Credibility intervals for Covid19 reproduction number from Nonsmooth Langevin-type Monte Carlo sampling

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Talk based on the papers:

- *Covid19 Reproduction Number: Credibility Intervals by Blockwise Proximal Monte Carlo samplers*
  by G. Fort, B. Pascal, P. Abry and N. Pustelnik
  HAL 03611079, submitted
- *Temporal evolution of the Covid19 pandemic reproduction number: Estimations from Proximal optimization to Monte Carlo sampling*
  by P. Abry, G. Fort, B. Pascal and N. Pustelnik
  EMBC 2022 (HAL 03565440)
- *Credibility intervals design for Covid19 reproduction number from nonsmooth Langevin-type Monte Carlo sampling*
  by H. Artigas, B. Pascal, G. Fort, P. Abry and N. Pustelnik
  EUSIPCO 2022 (HAL 03371837)
- *Estimation et intervalles de crédibilité pour le taux de reproduction de la Covid19 par échantillonnage Monte Carlo Langevin proximal*
  by P. Abry, G. Fort, B. Pascal and N. Pustelnik
  GRETSI 2022 (HAL 03611891)

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I. Reproduction number of the Covid19
The model

Credibility intervals for the Reproduction number R, why ?

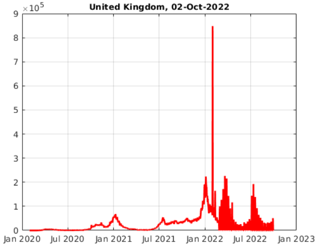
- Monitoring the Covid19 pandemic constitutes a critical societal stake: Covid19 pandemic caused/is causing unprecedented health, social, and economic crises.
- Need to assess the intensity of the / a pandemic, prerequisite for efficient sanitary policies.

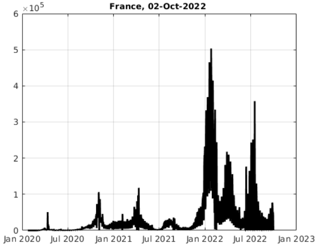
- The reproduction number measures
- the strength of the pandemic by quantifying rate of growth of daily new infections
- the number of second infections caused by one primary infection.

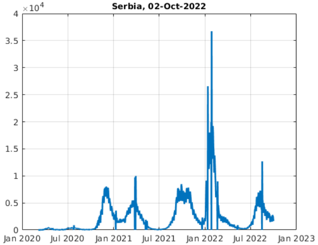
- Estimation of the daily R_t
- by a value of the index
- by credibility intervals: valuable information for the decision makers, notably in periods of rapid evolution or of changes in trends.

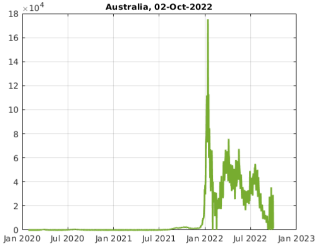
The data: daily new infections

- Real data, from Johns-Hopkins University repository
- Examples for UK, France, Serbia and Australia



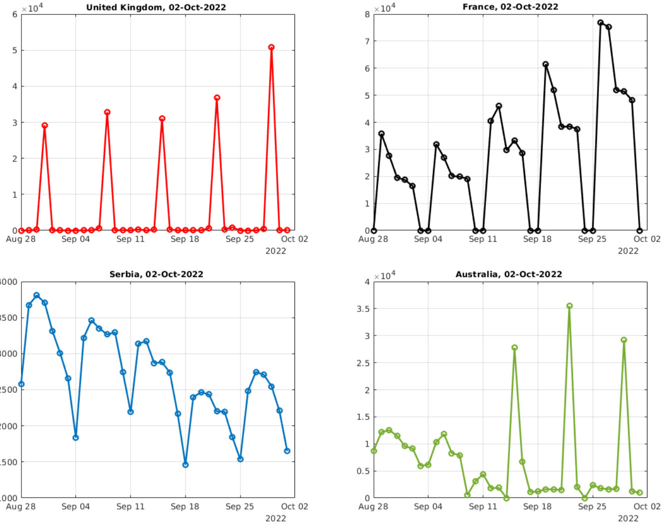






The data: daily new infections - zoom on the last 35 days

- Examples for UK, France, Serbia and Australia



The statistical model (1/4)

From Cori et al Cori et al (2013)

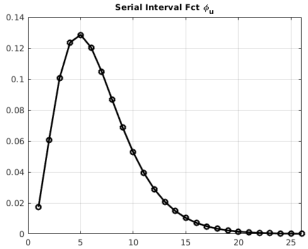
- The data $Z_1, \cdots, Z_T$: non negative integers
- Parameter: $(\mathsf{R}_1, \cdots, \mathsf{R}_T) \in (\mathbb{R}_+)^T$
- Conditionally to the past

$$Z_t \Big| Z_{1:(t-1)} \sim \mathcal{P}\left(\mathsf{R}_t \ \Phi_t^{\mathsf{Z}}\right) \qquad \text{where } \Phi_t^{\mathsf{Z}} := \sum_{u=1}^{\tau_\phi} \phi_u Z_{t-u}$$

- $\tau_\phi = 26$ days
- $\phi_u := \mathrm{PDF}_{\mathrm{Gamma}}(u)$

shape $= 1/0.28$, scale $= 1.87$

mean 6.68 days
std 3.53 days
mode 4.8 days



The statistical model (2/4)

From Abry et al Abry et al (2020)

- A priori distribution on the R_t's

- regularization needed as many parameters as observations
- piecewise linear time evolutions of $t \mapsto \mathsf{R}_t$

- L^1 penalization of the discrete time second derivative of $t \mapsto \mathsf{R}_t$

$$\ln \text{prior}: = -\lambda_{\mathsf{R}} \; \|\mathsf{D}_2 \, \mathsf{R}_{1:T}\|_1 \qquad \text{up to an additive constant}$$

where

$$\mathsf{D}_2 := \frac{1}{\sqrt{6}} \begin{bmatrix} 1 & -2 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & -2 & 1 & 0 & \cdots & 0 \\ \cdots & & & & & & \\ 0 & \cdots & & 0 & 1 & -2 & 1 \end{bmatrix} \in \mathbb{R}^{(T-2)\times T} \qquad \mathsf{R}_{1:T} := \begin{bmatrix} \mathsf{R}_1 \\ \mathsf{R}_2 \\ \cdots \\ \mathsf{R}_T \end{bmatrix}$$

- $\lambda_R := 3.5 \, \frac{\sqrt{6}}{4} \, \mathrm{std}(Z_1, \cdots, Z_T)$

The statistical model (3/4)

From Pascal et al Pascal et al (2021)

- The model from Cori et al + the regularized R_t's

- Model the errors on the counts via $\mathsf{O}_1, \cdots, \mathsf{O}_T$ in $\mathbb{R}^T$

- corrupted data, with pseudo-seasonalities, under-evaluations / over-evaluations ($\rightarrow$ negative counts)
- a priori distribution **and** modification of the likelihood

$$Z_t \Big| Z_{1:(t-1)} \sim \mathcal{P}\left(\mathsf{R}_t \ \Phi_t^{\mathsf{Z}} + \mathsf{O}_t\right) \qquad \text{where } \Phi_t^{\mathsf{Z}} := \sum_{u=1}^{\tau_\phi} \phi_u Z_{t-u}$$

$$\ln \text{prior:} = -\lambda_{\mathsf{R}} \ \|\mathsf{D}_2 \, \mathsf{R}_{1:T}\|_1 - \lambda_{\mathsf{O}} \ \|\mathsf{O}_{1:T}\|_1 \qquad \text{up to an additive constant}$$

- A constraint set

$$\mathcal{D} := \bigcap_t \{\mathsf{R}_{1:T}, \mathsf{O}_{1:T} \text{ s.t. positive intensity if } Z_t > 0 \text{ and non negative intensity if } Z_t \geq 0\}$$

- $\lambda_{\mathsf{O}} := 0.05$

The statistical model (4/4): hereafter

- A posteriori distribution of

$$\theta := (\mathsf{R}_{1:T}, \mathsf{O}_{1:T}) \in (\mathbb{R}_+)^T \times \mathbb{R}^T$$

given $Z_1, \cdots, Z_T$

- with log-density

$$\theta \mapsto \begin{cases} -f(\theta) - g(\mathsf{A}\,\theta) & \text{on } \mathcal{D} \\ -\infty & \text{otherwise} \end{cases} \qquad \mathsf{A} := \begin{bmatrix} \mathsf{D}_2 & 0_{(T-2)\times T} \\ 0_{T\times T} & \frac{\lambda_{\mathsf{O}}}{\lambda_{\mathsf{R}}} \mathrm{I}_T \end{bmatrix} \in \mathbb{R}^{(2T-2)\times(2T)}$$

where

$$f(\theta) := \sum_{t=1}^{T} \{(\mathsf{R}_t \Phi_t^{\mathsf{Z}} + \mathsf{O}_t) - Z_t \ln(\mathsf{R}_t \Phi_t^{\mathsf{Z}} + \mathsf{O}_t)\} \qquad g(\theta) := \lambda_{\mathsf{R}} \, \|\theta\|_1$$

II. Estimation of the R_t's via MCMC

Bayesian estimation

- Quantiles and other statistics Maximum a Posteriori, mean a posteriori, etc

- for each component R_t and O_t
- based on the marginal distributions of the distribution of $\theta := (R_{1:T}, O_{1:T})$

- Joint distribution of the form

$$-\ln \text{posterior:} \quad \theta \mapsto \begin{cases} f(\theta) + g(A\,\theta) & \text{on } \mathcal{D} \\ +\infty & \text{otherwise} \end{cases} \qquad A := \begin{bmatrix} D_2 & 0_{(T-2)\times T} \\ 0_{T\times T} & \frac{\lambda_O}{\lambda_R} I_T \end{bmatrix}$$

$$f(\theta) := \sum_{t=1}^{T} \{(R_t \Phi_t^Z + O_t) - Z_t \ln(R_t \Phi_t^Z + O_t)\} \qquad g(\theta) := \lambda_R \, \|\theta\|_1$$

Bayesian estimation via MCMC

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Markov Chain Monte Carlo samplers

Bayesian estimation via MCMC

- Quantiles and other statistics Maximum a Posteriori, mean a posteriori, etc

- for each component R_t and O_t
- based on the marginal distributions of the distribution of $\theta := (\mathsf{R}_{1:T}, \mathsf{O}_{1:T})$

- Joint distribution of the form

$$-\ln \text{posterior:} \quad \theta \mapsto \begin{cases} f(\theta) + g(\mathsf{A}\,\theta) & \text{on } \mathcal{D} \\ +\infty & \text{otherwise} \end{cases} \qquad \mathsf{A} := \begin{bmatrix} \mathsf{D}_2 & 0_{(T-2)\times T} \\ 0_{T\times T} & \frac{\lambda_{\mathsf{O}}}{\lambda_{\mathsf{R}}} \mathrm{I}_T \end{bmatrix}$$

- f is C^1, convex
- $g := \lambda_R \|\cdot\|_1$ is lower semi-continuous, proper, convex
  - has a unique proximity operator, which is explicit
  - the proximity operator of $g(\mathsf{A}\cdot)$ is not explicit
- A is a $(2T-2) \times 2T$ matrix
  - full row rank
  - $g(\mathsf{A}\theta) = \sum_{i=1}^{3} g_i(\mathsf{A}_i\theta)$ and $g_i(\mathsf{A}_i\cdot)$ has an explicit proximity operator.

The Maximum a Posteriori estimator of the (R_t, O_t)'s (1/2)

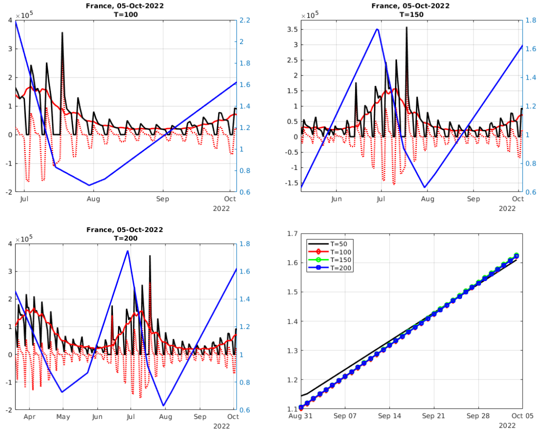
- Does it exist ? Unique ? Pascal et al (2022), Fort et al (2022)
  - If $\Phi_t^Z > 0$ and $\Phi_{t'}^Z$ for $t < t' \leq T$, and $Z_{t''} > 0$ then a MAP exists.
  - It two MAP, then: same Poisson intensity and same sign $O_t O'_t \geq 0; (D_2 R_t)\,(D_2 R'_t) \geq 0$.

- Computation: a Chambolle-Pock iterative algorithm proposed by Pascal et al (2022) see also Abry et al (2020)

- MAP for France, Serbia and Australia over the last 100 days:
(left y-axis) the data Z_t, (dots) $\hat{O}_t$ by MAP and (line) $Z_t - \hat{O}_t$
(right y-axis) $\hat{R}_t$ by MAP.

The Maximum a Posteriori estimator of the (R_t, O_t)'s (2/2)

- Role of T for the MAP estimate of the last R_t's:
  - [top,bottom left] The MAP estimate is computed from $T = 100$, $T = 150$ and $T = 200$ observations.
  - [bottom right] The three estimates for the last 35 days are displayed



Which MCMC sampler ?

$$-\ln \pi(\mathsf{R}_{1:T}, \mathsf{O}_{1:T}) = \sum_{t=1}^{T} \{(\mathsf{R}_t \Phi_t^{\mathsf{Z}} + \mathsf{O}_t) - Z_t \ln(\mathsf{R}_t \Phi_t^{\mathsf{Z}} + \mathsf{O}_t)\} + \lambda_{\mathsf{R}} \|\mathsf{A}\theta\|_1 \qquad \text{on } \mathcal{D}$$

$$\mathsf{A} := \begin{bmatrix} \mathrm{D}_2 & 0_{(T-2)\times T} \\ 0_{T\times T} & \frac{\lambda_{\mathsf{O}}}{\lambda_{\mathsf{R}}} \mathrm{I}_T \end{bmatrix}$$

- Hastings-Metropolis family
  - a Markov kernel for the update of $\theta := (\mathsf{R}_{1:T}, \mathsf{O}_{1:T})$
  - with a proposal mechanism using *first order* information on $\ln \pi$
- Gibbs family
  - update in turn $\mathsf{R}_{1:T}$ given $\mathsf{O}_{1:T}$, and then $\mathsf{O}_{1:T}$ given $\mathsf{R}_{1:T}$
  - the conditional distributions are not explicit → Metropolis-within-Gibbs methods

In both cases, faced with the design of Hastings-Metropolis sampler when the target distribution is of the form

$$-\ln \mathsf{target}(\tau) = f(\tau) + g(\mathsf{C}\tau) \qquad \tau \in \mathcal{D}$$

with C full row rank $\qquad f$ is $C^1 \qquad \mathrm{prox}_{\gamma\, g}$ exists and is unique

Which sampling space ?

$\pi(\tau) \propto \exp(-f(\tau) - g(\mathsf{C}\tau))\, 1_{\mathcal{D}}(\tau)$ where $\mathsf{C} \propto \mathrm{I}$ or C is a full row rank $(d-2) \times d$ matrix

- First strategy: Sample in the original space

- Obtain a Markov chain $\{\tau_n, n \geq 0\}$ with target π.

- Second strategy: Move to an *image space* and go back

$$\tilde{\tau} := \bar{\mathsf{C}}\,\tau \qquad \bar{\mathsf{C}} := \begin{bmatrix} \cdots \\ \cdots \\ \mathsf{C} \end{bmatrix} \text{invertible matrix,}$$

Image of π by $\bar{C}$:

$$\tilde{\pi}(\tilde{\tau}) \propto \exp(-f(\bar{\mathsf{C}}^{-1}\tilde{\tau}) - g(\tilde{\tau}_{3:d}))\, 1_{\mathcal{D}}(\bar{\mathsf{C}}^{-1}\tilde{\tau})$$

- Sample a Markov chain $\{\tilde{\tau}_n, n \geq 0\}$ with target distribution $\tilde{\pi}$

- Go back: $\tau_n := \bar{\mathsf{C}}^{-1}\,\tilde{\tau}_n$ is a Markov chain with target distribution π.

III. Proximal-Langevin based proposal distributions

$$-\ln \mathsf{target}(\tau) = f(\tau) + g(\mathsf{C}\tau) \qquad \tau \in \mathcal{D} \subseteq \mathbb{R}^d$$

with

- f is C^1
- $\mathrm{prox}_{\gamma g}$ exists, is unique and is explicit
- C is an **invertible** $d \times d$ matrix

Case $\ln \pi$ smooth: Langevin dynamic

- **Langevin proposal** Roberts and Tweedie (1996); scaling in Roberts and Rosenthal (2002)

$$\tau_{n+\frac{1}{2}} = \tau_n + \gamma_{n+1} \nabla \ln \pi(\tau_n) + \sqrt{2\gamma_{n+1}}\, \mathcal{N}(0, \mathrm{I})$$

- Tempered Langevin proposal Kent (1978), Roberts and Stramer (2002)

$$\tau_{n+\frac{1}{2}} = \tau_n + \gamma_{n+1}\, \Gamma\, \nabla \ln \pi(\tau_n) + \sqrt{2\gamma_{n+1}}\, \sqrt{\Gamma}\, \mathcal{N}(0, \mathrm{I})$$

Case $\ln \pi$ composite: a proximal-gradient approach $\rightarrow$ `PGdual`

Via the "move and go back" approach, we propose a proximal-gradient step

- From the result in convex optimization:

$$\tau_\star = \operatorname{argmin}(f(\mathsf{C}^{-1}\cdot) + g) \iff 0 \in \mathsf{C}^{-\top}\nabla f(\mathsf{C}^{-1}\tau_\star) + \partial g(\tau_\star)$$
$$\iff \operatorname{prox}_{\gamma g}(\tau_\star - \gamma\,\mathsf{C}^{-\top}\nabla f(\mathsf{C}^{-1}\tau_\star)) = \tau_\star \qquad \forall \gamma > 0.$$

- The proposal for sampling $\tilde{\pi}$

$$\tilde{\tau}_{n+\frac{1}{2}} = \operatorname{prox}_{\gamma g}(\tilde{\tau}_n - \gamma\,\mathsf{C}^{-\top}\nabla f(\mathsf{C}^{-1}\tilde{\tau}_n)) + \sqrt{2\gamma}\,\mathcal{N}(0, \mathrm{I})$$

- The proposal for sampling π: $\tau_{n+\frac{1}{2}} := \mathsf{C}^{-1}\tilde{\tau}_{n+\frac{1}{2}}$

Interpretation:

Tempered Langevin + variable metric Proximal = Variable Metric Proximal-Gradient

$$\tau_{n+\frac{1}{2}} = \operatorname{prox}^{\mathsf{C}^\top\mathsf{C}}_{\gamma g(\mathsf{C}\cdot)}(\tau_n - \gamma\,\mathsf{C}^{-1}\mathsf{C}^{-\top}\nabla f(\tau_n)) + \sqrt{2\gamma}\,\mathsf{C}^{-1}\mathcal{N}(0, \mathrm{I})$$

Case $\ln \pi$ composite: a Proximal-Gradient approach → PGdec

We propose a Proximal-Gradient (PG) step, when

- block-splitting $g(\mathsf{C}\tau) = \sum_{i=1}^{I} g_i(\mathsf{C}_i\tau)$
- explicit proximity operator $\mathrm{prox}_{\gamma\, g_i(\mathsf{C}_i \cdot)}$

- The proposal for sampling π:

(a) Sample uniformly $i \in \{1, \cdots, I\}$

(b) Update via a PG step which uses the component $\#i$ of the non-smooth fct

Langevin + Proximal = Proximal-Gradient

$$\tau_{n+\frac{1}{2}} = \mathrm{prox}_{\gamma\, g_i(\mathsf{C}_i \cdot)} \left(\tau_n - \gamma \nabla f(\tau_n)\right) + \sqrt{2\gamma}\, \mathcal{N}(0, \mathrm{I})$$

Case $\ln \pi$ composite: a Moreau envelope approach $\rightarrow$ MYULA Durmus et al (2018)

Via the "move and go back" approach, combined with MYULA

- From non-smooth convex optimization, the Moreau envelope

$$g_\rho(\tilde{\tau}) := \min_{\mathbb{R}^d} \left(\rho g(\cdot) + \frac{1}{2} \| \cdot - \tilde{\tau} \|^2 \right) \qquad \nabla g_\rho(\tilde{\tau}) = \frac{1}{\rho} \left(\tilde{\tau} - \mathrm{prox}_{\rho\, g}(\tilde{\tau}) \right)$$

- The proposal for sampling $\tilde{\pi}$

$$\tilde{\tau}_{n+\frac{1}{2}} = \tilde{\tau}_n - \gamma\, \mathsf{C}^{-\top} \nabla f(\mathsf{C}^{-1}\, \tilde{\tau}_n) - \frac{\gamma}{\rho} \left(\tilde{\tau}_n - \mathrm{prox}_{\rho\, g}(\tilde{\tau}_n) \right) + \sqrt{2\gamma}\, \mathcal{N}(0, \mathrm{I})$$

- The proposal for sampling π: $\quad \tau_{n+\frac{1}{2}} := \mathsf{C}^{-1} \tilde{\tau}_{n+\frac{1}{2}}$

Interpretation:

Tempered Langevin + variable metric Moreau envelope see e.g. Hiriart-Urruty and Lemaréchal (1996, Chapter XV)

$$\tau_{n+\frac{1}{2}} = \left(1 - \frac{\gamma}{\rho}\right) \tau_n - \gamma\, \mathsf{C}^{-1} \mathsf{C}^{-\top} \nabla f(\tau_n)) + \frac{\gamma}{\rho} \mathrm{prox}^{\mathsf{C}^\top \mathsf{C}}_{\rho\, g(\mathsf{C} \cdot)}(\tau_n) + \sqrt{2\gamma}\, \mathsf{C}^{-1}\, \mathcal{N}(0, \mathrm{I})$$

Case $\ln \pi$ composite: a Moreau envelope $\longrightarrow$ `semi-FBLMC` Luu et al (2021)

- Since $\mathrm{prox}_{\gamma g(\mathsf{C}\cdot)}$ not explicit:

- Do not use the Moreau envelope of $\tau \mapsto g(\mathsf{C}\tau)$

- Use the Moreau envelope of g, evaluated at $\mathsf{C}\tau$

$$\nabla[g_\rho(\mathsf{C}\cdot)](\tau) = \mathsf{C}^\top \nabla[g_\rho](\mathsf{C}\tau) = \frac{1}{\rho}\mathsf{C}^\top \left(\mathsf{C}\tau - \mathrm{prox}_{\rho g}(\mathsf{C}\tau)\right)$$

- The proposal for sampling π:

$$\tau_{n+\frac{1}{2}} = \left(\mathrm{I} - \frac{\gamma}{\rho}\mathsf{C}^\top\mathsf{C}\right)\tau_n - \gamma\,\nabla f(\tau_n) + \frac{\gamma}{\rho}\mathsf{C}^\top \mathrm{prox}_{\rho g}(\mathsf{C}\tau) + \sqrt{2\gamma}\,\mathcal{N}(0,\mathrm{I})$$

To summarize

- $-\ln \pi = f$ i.e. $\ln \pi$ is smooth

Algorithm	–	–	Drift	Cov	hyperparam
Langevin			$\tau - \gamma \nabla f(\tau)$	$2\gamma \mathrm{I}$	γ
Tempered Langevin			$\tau - \gamma\Gamma \, \nabla f(\tau)$	$2\gamma\Gamma$	γ, Γ

- $-\ln \pi = f + g(\mathsf{C}\cdot)$ i.e. $\ln \pi$ is non-smooth $\qquad g(\mathsf{C}\cdot) = \sum_{i=1}^{I} g_i(\mathsf{C}_i\cdot)$

Set $\quad \Gamma := \mathsf{C}^{-1}\mathsf{C}^{-\top}$

Algorithm	original space	move and go back	Drift	Cov	hyperparam
`PGdec`	✓		$\mathrm{prox}_{\gamma\, g_i(\mathsf{C}_i\cdot)}\left(\tau - \gamma \nabla f(\tau)\right)$	$2\gamma \mathrm{I}$	γ
`PGdual`		✓	$\mathrm{prox}^{\Gamma^{-1}}_{\gamma\, g(\mathsf{C}\cdot)}\left(\tau - \gamma\Gamma \, \nabla f(\tau)\right)$	$2\gamma\Gamma$	γ
`semi-FBLMC`	✓		$\tau - \gamma \nabla f(\tau) - \frac{\gamma}{\rho}\mathsf{C}^\top\left(\mathsf{C}\tau - \mathrm{prox}_{\rho\, g}(\mathsf{C}\tau)\right)$	$2\gamma \mathrm{I}$	γ, ρ
`MYULA`		✓	$\tau - \gamma\Gamma \, \nabla f(\tau) - \frac{\gamma}{\rho}\left(\tau - \mathrm{prox}^{\Gamma^{-1}}_{\rho\, g(\mathsf{C}\cdot)}(\tau)\right)$	$2\gamma\Gamma$	γ, ρ

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`semi-FBLMC`	✓		$\tau - \gamma \nabla f(\tau)$ $- \frac{\gamma}{\rho} \mathsf{C}^\top \left(\mathsf{C}\tau - \mathrm{prox}_{\rho\, g}(\mathsf{C}\tau)\right)$	$2\gamma \mathrm{I}$	γ, ρ
`MYULA`		✓	$\tau - \gamma \Gamma \nabla f(\tau)$ $- \frac{\gamma}{\rho} \left(\tau - \mathrm{prox}^{\Gamma^{-1}}_{\rho\, g(\mathsf{C}\cdot)}(\tau)\right)$	$2\gamma \Gamma$	γ, ρ

To summarize

- $-\ln \pi = f$ i.e. $\ln \pi$ is smooth

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Algorithm	original space	move and go back	Drift	Cov	hyperparam
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`PGdual`		✓	$\operatorname{prox}^{\Gamma^{-1}}_{\gamma\, g(\mathsf{C}\cdot)}(\tau - \gamma \Gamma \nabla f(\tau))$	$2\gamma\Gamma$	γ
`semi-FBLMC`	✓		$\tau - \gamma \nabla f(\tau) - \frac{\gamma}{\rho} \mathsf{C}^\top \left(\mathsf{C}\tau - \operatorname{prox}_{\rho\, g}(\mathsf{C}\tau)\right)$	$2\gamma \mathrm{I}$	γ, ρ
`MYULA`		✓	$\tau - \gamma \Gamma \nabla f(\tau) - \frac{\gamma}{\rho}\left(\tau - \operatorname{prox}^{\Gamma^{-1}}_{\rho\, g(\mathsf{C}\cdot)}(\tau)\right)$	$2\gamma\Gamma$	γ, ρ

Propose and then: with or without an accept-reject step ?

- Without an AR step → `Langevin MC` Parisi (1981); `ULA` Roberts and Tweedie (1996); `MYULA` Durmus et al (2018)

$$\tau_{n+1} = \tau_{n+\frac{1}{2}}$$

- With an AR step → `MALA` Roberts and Tweedie (1996); Bou-Rabee and Hairer (2012); Eberle (2014); Dwivedi et al (2019); `MYMALA` Durmus et al (2018)

$$\tau_{n+1} = \tau_{n+\frac{1}{2}} \quad \text{or} \quad \tau_{n+1} = \tau_n$$

- In our case: **with an AR step**
  - the support of π is $\mathcal{D}$. This "constraint" is managed by the AR step.
  - Better mixing time in TV, as a function of d and the tolerance ϵ for `MALA` vs `ULA` Dwivedi et al (2019)

Comparison on a toy example (1/2)

- Logistic regression:
  - $\mathsf{Y} \in \mathbb{R}^T$ collects the T $\{0,1\}$-valued response variables.
  - X is the $T \times d$ covariate matrix.
  - $\theta \in \mathbb{R}^d$: unknown regression vector.
  - a priori distribution: piecewise constant

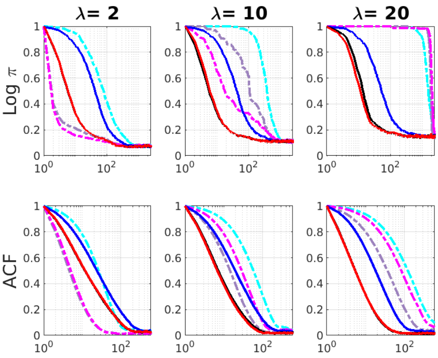
- A posteriori distribution, on $\mathbb{R}^d$

$$-\ln \pi(\theta) := -\mathsf{Y}^\top \mathsf{X}\theta + \sum_{t=1}^{T} \ln\left(1 + \exp((\mathsf{X}\theta)_t)\right) + \lambda \|\mathsf{D}_1\theta\|_1$$ up to an additive constant

- $T = 2\,000$; $d = 20$;

X: independent Rademacher and norm of the rows equal to one;
$\theta_\star$: piecewise constant
Y: independent Bernoulli with success probability $(1 + \exp(-(\mathsf{X}\theta_\star)_t))^{-1}$

Comparison on a toy example (2/2)



RW semi-FBLMC PGdec
RW "move/go back" MYULA PGdual

[top] Evolution $n \mapsto \frac{\ln \pi(\theta_n) - \max \ln \pi}{\ln \pi(\theta_1) - \max \ln \pi}$ along the first $2\,500$ samples.
[bottom] the ACF function, computed from $17\,500$ samples.

Show

- the benefit of first order informations on $\ln \pi$, mainly for small and medium λ

- PGdec < semi-FBLMC in the stationary phase

- PGdec: partial informations OK, when λ small

- PGdual and MYULA: robust to λ; and almost equivalent

- PGdual and MYULA: the best when λ medium to large.

Convergence analysis of PGdec and PGdual

Law of large numbers Proposition 13, Fort et al (2022)

Assume that π *is continuous on* $\mathcal{D}$; f *is continuously differentiable on* $\mathcal{D}$ *and* g *is convex, proper.*

PGdec. *Assume that* $\mathrm{prox}_{\gamma\, g_i(\mathsf{C}_i\cdot)}$ *exists and has a closed form expression.*
The sequence $\{\tau_n, n \geq 0\}$ *given by* `PGDec` *is a Markov chain, taking values on* $\mathcal{D}$ *and with unique invariant distribution* π. *In addition, for all initial point* $\tau_0 \in \mathcal{D}$ *and any measurable function* h *such that* $\int |h(\tau)|\, \pi(\tau) \mathrm{d}\tau < \infty$

$$\lim_{N \to +\infty} \frac{1}{N} \sum_{n=1}^{N} h(\tau_n) = \int h(\tau)\, \pi(\tau) \mathrm{d}\tau \qquad \textit{a.s.}$$

PGdual. *Assume that* $\mathrm{prox}_{\gamma\, g}$ *exists and has a closed form expression.*
The same ergodicty result holds for the sequence $\{\tau_n, n \geq 0\}$ *given by* `PGdual`.

Proof: under the stated assumptions, the chain is irreducible, aperiodic and π is its unique invariant distribution. In addition, the chain is positive Harris-recurrent.

Case log-π composite: other MCMC samplers in the literature

- `Perturbed Langevin MC`: bases on Gaussian smoothing of g (Chatterji et al (2020))

$$g_\mu(\cdot) := \mathbb{E}\left[g(\cdot + \mu \mathcal{N}(0, \mathrm{I}))\right]$$

When g is Hölder-continuous.

- AXDA (Vono et al (2019))
  - From the result

$$\min_{\tau} \left(f(\tau) + g(\mathsf{C}\tau)\right) \Longleftrightarrow \min_{\tau=\tau'} \left(f(\tau) + g(\mathsf{C}\tau')\right)$$

  - Data Augmentation scheme + Metropolis-within-Gibbs

$$-\ln \pi_{\mathrm{da}}(\tau, \tau') \propto f(\tau) + g(\mathsf{C}\tau') + \phi(\tau, \tau')$$

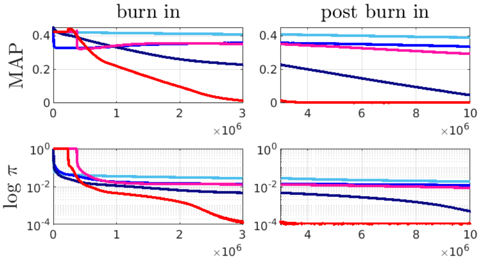
IV. Credibility intervals for the Covid19 reproduction number

Design parameters

$$-\ln \pi(\theta) = f(\theta) + \lambda_{\mathsf{R}} \|\mathsf{A}\theta\|_1 \qquad \mathsf{A} := \begin{bmatrix} \mathsf{D}_2 & 0_{(T-2)\times T} \\ 0_{T\times T} & \frac{\lambda_{\mathsf{O}}}{\lambda_{\mathsf{R}}} \mathrm{I}_T \end{bmatrix} \in \mathbb{R}^{(2T-2)\times(2T)}$$

- Augmentation of D_2 into a $T \times T$ invertible matrix: we compare two strategies "invert" and "ortho"
- Step size γ_{n+1} adapted during the first iterations (burn-in) and then fixed.
- Estimation of the quantiles: empirical quantiles, from the output of the Markov chain after a burn-in period.
- For the Moreau envelope: $\rho = \gamma$.

From D_2 to $\bar{D}_2$ (1/2)



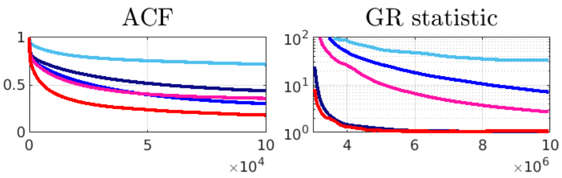
RW RW Invert RW Ortho
PGdual Invert PGdual Ortho.

During the burn in period [left] and after [right],
(top) evolution of the distance from θ_n to the MAP along iterations
(bottom) evolution of the distance from $\ln \pi(\theta_n)$ to $\max \ln \pi$ along iterations

Show

- Among the RW's, more efficient when the Gaussian noise is not i.i.d.
- $\bar{D}_2^{(o)}$ more efficient for RW's and PGdual
- PGdual more efficient than the RW's

From D_2 to $\bar{D}_2$ (2/2)



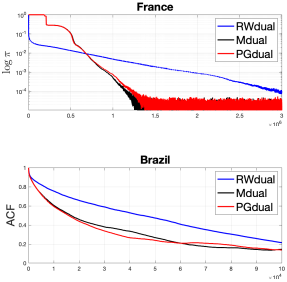
RW RW Invert RW Ortho
PGdual Invert PGdual Ortho.

(left) Mean absolute value of the ACF vs the first 10^5 lags
(right) The Gelman-Rubin statistic vs iterations.

Show

- same conclusions as before
it promotes PGdual "ortho"

PGdual and MYMALA (1/2)



Show

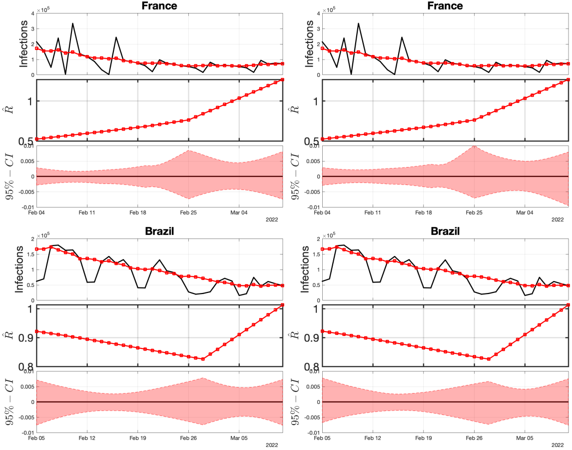
- the benefit of first-order methods

- equivalent results for PGdual and MYMALA

RW "ortho" when the Gaussian noise is no iid
MYMALA "ortho"
PGdual "ortho"

(top) $n \mapsto (\log \pi(\theta_n) - \max \log \pi)/(\log \pi(\theta_1) - \max \log \pi)$
(bottom) ACF

PGdual and MYMALA (2/2)

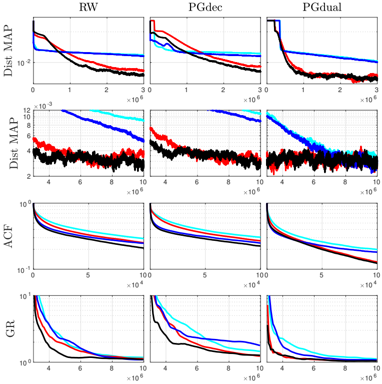


Show

- Same conclusions as before:

equivalent results for PGdual and MYMALA.

Gibbs or Hastings-Metropolis ?



HM "Invert" HM "Ortho"
Gibbs "Invert" Gibbs "Ortho"

Show

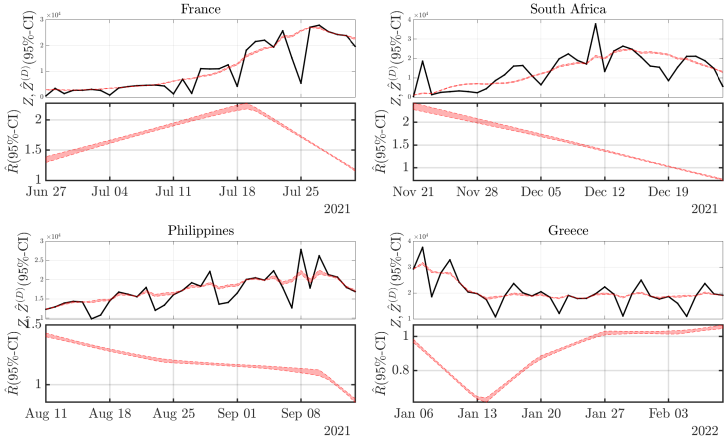
- the benefit of "ortho" w.r.t. "invert"

- Gibbs "ortho" among the best, whatever the criterion and the algorithm

- `PGdual` looks the best.

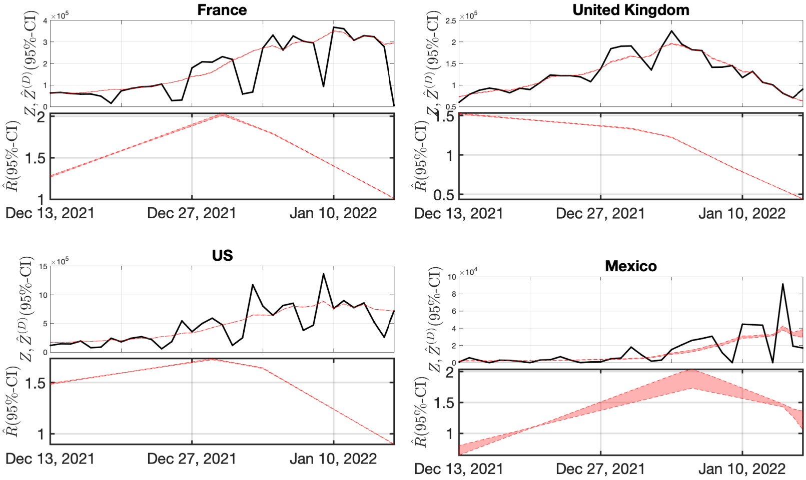
(rows 1 & 2) evolution of the distance from θ_n to the MAP along iterations
(row 3) ACF
(row 4) Gelman Rubin statistics

Credibility intervals for different countries and different time periods



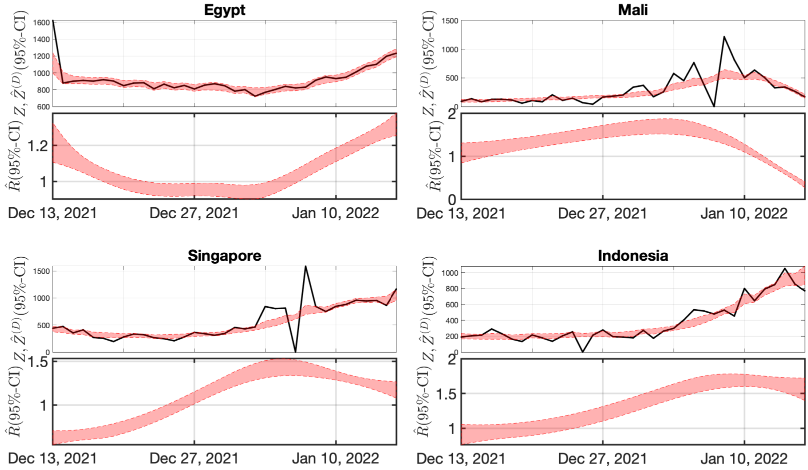
For each country,
(top) observed counts $Z_1, \cdots, Z_T$; and 95% credibility interval for the $Z_t - \hat{O}_t$'s
(bottom) 95% credibility interval for $\mathsf{R}_1, \cdots, \mathsf{R}_T$
With `PGdual` "ortho"

Credibility Intervals for different countries



With Gibbs + `PGdual` "ortho"

Credibility Intervals for different countries



With Gibbs + `PGdual` "ortho"

V. Conclusions and future works

- In Fort et al. (2022), extension of `PGdec` and `PGdual` to blockwise structure of the non-smooth component:

$$g(\mathsf{C}\theta) = \sum_{j=1}^{J} \sum_{i=1}^{I_j} g_{ij}(\mathsf{C}_{ij}\theta^{(j)}) \qquad \theta = \mathsf{blocks}(\theta^{(1)}, \cdots, \theta^{(J)})$$

- How to choose λ_R and λ_O ?
- What about sequential analysis $\rightarrow$ SMC
- Noisy data $\rightarrow$ noisy data and missing data.

VI. Bibliography

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