Stochastic Approximation: Finite-time analyses and Variance Reduction

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Publications:

Stochastic Approximation Beyond Gradient for Signal Processing and Machine Learning

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A Stochastic Path Integrated Differential Estimator Expectation Maximization Algorithm

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Outline

1. Stochastic Approximation: the algorithm

   > Stochastic Approximation:
   > an iterative stochastic algorithm, for finding zeros of a vector field.

2. Examples of SA: stochastic gradient and beyond

   Stochastic Gradient is an example of SA, but SA encompasses broader scenarios

3. Non-asymptotic analysis

   best strategy after T iterations, complexity analysis

4. Variance reduction

5. Conclusion

Stochastic Approximation

Stochastic Approximation

Examples of SA: Stochastic Gradient and beyond

Non-asymptotic analysis

Variance Reduction within SA

Conclusion

Stochastic Approximation: a root-finding method

Robbins and Monro (1951) **Wolfowitz (1952), Kiefer and Wolfowitz (1952), Blum (1954), Dvoretzky (1956)**

Problem:

Given a vector field $h : \mathbb{R}^d \to \mathbb{R}^d$, solve

$$\omega \in \mathbb{R}^d \qquad \text{s.t.} \quad h(\omega) = 0$$

Available: for all ω, stochastic oracles of $h(\omega)$.

The *Stochastic Approximation* **method**:

Choose: a sequence of positive step sizes $\{\gamma_k\}_k$ and an initial value $\omega_0 \in \mathbb{R}^d$.
Repeat:

$$\omega_{k+1} = \omega_k + \gamma_{k+1} \; H(\omega_k, X_{k+1})$$

where $H(\omega_k, X_{k+1})$ is a stochastic oracle of $h(\omega_k)$.

Rmk: here, the field h is defined on $\mathbb{R}^d$; and for all $\omega \in \mathbb{R}^d$.

Example: $h(\omega)$ is an expectation; $H(\omega, X_{k+1})$ is a Monte Carlo approximation.

Examples of SA: Stochastic Gradient and beyond

Stochastic Approximation

Examples of SA: Stochastic Gradient and beyond

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Stochastic Gradient is a SA method (1/2)

Find a root of h: $\omega_{k+1} = \omega_k + \gamma_{k+1}\ H(\omega_k, X_{k+1})$ **where** $H(\omega_k, X_{k+1}) \approx h(\omega_k)$

SG is a root finding algorithm

- designed to solve $\nabla R(\omega) = 0$
- for convex and **non-convex** optimization.

SG is a SA algorithm

$$\omega_{k+1} = \omega_k - \gamma_{k+1}\ \widehat{\nabla R(\omega_k})$$

see e.g. survey by Bottou (2003, 2010); Lan (2020). Non-convex case: Bottou et al (2018); Ghadimi and Lan (2013)

Empirical Risk Minimization for batch data $R(\omega) = \frac{1}{n}\sum_{i=1}^{n} \ell(\omega, Z_i)$

Vector field: $h(\omega) = -\dfrac{1}{n}\displaystyle\sum_{i=1}^{n} \nabla_\omega \ell(\omega, Z_i)$

Oracle: $H(\omega, X_{k+1}) = -\dfrac{1}{\mathsf{b}}\displaystyle\sum_{i \in X_{k+1}} \nabla_\omega \ell(\omega, Z_i);$ X_{k+1} is a random mini-batch, cardinal b.

Unbiased oracles: $\mathbb{E}[H(\omega, X_{k+1})] = h(\omega)$

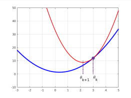
Stochastic Gradient is a SA method (2/2)

		oracles given by the system	oracles built by the user	Biased oracle	Unbiased oracle
SGD	batch		✓		✓
	online	✓		(✓)	✓

Batch learning: $\text{argmin}_\omega \frac{1}{n} \sum_{i=1}^n \ell(\omega, Z_i)$

Online learning: $\text{argmin}_\omega \mathbb{E}\left[\ell(\omega, Z)\right]$ from examples $Z_1, Z_2, \cdots$

Majorization-Minimization algorithms (Expectation-Maximization algorithms) with structured majorizing functions (1/3)

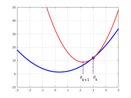


MM algorithms for the minimization of $F : \mathbb{R}^p \to \mathbb{R}$

$$F(\cdot) \leq \mathcal{Q}(\cdot, \tau), \qquad \forall \tau, \qquad\qquad F(\tau) = \mathcal{Q}(\tau, \tau)$$

Structured majorizing fcts: parametric family, $\mathcal{Q}(\cdot, \tau) = \langle \mathbb{E}_\tau \left[\mathsf{S}(X) \right], \phi(\cdot) \rangle$

Majorization-Minimization algorithms (Expectation-Maximization algorithms) with structured majorizing functions (1/3)



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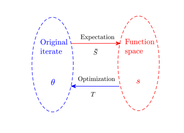
Structured majorizing fcts: parametric family, $\mathcal{Q}(\cdot, \tau) = \langle \mathbb{E}_\tau [\mathsf{S}(X)], \phi(\cdot) \rangle$

$$w_k \xrightarrow{\text{Minimize}} \mathsf{T}(w_k) := \mathrm{argmin}_\theta \langle w_k, \phi(\theta) \rangle$$
$$\xrightarrow{\text{Majorize}} w_{k+1} := \mathbb{E}_{\mathsf{T}(w_k)} [\mathsf{S}(X)]$$
$$\xrightarrow{\text{Minimize}} \mathsf{T}(w_{k+1}) := \mathrm{argmin}_\theta \langle w_{k+1}, \phi(\theta) \rangle$$
$$\ldots$$



A root-finding algorithms: $\mathbb{E}_{\mathsf{T}(\omega)} [\mathsf{S}(X)] - \omega = 0$

SA-MM The oracles are Monte Carlo approximations of the intractable expectations.

Majorization-Minimization algorithms (Expectation-Maximization algorithms) with structured majorizing functions (2/3)

EM algorithm for the maximization of $F : \mathbb{R}^p \to \mathbb{R}$

$$F(\omega) := \frac{1}{n} \sum_{i=1}^{n} \log \int p_{\mathsf{joint}}(Z_i, h; \omega)\, \nu(\mathrm{d}h)$$

Structured minorizing functions (curved exponential family)

$$\mathcal{Q}(\cdot, \tau) := \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}_\tau \left[\log p_{\mathsf{joint}}(Z_i, H; \cdot)\right] \qquad \text{w.r.t.} \quad p_{\mathsf{joint}}(\cdot; \tau)$$

$$\log p_{\mathsf{joint}}(Z_i, H; \cdot) = \left\langle \frac{1}{n} \sum_{i=1}^{n} \mathsf{S}_i(H), \phi(\cdot) \right\rangle$$

A root-finding algorithms: $\mathbb{E}_{\mathsf{T}(\omega)} \left[\frac{1}{n} \sum_{i=1}^{n} \mathsf{S}_i(H)\right] - \omega = 0$

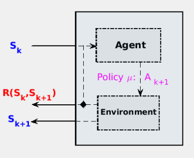
SA within EM The oracles are Monte Carlo approximations of the intractable expectations.

Expectation-Maximization, for curved exponential family	Dempster et al (1977)
- `SAEM`, SA with biased or unbiased oracles	Delyon et al (1999)
- `Mini-batch EM`, SA with unbiased oracles	adapted from Online EM - Cappé and Moulines (2009)

Majorization-Minimization algorithms (Expectation-Maximization algorithms) with structured majorizing functions (3/3)

		oracles given by the system	oracles built by the user	Biased oracles	Unbiased oracles
EM	batch		✓	✓	(✓)
	online	✓	(✓)	✓	(✓)

Value function in a Reward Markov process:



- Markov process $(s_t)_t$ with stationary distribution π
- taking values in $\mathcal{S}$, $\quad \mathrm{Card}(\mathcal{S}) = n$.
- Reward $\mathsf{R}(s, s')$
- Value function: $\qquad \lambda \in (0,1)$

$$\forall\ s \in \mathcal{S}, \qquad V_\star(s) := \sum_{t \geq 0} \lambda^t\, \mathbb{E}\left[\mathsf{R}(S_t, S_{t+1}) \middle| S_0 = s\right].$$

The Bellman equation $\qquad \mathsf{B}[V] - V = 0$

$$\mathbb{E}\left[\,\mathsf{R}(S_0, S_1) + \lambda V(S_1) \,|\, S_0 = s\,\right] - V(s) = 0, \qquad \forall s \in \mathcal{S}$$

with linear fct approximation: $V^\omega := \Phi\omega = \omega_1 \Phi_1(\cdot) + \cdots + \omega_d \Phi_d(\cdot)$

Algorithm TD(0):

TD(0) is a SA — Sutton (1987); Tsitsiklis and Van Roy (1997)

Oracle: $\quad H(\omega, (S_k, S_{k+1}, R(S_k, S_{k+1}))) := \left(\mathsf{R}(S_k, S_{k+1}) + \lambda V^\omega(S_{k+1}) - V^\omega(S_k)\right) (\Phi_{S_k,:})'$

Value function in a Reward Markov process via Bellman equation (2/2)

Which mean field h ? under stationarity $S_k \sim \pi$,

$$h(\omega) := \Phi' \operatorname{diag}(\pi) \left(\mathsf{B}[\Phi\omega] - \Phi\omega\right)$$

Which roots ? $\omega_\star$ s.t.

$$\langle \Phi\omega, \mathsf{B}[\Phi\omega_\star] - \Phi\omega_\star \rangle_{\operatorname{diag}(\pi)} = 0 \quad \Longleftrightarrow \quad \operatorname{Proj}\mathsf{B}[\Phi\omega_\star] = \Phi\omega_\star.$$

	oracles given by the system	oracles built by the user	Biased oracle	Unbiased oracle
TD(0)	✓		✓	(✓)

Conclusions

SA beyond the gradient case

Understanding the behavior of SA algorithms and designing improved algorithms require new insights that depart from the study of *traditional SG* algorithms.

What is the "gradient case" ?

- the mean field h is a gradient: $h(\omega) = -\nabla R(\omega)$
- the oracle is unbiased: $\mathbb{E}\left[H(\omega, X)\right] = h(\omega)$

From time homogeneous iterative algorithm to SA

$$\omega_{k+1} = \mathsf{M}(\omega_k) \qquad \Longrightarrow \qquad \text{the fixed points:} \quad \mathsf{M}(\omega) - \omega = 0.$$

When M is not explicit but stochastic oracles are available, run

$$\omega_{k+1} = \omega_k + \gamma_{k+1} \left(\mathcal{M}(\omega_k, X_{k+1}) - \omega_k\right)$$

Does it converge to the same limiting points ? $\cdots$ Lyapunov function !

Non-asymptotic analysis

Stochastic Approximation

Examples of SA: Stochastic Gradient and beyond

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Conclusion

Analyses

- Asymptotic convergence analysis, when the horizon tends to infinity

  Benveniste et al (1987/2012), Benaïm (1999), Kushner and Yin (2003), Borkar (2009)

  - almost-sure convergence of the sequence $\{\omega_k, k \geq 0\}$
  - to (a connected component of) the set $\mathcal{L} := \{\omega : \langle \nabla V(\omega), h(\omega) \rangle = 0\}$
  - CLT, $\cdots$

- Non-asymptotic analysis

Given a total number of iterations T

- After T calls to an oracle, what can be obtained ?

ϵ-approximate stationary point and sample complexity

- How many iterations to reach an ϵ-approximate stationary point

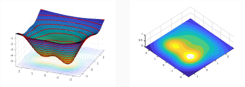
$$\forall \epsilon > 0, \quad \mathbb{E}\left[W(\omega_\bullet)\right] \leq \epsilon$$

Stochastic Approximation: root-finding method in a Lyapunov setting

SA: $\omega_{k+1} = \omega_k + \gamma_{k+1} H(\omega_k, X_{k+1})$ **with an oracle** $H(\omega_k, X_{k+1}) \approx h(\omega_k)$

A Lyapunov function. $V : \mathbb{R}^d \to \mathbb{R}_{>0}$, C^1 and inf-compact s.t.

$$\langle \nabla V(\omega), h(\omega) \rangle \leq 0$$



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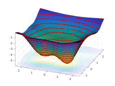
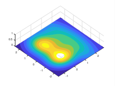

- **Key property**

> A Robbins-Siegmund type inequality — Robbins and Siegmund (1971)
>
> $$\mathbb{E}\left[V(\omega_{k+1}) | \mathrm{past}_k\right] \leq V(\omega_k) + \gamma_{k+1} \langle \nabla V(\omega_k), h(\omega_k) \rangle + \gamma_{k+1}\, \rho_k$$

ρ_k depends on the conditional L^2-moment (bias and variance) of the oracles.

- The Lyapunov fct is **not monotone** along the random path $\{\omega_k, k \geq 0\}$
- Key property for the (a.s.) boundedness of the random path, and its convergence.

- SA is an *optimization* method for the minimization of V

... but, converges to $\{\langle \nabla V(\cdot), h(\cdot) \rangle = 0\}$.

The assumptions

$\omega_{k+1} = \omega_k + \gamma_{k+1} H(\omega_k, X_{k+1})$

Lyapunov function V and control W

There exist $V : \mathbb{R}^d \to [0, +\infty)$, $W : \mathbb{R}^d \to [0, +\infty)$ and positive constants s.t.

- V and W: $\forall \omega \quad \langle \nabla V(\omega), h(\omega) \rangle \leq -\rho \, W(\omega)$
- V smooth $\forall \omega, \omega' \quad \| \nabla V(\omega) - \nabla V(\omega') \| \leq L_V \| \omega - \omega' \|$

		$h(\omega)$	$V(\omega)$	$W(\omega)$
Gradient case		$-\nabla R(\omega)$	$R(\omega)$	$\|h(\omega)\|^2$
and R convex	$\omega_\star$ solution	$-\nabla R(\omega)$	$0.5 \|\omega - \omega_\star\|^2$	$-\langle \omega - \omega_\star, h(\omega) \rangle$
and R strongly cvx	$\omega_\star$ solution	$-\nabla R(\omega)$	$0.5 \|\omega - \omega_\star\|^2$	$W = V$ or, as above
Stochastic EM		$\bar{s}(\mathsf{T}(\omega)) - \omega$	$F(\mathsf{T}(\omega))$	$\|h(\omega)\|^2$
TD(0)	$\Phi\omega_\star$ solution	$\Phi' D(\mathrm{B}\Phi\omega - \Phi\omega)$	$0.5 \|\omega - \omega_\star\|^2$	$(\omega - \omega_\star)' \Phi' D \Phi (\omega - \omega_\star)$

The assumptions

$\omega_{k+1} = \omega_k + \gamma_{k+1} H(\omega_k, X_{k+1})$

On the oracles and the mean field

There exist non-negative constants s.t.

- The mean field $\forall \omega \quad \|h(\omega)\|^2 \leq c_0 + c_1 W(\omega)$

for all k, almost-surely,

- Bias $\|\mathbb{E}\left[H(\omega_k, X_{k+1}) \middle| \mathcal{F}_k\right] - h(\omega_k)\|^2 \leq \tau_0 + \tau_1 W(\omega_k)$
- Variance $\mathbb{E}\left[\|H(\omega_k, X_{k+1}) - \mathbb{E}\left[H(\omega_k, X_{k+1}) \middle| \mathcal{F}_k\right]\|^2 \middle| \mathcal{F}_k\right] \leq \sigma_0^2 + \sigma_1^2 W(\omega_k)$

- If **biased** oracles i.e. $\tau_0 + \tau_1 > 0$,

$$\sqrt{c_V}\ (\sqrt{\tau_0}/2 + \sqrt{\tau_1}) < \rho, \qquad c_V := \sup_\omega \frac{\|\nabla V(\omega)\|^2}{W(\omega)} < \infty.$$

Includes cases:

- Biased oracles, unbiased oracles
- Bounded variance of the oracles, unbounded variance of the oracles

A non-asymptotic convergence bound in expectation

Theorem 1, Dieuleveut-F.-Moulines-Wai (2023)

Assume also that $\gamma_k \in (0, \gamma_{\max})$, $\eta_1 \geq \sigma_1^2 + c_1 > 0$

$$\gamma_{\max} := \frac{2(\rho - \mathtt{b}_1)}{L_V \, \eta_1}$$

Then, there exist non-negative constants s.t. for any $T \geq 1$

$$\sum_{k=1}^{T} \frac{\gamma_k \mu_k}{\sum_{\ell=1}^{T} \gamma_\ell \mu_\ell} \, \mathbb{E}\left[W(\omega_{k-1})\right] \leq 2 \, \frac{\mathbb{E}\left[V(\omega_0)\right]}{\sum_{\ell=1}^{T} \gamma_\ell \mu_\ell} + L_V \;\eta_0 \, \frac{\sum_{k=1}^{T} \gamma_k^2}{\sum_{\ell=1}^{T} \gamma_\ell \mu_\ell} + c_V \sqrt{\tau_0} \, \frac{\sum_{k=1}^{T} \gamma_k}{\sum_{\ell=1}^{T} \gamma_\ell \mu_\ell}$$

$\mu_\ell = 2(\rho - \mathtt{b}_1) - \gamma_\ell L_V \eta_1 > 0$

- η_ℓ depends on the bias and variance of the oracles; $\eta_0 > 0$.
- For unbiased oracles: $\tau_0 = \mathtt{b}_1 = 0$
- Better bounds when $V = W$; not discussed here — ex.: SGD for strongly cvx fct; TD(0)

Case `unbiased oracles`: After T iterations

The strategy

- Choose a constant stepsize $\gamma_k = \gamma := \frac{\gamma_{\max}}{2} \wedge \frac{\sqrt{2\mathbb{E}[V(\omega_0)]}}{\sqrt{\eta_0 L_V}\sqrt{T}}$
- Random stopping: return $\omega_{\mathcal{R}_T}$ where $\mathcal{R}_T \sim \mathcal{U}(\{0, \cdots, T-1\})$

  or when W is convex: return the averaged iterate $T^{-1}\sum_{k=0}^{T-1}\omega_k$

yields

$$\mathbb{E}\left[W(\omega_{\mathcal{R}_T})\right] \leq \frac{2\sqrt{2L_V\eta_0}\sqrt{\mathbb{E}\left[V(\omega_0)\right]}}{(\rho - b_1)\sqrt{T}} \vee \frac{8\mathbb{E}\left[V(\omega_0)\right]}{\gamma_{\max}(\rho - b_1)T}$$

- The left hand side comes from $T^{-1}\sum_{t=0}^{T-1}\mathbb{E}\left[W(\omega_t)\right] = \mathbb{E}\left[W(\omega_{\mathcal{R}_T})\right]$
- The right hand side: it is an *optimal* control in expectation.

Case `unbiased oracles`: ϵ-approximate stationary point

For all $\epsilon > 0$, let $\mathcal{T}(\epsilon) \subset \mathbb{N}$ s.t. for all $T \in \mathcal{T}(\epsilon)$, $\quad \mathbb{E}\left[W(\omega_{\mathcal{R}_T})\right] \leq \epsilon$.

For unbiased oracles,

$\mathcal{T}(\epsilon) = [T_\epsilon, +\infty)$ with

$$T_\epsilon := 8\,\mathbb{E}[V(\omega_0)]\,\frac{\eta_0 L_V}{\rho^2}\left(\frac{1}{\epsilon^2} \vee \frac{\eta_1}{2\eta_0 \epsilon}\right)$$

- Low precision regime: $\epsilon > 2\eta_0/\eta_1$,

$$T_\epsilon = 4\,\mathbb{E}[V(\omega_0)]\frac{\eta_1 L_V}{\rho^2\,\epsilon}, \qquad \gamma = \frac{\gamma_{\max}}{2}$$

- High precision regime: $\epsilon \in (0, 2\eta_0/\eta_1]$,

$$T_\epsilon = 8\,\mathbb{E}[V(\omega_0)]\frac{\eta_0 L_V}{\rho^2\,\epsilon^2}, \qquad \gamma = \frac{\rho\,\epsilon}{2\eta_0 L_V}$$

Case biased oracles

"Biased oracles" mean :

$$\|\mathbb{E}\left[H(\omega_k, X_{k+1})\middle|\mathcal{F}_k\right] - h(\omega_k)\|^2 \leq \tau_0 + \tau_1 W(\omega_k)$$

Specific assumptions: $$\rho > (\sqrt{\tau_0}/2 + \sqrt{\tau_1}) \left(\sup \|\nabla V\|/\sqrt{W}\right)$$

where

$$\exists \rho > 0, \qquad \forall \omega, \qquad \langle \nabla V(\omega), h(\omega)\rangle \leq -\rho\, W(\omega)$$

When $\tau_0 \neq 0$

- Difficult !
- The previous strategy "constant step size, uniform random stopping time" does not hold: the RHS can not be made small by any choice of γ.

Example. SAEM with self-normalized Importance Sampling (m draws per iterations):

- $\tau_0 = O(1/m)$
- ϵ-approximate stationary point with $m \leftarrow m_\epsilon$, $T \leftarrow T_\epsilon$, $\mathrm{cost}_\epsilon = O(T_\epsilon m_\epsilon)$

Case biased oracles: ϵ-approximate stationary point on an example

EM $\quad h(\omega) = \frac{1}{n}\sum_{i=1}^{n} \bar{\mathsf{S}}_i(\mathsf{T}(\omega)) - \omega$ where $\bar{\mathsf{S}}_i(\tau) := \int_{\mathcal{X}} S_i(x)\pi(x;\tau)\mathsf{d}x$

The SA-EM oracle

- Monte Carlo sum with m points,
- case "Self-normalized Importance Sampling": bias β_0/m and variance β_1/m.

Make the bias small by choosing $m = m(\epsilon)$.

Complexity

For all $\epsilon > 0$, let $\mathcal{T}(\epsilon) \subset \mathbb{N}^2$ s.t. for all $(T, m) \in \mathcal{T}(\epsilon)$, $\quad \mathbb{E}\left[W(\omega_{\mathcal{R}_T})\right] \leq \epsilon$.

$$T \geq \frac{16\mathbb{E}[V(\omega_0)](1+\bar{\sigma_1}^2/m)}{v_{\min}^2 \kappa \epsilon} \vee \frac{32\mathbb{E}[V(\omega_0)]\bar{\sigma}_0^2 L_V}{m v_{\min}^2 \kappa^2 \epsilon^2} \qquad m \geq \frac{4c_b}{(1-\kappa)v_{\min}\epsilon}$$

For low precision regime,

$$T_\epsilon = \frac{C_1}{\epsilon}, \qquad m_\epsilon = \frac{C_2}{\epsilon}, \qquad \mathrm{cost}_{\mathrm{comp}} = T_\epsilon \left(n m_\epsilon \,\mathrm{cost}_{\mathrm{MC}} + \mathrm{cost}_{\mathrm{opt}}\right)$$

Other rates for high precision regime.

Variance Reduction within SA

Stochastic Approximation

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Conclusion

Control variates for variance reduction

- Add a random variable to the *natural oracle* $H(\omega, X)$
- *Control variates* U, classical in Monte Carlo:

$$\mathbb{E}\left[H(\omega, X) + U\right] = \mathbb{E}\left[H(\omega, X)\right] \qquad \mathrm{Var}\left(H(\omega, X) + U\right) < \mathrm{Var}\left(H(\omega, X)\right).$$

Introduced in Stochastic Gradient, in the case *finite sum*

$$h(\omega) = \frac{1}{n}\sum_{i=1}^{n} h_i(\omega)$$

Extended to SA

Survey on Variance Reduction in ML: Gower et al (2020)
Gradient case: Johnson and Zhang (2013), Defazio et al (2014), Nguyen et al (2017), Fang et al (2018), Wang et al (2018), Shang et al (2020)
Riemannian non-convex optimization: Han and Gao (2022)
Mirror Descent: Luo et al (2022)
Stochastic EM: Chen et al (2018), Karimi et al (2019), Fort et al. (2020, 2021), Fort and Moulines (2021,2023)

Finite sum case: which "control variate" ?

"Finite sum" case. $h(\omega) = n^{-1} \sum_{i=1}^{n} h_i(\omega)$ and h_i globally Lipschitz.

Usual oracle for $h(\omega_k)$.

$$H(\omega_k, X_{k+1}) := \frac{1}{b} \sum_{i \in X_{k+1}} h_i(\omega_k) \qquad X_{k+1} \text{ mini batch of size } b \text{ in } \{1, \cdots, n\}$$

SVRG approach fix ℓ and for $k \geq \ell$: add the term

$$\frac{1}{n} \sum_{i=1}^{n} h_i(\omega_\ell) - \frac{1}{b} \sum_{i \in X_{k+1}} h_i(\omega_\ell)$$

SAGA approach Add the term

$$\frac{1}{n} \sum_{i=1}^{n} \tilde{h}_{i,k} - \frac{1}{b} \sum_{i \in X_{k+1}} \tilde{h}_{i,k}$$

and update the auxiliary quantity

$$\tilde{h}_{i,k+1} := h_i(\omega_k) \quad i \in X_{k+1}, \qquad \tilde{h}_{i,k+1} := \tilde{h}_{i,k} \quad \text{otherwise}.$$

The SPIDER control variate when h is a finite sum

Adapted from the gradient case: Stochastic Path-Integrated Differential EstimatoR

Nguyen et al (2017), Fang et al (2018), Wang et al (2019)

The SPIDER oracle is

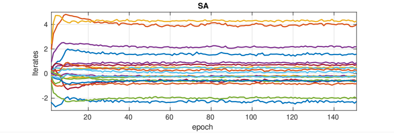
$$H^{\text{sp}}_{k+1} := \frac{1}{\mathsf{b}} \sum_{i \in X_{k+1}} h_i(\omega_k) + \underbrace{H^{\text{sp}}_k}_{\substack{\text{oracle} \\ \text{for } h(\omega_{k-1})}} - \underbrace{\frac{1}{\mathsf{b}} \sum_{i \in X_{k+1}} h_i(\omega_{k-1})}_{\substack{\text{oracle} \\ \text{for } h(\omega_{k-1})}}$$

- Implementation: Run $K_{\text{out}} K_{\text{in}}$ iterations and *refresh* the control variate every K_{in} iterations
- It holds: at outer loop $\# t$,

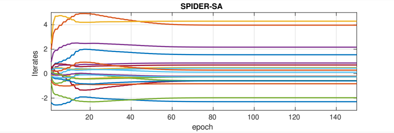
$$\left(1 - 2\gamma^2 L^2 \frac{K_{\text{in}}}{b}\right) \sum_{k=0}^{K_{\text{in}}} \mathbb{E}\left[\| H^{\text{sp}}_{t,k} - h(\omega_{t,k-1}) \|^2 | \text{past}_{t,0}\right] \leq 2\gamma^2 L^2 \frac{K_{\text{in}}}{b} \sum_{k=1}^{K_{\text{in}}} \mathbb{E}\left[\| h(\omega_{t,k}) \|^2 | \text{past}_{t,0}\right]$$

Efficiency ... via plots (here)

Application: Stochastic EM with ctt step size, mixture of twelve Gaussian in $\mathbb{R}^{20}$; unknown weights, means and covariances.

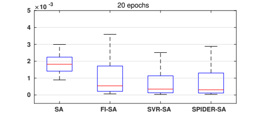


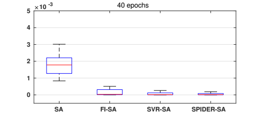
Estimation of 20 parameters, one path of SA



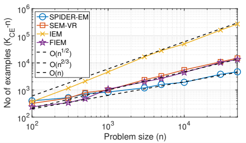
Estimation of 20 parameters, one path of SPIDER-SA

Squared norm of the mean field h, after 20 and 40 epochs; for SA and three variance reduction methods





Application: Stochastic EM with ctt step size, mixture of two Gaussian in $\mathbb{R}$, unknown means.



For a fixed accuracy level, for different values of the problem size n, display the number of examples processed to reach the accuracy level (mean nbr over 50 indep runs).

Conclusion

Stochastic Approximation

Examples of SA: Stochastic Gradient and beyond

Non-asymptotic analysis

Variance Reduction within SA

Conclusion

Conclusion

- SA methods with non-gradient mean field and/or biased oracles - in ML and compurational statistics.
- A non-asymptotic analysis for *general Stochastic Approximation schemes*
- For *finite sum field* h*:* variance reduction within SA via control variates.
- Oracles, from *Markovian* examples
- Roots of $h = 0$, on $\Omega \subset \mathbb{R}^d$
- Federated SA: compression, control variateS, partial participation, heterogeneity, local iterations, ...