathcal{C}})$ which is bounded below on $\mathcal{C}$. It follows from the extension property that there exists $\varphi \in PSH(V, \theta)$ such that $\varphi|_{\mathcal{C}} = \phi$. Shifting φ, ϕ by a constant, we can assume that $\varphi \leq 0$, hence $W_\theta(x) \geq \varphi(x) = \phi(x) > -\infty$. In particular W_θ has zero Lelong numbers at all points $x \in V$, so by Demailly's regularization theorem [D92] one can find smooth functions $v_\varepsilon \in PSH(V, \theta + \varepsilon \omega_V)$ decreasing to W_θ. Thus α is a nef class, limit of the Kähler classes $\alpha + 2\varepsilon\{\omega_V\}$.

Step 2: Transcendental Nakai-Moishezon criterion. It follows now from the main result of [DP04] that α is Kähler if and only if the volume of α is positive on any k-dimensional irreducible subvariety X of V, i.e.

$$\mathrm{Vol}(\alpha|_X) = \int_X \theta^k > 0.$$

We thus fix X such a subvariety and reproduce the final step of the proof of [Mat13, Theorem 1.2]. Fix $p \in X_{reg}$ and $B(p, r) \subset V$ a small ball around p in a local holomorphic chart where $p = 0$. Let $A > 0$ such that $Add^c\|z\|^2 + \theta \geq \omega_V$ in $B(p, r)$.

If $q \in \partial B(p, r)$ we let $\mathcal{C}$ be an irreducible curve as above passing through p and q. Since $\alpha|_{\mathcal{C}}$ is Kähler, can find u a θ-psh function on $\mathcal{C}$ such that $u(p) = -\infty$ while $u(q) \geq 0$. By the extension property u is the restriction of an ambient θ-psh function U. We regularize U by Demailly's technique, thus obtaining $U_\varepsilon \in PSH(V, \theta + \varepsilon \omega_V)$ with analytic singularities such that $U_\varepsilon > 0$ near q and $U_\varepsilon \leq -2Ar^2$ in some smaller ball $B(p, \delta_q)$, if ε is small enough. Using the compactness of $\partial B(p, r)$ we can find finitely many such functions and consider their maximum to end up with a $(\theta + \varepsilon \omega_V)$-psh function ϕ_ε such that $\phi_\varepsilon \geq 0$ on $\partial B(p, r)$ and $\phi_\varepsilon < -2Ar^2$ in some ball $B(p, \delta) \Subset B(p, r)$. Moreover, ϕ_ε is continuous outside the analytic set $\{\phi_\varepsilon = -\infty\}$. By Step 1, there exists a smooth function $v_\varepsilon \in PSH(V, \theta + \varepsilon \omega_V)$. Replacing ϕ_ε by $\max\{\phi_\varepsilon, v_\varepsilon - M\}$ for a large constant M, we obtain a continuous $(\theta + \varepsilon \omega_V)$-psh function ϕ_ε such that $\phi_\varepsilon \geq 0$ on $\partial B(p, r)$ and $\phi_\varepsilon < -2Ar^2$ in $B(p, \delta)$.

We finally use a gluing construction: the function $\psi_\varepsilon = \phi_\varepsilon$ in $V \setminus B(p, r)$ and

$$\psi_\varepsilon = \max \left\{ A(\|z\|^2 - r^2), \phi_\varepsilon \right\} \text{ in } B(p, r).$$

is continuous $(\theta + \varepsilon\omega_V)$-psh and coincides with $A(\|z\|^2 - r^2)$ in $B(p, \delta)$. Let $[X]$ denote the current of integration on X. Since ψ_ε is continuous, $(\theta + \varepsilon\omega_V + dd^c\psi_\varepsilon)^k \wedge [X]$ is a well-defined positive measure supported in X and

$$\int_X (\theta + \varepsilon\omega_V)^k = \int_V (\theta + \varepsilon\omega_V + dd^c\psi_\varepsilon)^k \wedge [X]$$

$$\geq \int_{B(p,\delta)} (\theta + \varepsilon\omega_V + A dd^c\|z\|^2)^k \wedge [X] \geq \int_{X \cap B(p,\delta)} \omega_V^k.$$

As the lower bound is independent of ε, the conclusion follows by letting $\varepsilon \to 0$. $\qquad \square$

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