Rigorous derivation of the nonlocal reaction-diffusion FitzHugh-Nagumo system

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December 12, 2018

Abstract

We introduce a spatially extended transport kinetic FitzHugh-Nagumo model with forced local interactions and prove that its hydrodynamic limit converges towards the classical nonlocal reaction-diffusion FitzHugh-Nagumo system. Our approach is based on a relative entropy method, where the macroscopic quantities of the kinetic model are compared with the solution to the nonlocal reaction-diffusion system. This approach allows to make the rigorous link between kinetic and reaction-diffusion models.

1 Introduction

Generally, neuron models focus on the regulation of the electrical potential of the membrane of a nerve cell depending on the input it receives. This regulation is the result of ionic exchanges between the neuron and its environment through its cellular membranes. A very precise modeling of these ion exchanges led to the well-known Hodgkin-Huxley model [25]. In this paper, we shall rather focus on a simplified version, called the FitzHugh-Nagumo (FHN) model [22, 35], which keeps its most valuable aspects, but remains relatively simple mathematically. More precisely, the FHN model accounts for the variations of the membrane potential v of a neuron coupled to an auxiliary variable w called the adaptation variable. It is usually written as follows

$$
\begin{cases}
\dfrac{\mathrm{d}v}{\mathrm{d}t} = N(v) \, - \, w \, + \, I_{\text{ext}}, \\[2mm]
\dfrac{\mathrm{d}w}{\mathrm{d}t} = \tau \, (v \, + \, a \, - \, b \, w),
\end{cases}
\tag{1.1}
$$

where I_{ext} stands for the input current the neuron receives, $N(v)$ is a bistable nonlinearity which models the cell excitability, and $\tau \geq 0$, $a \in \mathbb{R}$ and $b \geq 0$ are some given constants. Without loss of generality [2, 20, 34, 35], we assume that $N(v)$ is given by the following cubic nonlinearity[1]

$$
N(v) := v - v^3, \quad v \in \mathbb{R}.
\tag{1.2}
$$

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[1]All our results remain true for $N(v) = v(\alpha - \beta|v|^2)$ with any $\alpha, \beta > 0$.

Another mathematical reason for looking at system (1.1) is that it is a prototypical model of excitable kinetics. Interest in such systems stems from the fact that although the kinetics are very simple spatial coupling can produce complex dynamics, where well-known examples are the propagation of excitatory pulses, spiral waves in two-dimensions, and spatio-temporal chaos. Here, we introduce spatial coupling through the input current I_{ext} which models how the membrane potentials of the other neurons influence the membrane potential of one given neuron. More specifically, following [2, 34], we consider a network of n neurons interacting through their synapses, which we suppose to be of electrical type, such that the evolution of the pair voltage-adaptation $(v_i, w_i)_{1 \leq i \leq n}$ satisfies the system of equations

$$
(\textbf{Micro}) \quad
\begin{cases}
\dfrac{\mathrm{d}x_i}{\mathrm{d}t} = 0, \\[2mm]
\dfrac{\mathrm{d}v_i}{\mathrm{d}t} = N(v_i) - w_i - \dfrac{1}{n}\sum_{j=1}^{n} \Phi_\varepsilon(\mathbf{x}_i - \mathbf{x}_j)(v_i - v_j), \\[2mm]
\dfrac{\mathrm{d}w_i}{\mathrm{d}t} = \tau(v_i + a - bw_i),
\end{cases}
\qquad (1.3)
$$

for $t > 0$, where we choose to associate to each neuron $i \in \{1, ..., n\}$ its position in the network $\mathbf{x}_i \in \mathbb{R}^d$ with $d \in \{1, 2, 3\}$, that is fixed, as in [33]. The connectivity kernel $\Phi_\varepsilon : \mathbb{R}^d \to \mathbb{R}$ models the effect of the distance between two neurons on their electro-chemical interactions, and here $\varepsilon > 0$ is a scaling parameter as explained later. System (1.3) is a natural spatial extension of the FHN models introduced and studied in [2, 34].

The main purpose of this article is to establish a macroscopic description of the neural network (1.3) in the limit $n \to +\infty$, choosing an appropriate scaling for the connectivity function Φ_ε. Our strategy is to introduce an intermediate modeling scale, namely a mesoscopic description of system (1.3). We thus associate to system (1.3) the following spatially-extended kinetic equation

$$
(\textbf{Kinetic})_\epsilon \quad \partial_t f^\varepsilon + \partial_v \left(f^\varepsilon N(v) - f^\varepsilon w - f^\varepsilon \mathcal{K}_{\Phi_\varepsilon}[f^\varepsilon] \right) + \partial_w \left(A(v, w) f^\varepsilon \right) = 0, \qquad (1.4)
$$

for $(t, \mathbf{x}, v, w) \in (0, \infty) \times \mathbb{R}^{d+2}$, where the operator $\mathcal{K}_{\Phi_\varepsilon}[f^\varepsilon]$ and the convective term A are given by

$$
\begin{cases}
\mathcal{K}_{\Phi_\varepsilon}[f^\varepsilon](t, \mathbf{x}, v) := \displaystyle\int_{\mathbb{R}^{d+2}} \Phi_\varepsilon(\mathbf{x} - \mathbf{x}')(v - v')\, f^\varepsilon(t, \mathbf{x}', v', w')\, \mathrm{d}w' \mathrm{d}v' \mathrm{d}\mathbf{x}', \\[4mm]
A(v, w) := \tau(v + a - b\,w).
\end{cases}
$$

Such an equation models the evolution of the density function $f^\varepsilon(t, \mathbf{x}, v, w)$ of finding neurons with a potential membrane $v \in \mathbb{R}$ and an adaptation variable $w \in \mathbb{R}$ at time $t \geq 0$ and position $\mathbf{x} \in \mathbb{R}^d$ within the cortex. The term $\mathcal{K}_\varepsilon[f^\varepsilon]$ describes nonlocal interactions through the whole field, whereas the other terms account for the local reactions due to the excitability of nerve cells.

Using classical arguments [10, 23] and under some mild assumptions on Φ_ε, it can actually be proved that the kinetic equation (1.4) is the mean-field limit of (1.3) as n goes to infinity [17]. In fact, the mean-field limit of the FHN system (1.3) with stochastic fluctuations towards a kinetic PDE similar to (1.4) has already been studied for conductance-based connectivity without spatial dependance [2, 4], for homogeneous interactions [34] and for spatially organized neural network with compactly supported communication weight [33]. Various other types of kinetic models have been derived during the past decades depending on the hypotheses assumed

for the dynamics of the emission of an action potential. They include for example integrate-and-fire neural networks [8, 9, 12] and time-elapsed neuronal models [13–15, 36].

Coming back to our problem of establishing a macroscopic description of the neural network (1.3), we introduce the following macroscopic quantities for f^ε a solution of (1.4):

$$
\begin{cases}
\rho^\varepsilon(t, \mathbf{x}) := \displaystyle\int_{\mathbb{R}^2} f^\varepsilon(t, \mathbf{x}, v, w) \, \mathrm{d}v\mathrm{d}w, \\[2mm]
\rho^\varepsilon(t, \mathbf{x}) V^\varepsilon(t, \mathbf{x}) = j^\varepsilon(t, \mathbf{x}) := \displaystyle\int_{\mathbb{R}^2} v f^\varepsilon(t, \mathbf{x}, v, w) \, \mathrm{d}v\mathrm{d}w, \\[2mm]
\rho^\varepsilon(t, \mathbf{x}) W^\varepsilon(t, \mathbf{x}) = q^\varepsilon(t, \mathbf{x}) := \displaystyle\int_{\mathbb{R}^2} w f^\varepsilon(t, \mathbf{x}, v, w) \, \mathrm{d}v\mathrm{d}w,
\end{cases}
\tag{1.5}
$$

where ρ^ε stands for the averaged neuron density, V^ε is the average membrane potential, and W^ε is the average adaptation variable. Formally, it is not difficult to check that the equation satisfied by j^ε is not closed because of the presence of the reaction term $\partial_v\left(f^\varepsilon(N(v))\right)$ in (1.4), which introduces higher moments of f^ε in v. To circumvent this difficulty and obtain a closed macroscopic system, we consider a specific form for the connectivity function Φ_ε. We will assume that it can be decomposed by the superposition of strong local interactions modeled by $\frac{1}{\varepsilon}\delta_0$, where δ_0 is the Dirac distribution centered at 0 and weak lateral interactions described by a non-negative connectivity kernel Ψ. As a consequence, for all $\varepsilon > 0$, we work with

$$
\Phi_\varepsilon(\mathbf{x}) = \Psi(\mathbf{x}) + \frac{1}{\varepsilon}\delta_0(\mathbf{x}), \quad \forall \mathbf{x} \in \mathbb{R}^d.
\tag{1.6}
$$

Note that such an assumption of strong local interactions and weak lateral interactions is often used when modeling visual cortex [7]. We can then rewrite the kinetic equation (1.4) as

$$
\partial_t f^\varepsilon + \partial_v\left(f^\varepsilon N(v) - f^\varepsilon w - f^\varepsilon \mathcal{K}_\Psi[f^\varepsilon]\right) - \frac{1}{\varepsilon}\partial_v\left(f^\varepsilon(\rho^\varepsilon v - j^\varepsilon)\right) + \partial_w\left(A(v, w) f^\varepsilon\right) = 0
\tag{1.7}
$$

for $(t, \mathbf{x}, v, w) \in (0, \infty) \times \mathbb{R}^{d+2}$, and the macroscopic quantities $(\rho^\varepsilon, \rho^\varepsilon V^\varepsilon, \rho^\varepsilon W^\varepsilon)$ formally verify the system of equations:

$$
(\mathbf{Macro})_\epsilon \quad
\begin{cases}
\partial_t \rho^\varepsilon = 0, \\[2mm]
\partial_t\left(\rho^\varepsilon V^\varepsilon\right) - \rho^\varepsilon \mathcal{L}_{\rho^\varepsilon}(V^\varepsilon) = \rho^\varepsilon\left[N(V^\varepsilon) - W^\varepsilon\right] + \mathcal{E}(f^\varepsilon), \\[2mm]
\partial_t\left(\rho^\varepsilon W^\varepsilon\right) = \rho^\varepsilon A(V^\varepsilon, W^\varepsilon),
\end{cases}
\tag{1.8}
$$

for $(t, \mathbf{x}) \in (0, \infty) \times \mathbb{R}^d$, where the spatial nonlocal operator $\mathcal{L}_\rho$ is defined through

$$
\mathcal{L}_\rho(V) := -(\Psi * \rho)V + \Psi * [\rho V],
\tag{1.9}
$$

whereas $*$ stands for the convolution product in $\mathbb{R}^d$ only with respect to space. Furthermore, we have set

$$
\mathcal{E}(f^\varepsilon)(t, \mathbf{x}) := \int_{\mathbb{R}^2} f^\varepsilon(t, \mathbf{x}, v, w)\left[N(v) - N(V^\varepsilon(t, \mathbf{x}))\right] \mathrm{d}v\mathrm{d}w.
\tag{1.10}
$$

It is worth noticing that v is a collisional invariant for the operator $\partial_v\left(f^\varepsilon(\rho^\varepsilon v - j^\varepsilon)\right)$ such that the macroscopic system (1.8) does not naturally present terms of order $\mathcal{O}(1/\epsilon)$.

Let us formally determine the hydrodynamic limit of the equation (1.7) as $\varepsilon \to 0$. If we multiply (1.7) by ε and if we take $\varepsilon \to 0$, the local interaction term $\rho^\varepsilon(v - V^\varepsilon) f^\varepsilon$ vanishes. Consequently, f^ε should converge in some weak sense to the mono-kinetic distribution in v

$$
f^\varepsilon(t, \mathbf{x}, v, w) \underset{\varepsilon \to 0}{\rightharpoonup} F(t, \mathbf{x}, w) \otimes \delta_0(v - V(t, \mathbf{x})),
\tag{1.11}
$$

3

where the limit (F, V) has to be determined. First we set

$$\rho(t, \mathbf{x}) := \int_{\mathbb{R}} F(t, \mathbf{x}, w) \mathrm{d}w, \quad \rho(t, \mathbf{x}) W(t, \mathbf{x}) := \int_{\mathbb{R}} F(t, \mathbf{x}, w) \, w \, \mathrm{d}w,$$

hence the macroscopic equation satisfied by V is formally obtained by observing that $\mathcal{E}(f^\varepsilon) \to 0$ as $\varepsilon \to 0$ in the second equation of (1.8), hence we get

$$\partial_t (\rho V) - \rho \mathcal{L}_\rho(V) = \rho \left[N(V) - W \right].$$

Then substituting (1.11) in the kinetic equation (1.7) and integrating with respect to $v \in \mathbb{R}$, it yields that F is solution to

$$\partial_t F + \partial_w \left(A(V, w) F \right) = 0.$$

Therefore, the limit function $\mathcal{Z} := (\rho, \rho V, \rho W)$ is expected to verify the following nonlocal reaction diffusion system:

$$\partial_t \mathcal{Z} = \mathcal{F}(\mathcal{Z}), \quad t > 0, \, \mathbf{x} \in \mathbb{R}^d, \tag{1.12}$$

with $\mathcal{F}$ given by

$$\mathcal{F}(\mathcal{Z}) := \rho \begin{pmatrix} 0 \\ \mathcal{L}_\rho(V) + N(V) - W \\ A(V, W) \end{pmatrix}. \tag{1.13}$$

It is interesting to note that although we obtain a mono-kinetic distribution in the v variable only, it is enough to obtain a closed macroscopic model. Indeed, the kinetic equation (1.7) is linear in the w variable which allows to close the system when integrating in w.

Let us first make some comments on the structure of the macroscopic model (1.12)-(1.13). For all $\mathbf{x} \in \mathbb{R}^d$ such that $\rho(t, \mathbf{x}) = \rho_0(\mathbf{x}) > 0$, the system (1.12) reduces to the usual nonlocal reaction-diffusion system of FitzHugh-Nagumo type

$$\begin{cases} \partial_t V - \mathcal{L}_{\rho_0}(V) = N(V) - W, \\ \\ \partial_t W = \tau(V + a - bW), \end{cases} \tag{1.14}$$

for $\mathbf{x} \in \mathbb{R}^d$ and $t > 0$, where $\mathcal{L}_{\rho_0}(V)$ can be interpreted as a nonlocal diffusion operator in $\mathbf{x}$. In the limiting case $\rho_0 \equiv 1$[2], such a system has already been well studied especially regarding the formation of propagating waves (traveling fronts and pulses) in both cases $\tau = 0$ and $0 < \tau \ll 1$ [3, 20]. We also mention the classical works regarding the local FitzHugh-Nagumo system, that is when $\mathcal{L}_{\rho_0}(V)$ is replaced by the standard diffusion operator [11, 24, 28], and the more recent advances for the discrete case [26, 27]. The present work is then a rigorous justification of the nonlocal reaction-diffusion system of FitzHugh-Nagumo type (1.12) that is obtained as the hydrodynamic limit of the kinetic equation (1.7) as $\varepsilon \to 0$.

The main challenge towards a rigorous proof of this hydrodynamic limit stems from the Dirac singularity of the mono-kinetic distribution which prevents us from using a classical entropy of the form $f \log(f)$ since it would not be well-defined. Following ideas from [21], who proved a similar hydrodynamic limit of the kinetic Cucker-Smale model for collective motion with forced local alignment towards the pressureless Euler equation with a nonlocal force, we shall overcome this problem by the mean of a relative entropy argument. Let us also mention the work of [29] in which the hydrodynamic limit of a collisionless and non-diffusive kinetic equation under strong local alignment regime is rigorously established *via* a relative entropy argument. The

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[2]In our setting, ρ_0 is probability density function such that the case $\rho_0 \equiv 1$ is excluded from our hypotheses. Nevertheless, system (1.14) is still well defined for $\rho_0 \equiv 1$, see Proposition 2.3.

specific difficulty here is that, instead of having a transport term as in [21], the presence of the reaction term $\partial_v(f^\varepsilon N(v))$ in (1.7) introduces higher order moments of f^ε in v which we will need to control. In fact, it will be enough to have a priori estimates of second and fourth order moments of f^ε in (v, w) to circumvent this difficulty.

We conclude this introduction by comparing the kinetic model (1.4) to the pioneer works of Amari, Wilson and Cowan in the 1970s [1, 38, 39] who heuristically derived spatially extended models describing the macroscopic activity of large assemblies of neurons. Such models are referred to as neural field equations in the literature and account for the time evolution of the spatial averaged membrane potential of a population of neurons. Such a class of models has received much attention and has been very successful at reproducing a number of biological phenomena, including in particular visual hallucinations, binocular rivalry or working memory. We refer to the recent surveys [5, 6, 16] for more developments on neural field models and applications in neuroscience. By derivation, neural field equations are macroscopic models and it is still an open problem to rigorously justify such models from conductance-based neural networks. In a slightly different direction, we mention the recent work of Chevallier *et al.* [15] on mean-field limits for nonlinear spatially extended Hawkes processes with exponential memory kernels who are able to recover at the limit a process whose law is an inhomogeneous Poisson process having an intensity which solves a scalar neural field equation. In this article, we have addressed this open question in the case of the FHN model.

The rest of this article is organized as follows. In Section 2, we state our main result about the hydrodynamic limit of (1.7) towards (1.12). Then, in Section 3, we derive some a priori estimates that will be crucial for our relative entropy argument. The proof of our main result is contained in Section 4.

2 Preliminaries and main result

In this section, we present our main result on the hydrodynamic limit from a weak solution $(f^\varepsilon)_{\varepsilon>0}$ of the kinetic equation (1.7) to a classical solution $(\rho_0, \rho_0 V, \rho_0 W)$ of the asymptotic system (1.12). For that, we first need to present the existence result for the weak solution of (1.7) and the classical solution of (1.12).

First we set the hypotheses we make for the study of the kinetic FHN equation (1.7) and the limit system (1.12). We consider a connectivity kernel Ψ in (1.6) which is non-negative, symmetric and satisfies

$$\Psi \in L^1(\mathbb{R}^d). \tag{2.1}$$

This last assumption models the fact that if two neurons are far away from each other, they have weak mutual interactions. The other condition is a natural biological assumption and expresses the symmetric and excitatory nature of the considered underlying neural network.

2.1 Existence of weak solution to (1.7)

We here say that f^ε is a weak solution of (1.7) if for any $T > 0$, $f^\varepsilon(0, .) = f_0^\varepsilon \geq 0$ in $\mathbb{R}^{d+2}$,

$$f^\varepsilon \in \mathscr{C}^0([0, T], L^1(\mathbb{R}^{d+2})) \cap L^\infty((0, T) \times \mathbb{R}^{d+2}),$$

and (1.7) holds in the sense of distribution, that is, for any $\varphi \in \mathscr{C}_c^\infty([0, T) \times \mathbb{R}^{d+2})$, the weak formulation holds

$$\int_0^T \int f^\varepsilon \left[\partial_t \varphi + \left(N(v) - w - \mathcal{K}_\Psi[f^\varepsilon] - \frac{1}{\varepsilon}(\rho^\varepsilon v - j^\varepsilon) \right) \partial_v \varphi + A(v, w) \partial_w \varphi \right] d\mathbf{z} \, dt$$

$$+ \int f_0^\varepsilon \varphi(0) \, d\mathbf{z} = 0 \tag{2.2}$$

where $\mathbf{z} = (\mathbf{x}, v, w) \in \mathbb{R}^{d+2}$.

Remark 2.1. *Using the mass conservation property of equation* (1.7)*, we can easily check that the time varying macroscopic quantities* $(\rho^\varepsilon, V^\varepsilon, W^\varepsilon)$ *defined in* (1.5) *simplify to*

$$\rho^\varepsilon(t, \mathbf{x}) := \int_{\mathbb{R}^2} f^\varepsilon(t, \mathbf{x}, v, w)\, \mathrm{d}v\, \mathrm{d}w = \int_{\mathbb{R}^2} f_0^\varepsilon(\mathbf{x}, v, w)\, \mathrm{d}v\, \mathrm{d}w = \rho_0^\varepsilon(\mathbf{x}), \qquad (2.3)$$

hence we have

$$\begin{cases} \rho_0^\varepsilon(\mathbf{x}) V^\varepsilon(t, \mathbf{x}) := \displaystyle\int_{\mathbb{R}^2} v\, f^\varepsilon(t, \mathbf{x}, v, w)\, \mathrm{d}v\, \mathrm{d}w, \\[3mm] \rho_0^\varepsilon(\mathbf{x}) W^\varepsilon(t, \mathbf{x}) := \displaystyle\int_{\mathbb{R}^2} w\, f^\varepsilon(t, \mathbf{x}, v, w)\, \mathrm{d}v\, \mathrm{d}w, \end{cases}$$

for all $\mathbf{x} \in \mathbb{R}^d$ *and all* $t > 0$ *where* f^ε *is well-defined.*

First, let us prove the well-posedness of the kinetic equation (1.7).

Proposition 2.2. *For any* $\varepsilon > 0$ *we choose* Ψ *to be non-negative, symmetric and satisfies* (2.1)*, we also assume that* f_0^ε *satisfies*

$$f_0^\varepsilon \geq 0, \quad f_0^\varepsilon \in L^1(\mathbb{R}^{d+2}), \quad f_0^\varepsilon, \nabla_{\mathbf{u}} f_0^\varepsilon \in L^\infty(\mathbb{R}^{d+2}), \qquad (2.4)$$

where $\mathbf{u} = (v, w)$ *and for all* $\mathbf{x} \in \mathbb{R}^d$,

$$\mathrm{Supp}(f_0^\varepsilon(\mathbf{x}, \cdot, \cdot)) \subseteq B(0, R_0^\varepsilon) \subset \mathbb{R}^2. \qquad (2.5)$$

Then for any $T > 0$*, there exists a unique* f^ε *weak solution to* (1.7) *in the sense of* (2.2)*, which is compactly supported in* $\mathbf{u} = (v, w) \in \mathbb{R}^2$.

The proof relies on a classical fixed point argument, but for the sake of completeness, it is postponed to the Appendix A.

2.2 Existence of classical solution to the nonlocal FitzHugh-Nagumo system

Let us now state the result of existence and uniqueness for the nonlocal reaction-diffusion FitzHugh-Nagumo system defined as

$$\begin{cases} \partial_t V - \mathcal{L}_{\rho_0}(V) = N(V) - W, \\[2mm] \partial_t W = \tau(V + a - bW), \qquad\qquad\qquad t > 0 \text{ and } \mathbf{x} \in \mathbb{R}^d, \\[2mm] V(0, \mathbf{x}) = V_0(\mathbf{x}), \quad W(0, \mathbf{x}) = W_0(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^d. \end{cases} \qquad (2.6)$$

Before describing precisely the existence and uniqueness of solution (V, W) to the hydro-dynamical system (2.6), let us emphasize that this system is more convenient to analyse than (1.12)-(1.13) verified by $\mathcal{Z} = (\rho_0, \rho_0 V, \rho_0 W)$. Indeed, as we will see, both solutions coincide in the region of interest where $\rho_0 > 0$, but the study of (2.6) allows to construct a solution such that for all $t \in [0, T]$

$$V(t), W(t) \in L^\infty(\mathbb{R}^d).$$

This property is crucial to apply the relative entropy method in the asymptotic analysis of (1.7) when $\varepsilon \to 0$.

Proposition 2.3. *We choose Ψ to be non-negative, symmetric and satisfies (2.1), we also suppose that ρ_0 and the initial data (V_0, W_0) satisfies,*

$$\rho_0 \geq 0, \quad \rho_0 \in L^1 \cap L^\infty(\mathbb{R}^d), \quad V_0, W_0 \in L^\infty(\mathbb{R}^d). \tag{2.7}$$

Then for any $T > 0$, there exists a unique classical solution $(V, W) \in \mathscr{C}^1([0, T], L^\infty(\mathbb{R}^d))$ to the equation (2.6). Furthermore, $\mathcal{Z} = (\rho_0, \rho_0 V, \rho_0 W)$ is a solution to (1.12)-(1.13).

The proof of this proposition also relies on a classical fixed point argument. For the sake of completeness, it is postponed to the Appendix B. Then as a direct consequence, for a given initial data (F_0, V_0), it leads to the existence and uniqueness of solution (F, V) to the following system of equations

$$\begin{cases} \partial_t F + \partial_w \left(A(V, w) F \right) = 0, \\[2mm] \rho_0 \partial_t V - \rho_0 \mathcal{L}_{\rho_0}(V) = \rho_0 \left[N(V) - W \right], \\[2mm] \rho_0(\mathbf{x}) = \displaystyle\int_{\mathbb{R}} F(t, \mathbf{x}, w) \, \mathrm{d}w, \\[2mm] V(t, \mathbf{x}) = 0, \quad t > 0 \text{ and } \mathbf{x} \in \mathbb{R}^d \setminus \mathrm{Supp}_{\mathrm{ess}}(\rho_0), \\[2mm] W(t, \mathbf{x}) = \begin{cases} \dfrac{1}{\rho_0(\mathbf{x})} \displaystyle\int_{\mathbb{R}} F(t, \mathbf{x}, w) \, w \, \mathrm{d}w, & \text{if } \rho_0(\mathbf{x}) > 0, \\[3mm] 0, & \text{else.} \end{cases} \end{cases} \tag{2.8}$$

More precisely we have the following result.

Corollary 2.4. *Consider $V_0 \in L^\infty(\mathbb{R}^d)$ and $F_0 \in \mathcal{M}(\mathbb{R}^{d+1})$ such that*

$$\int_{\mathbb{R}^{d+1}} w^2 F_0(\mathrm{d}\mathbf{x}, \mathrm{d}w) < \infty$$

and for almost every $\mathbf{x} \in \mathbb{R}^d$,

$$\rho_0(\mathbf{x}) = \int_{\mathbb{R}} F_0(\mathbf{x}, w) \, \mathrm{d}w, \quad W_0(\mathbf{x}) := \begin{cases} \dfrac{1}{\rho_0(\mathbf{x})} \displaystyle\int_{\mathbb{R}} F_0(\mathbf{x}, w) \, w \, \mathrm{d}w, & \text{if } \rho_0(\mathbf{x}) > 0, \\[3mm] 0, & \text{else,} \end{cases}$$

where $\rho_0 \in L^1 \cap L^\infty(\mathbb{R}^d)$ and $W_0 \in L^\infty(\mathbb{R}^d)$. We further assume that $V_0 = 0$ on $\mathbb{R}^d \setminus \mathrm{Supp}_{ess}(\rho_0)$. Then for any $T > 0$, there exists a unique couple (F, V) solution to (2.8), where

$$(F, V) \in L^\infty((0, T), \mathcal{M}(\mathbb{R}^{d+1})) \times \mathscr{C}^1([0, T], L^\infty(\mathbb{R}^d)),$$

and F is a measure solution to the first equation (2.8), that is, for any $\varphi \in \mathscr{C}_c^1(\mathbb{R}^{d+1})$

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^{d+1}} \varphi(\mathbf{x}, w) F(t, \mathrm{d}\mathbf{x}, \mathrm{d}w) - \int_{\mathbb{R}^{d+1}} A(V(t, \mathbf{x}), w) \, \partial_w \varphi(\mathbf{x}, w) F(t, \mathrm{d}\mathbf{x}, \mathrm{d}w) = 0 \tag{2.9}$$

such that there exists a constant $C_T > 0$,

$$\int_{\mathbb{R}^{d+1}} w^2 F(t, \mathrm{d}\mathbf{x}, \mathrm{d}w) < C_T, \quad t \in [0, T].$$

Proof. We first apply Proposition 2.3 with (ρ_0, V_0, W_0), where (ρ_0, W_0) is computed from the moments with respect to $(1, w)$ of the initial distribution F_0, that is, for almost every $\mathbf{x} \in \mathbb{R}^d$,

$$\rho_0(\mathbf{x}) = \int_{\mathbb{R}} F_0(\mathbf{x}, w) \, \mathrm{d}w, \quad W_0(\mathbf{x}) := \begin{cases} \dfrac{1}{\rho_0(\mathbf{x})} \displaystyle\int_{\mathbb{R}} F_0(\mathbf{x}, w) \, w \, \mathrm{d}w, & \text{if } \rho_0(\mathbf{x}) > 0, \\[2ex] 0, & \text{else.} \end{cases}$$

We then denote by $(\widetilde{V}, \widetilde{W})$ the corresponding unique classical solution to (2.6) starting from such an initial condition. Finally, we define

$$V(t, \mathbf{x}) := \begin{cases} \widetilde{V}(t, \mathbf{x}), & \text{if } \rho_0(\mathbf{x}) > 0, \\[2ex] 0, & \text{else.} \end{cases} \tag{2.10}$$

As ρ_0 is independent of time, we note that $V \in \mathscr{C}^1\left([0, T], L^\infty(\mathbb{R}^d)\right)$.

We now prove that for any couple solution $(\overline{F}, \overline{V})$ of (2.8) then $\overline{V}$ is precisely given by (2.10). Thus, let us suppose that $(\overline{F}, \overline{V})$ is a well-defined solution of (2.8) on $[0, T]$ with finite second moment. We necessarily get that $\rho_0 \overline{W}$ has to satisfy

$$\partial_t(\rho_0 \overline{W}) - \rho_0 \, A(\overline{V}, \overline{W}) = 0,$$

together with

$$\rho_0 \partial_t \overline{V} - \rho_0 \mathcal{L}_{\rho_0}(\overline{V}) = \rho_0 \left[N(\overline{V}) - \overline{W} \right].$$

Thus, for any $\mathbf{x} \in \mathbb{R}^d$ such that $\rho_0(\mathbf{x}) > 0$, the couple $(\overline{V}, \overline{W})$ coincides with the unique solution to (2.6) with initial condition (ρ_0, V_0, W_0). This is ensured from the fact that the convolution in the nonlocal part $\Psi * (\rho_0 \overline{V})$ of the linear operator $\mathcal{L}_{\rho_0}(\overline{V})$ is only evaluated on the regions where $\rho_0 > 0$. Then, this uniquely defines $\overline{V}(t, \mathbf{x}) = \widetilde{V}(t, \mathbf{x})$ on $\rho_0 > 0$. By definition of a solution to (2.8), whenever $\rho_0 = 0$, we have that $\overline{V}(, \mathbf{x}) = 0$. As a conclusion, we have just shown that if $(\overline{F}, \overline{V})$ is a well-defined solution of (2.8) on $[0, T]$ with finite second moment then necessarily $\overline{V} = V$ where V is uniquely defined in (2.10).

For V defined as above in (2.10), we consider the transport equation

$$\begin{cases} \partial_t F + \partial_w \left(A(V, w) \, F \right) = 0, \\[2ex] F(0) = F_0. \end{cases} \tag{2.11}$$

Then, for almost every $\mathbf{x} \in \mathbb{R}^d$ and all $(t, w) \in [0; T] \times \mathbb{R}$, we introduce the system of characteristic curves associated to (2.11), that is,

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}s} \mathcal{W}(s) = A\left(V(s, \mathbf{x}), \mathcal{W}(s) \right), \\[2ex] \mathcal{W}(t) = w, \end{cases} \tag{2.12}$$

where V is defined in (2.10). From the regularity with respect to $(t, w) \in [0, T] \times \mathbb{R}$ of the functions $A(.,.)$ and $V(.,.)$ and since $A(.,.)$ grows at most linearly with respect to w, we get global existence and uniqueness of a solution to (2.12). This solution is denoted by $\mathcal{W}(s, t, \mathbf{x}, w)$, then we verify using the theory of characteristics that the unique solution to the transport equation (2.11) is given by

$$F(t, \mathbf{x}, w) = F_0(\mathbf{x}, \mathcal{W}(0, t, \mathbf{x}, w)) \, e^{\tau b t}.$$

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From the above expression, it easy to compute the second order moment of F with respect to w and get that there exists $C_T > 0$ such that

$$\int_{\mathbb{R}^{d+1}} w^2 F(t, \mathrm{d}\mathbf{x}, \mathrm{d}w) < C_T, \quad t \in [0, T].$$

As a final consequence of the above computations, we have that

$$\frac{1}{\rho_0(\mathbf{x})} \int_{\mathbb{R}} F(t, \mathbf{x}, w)\, w\, \mathrm{d}w = \widetilde{W}(t, \mathbf{x}), \text{ for all } \mathbf{x} \in \mathbb{R}^d \text{ such that } \rho_0(\mathbf{x}) > 0,$$

by uniqueness of the solutions of (2.6). This shows that the couple (F, V) is the unique solution to (2.8) where V is given in (2.10) and F is the unique measure solution of (2.11). ∎

2.3 Main result

Now, we are ready to state our main result about the hydrodynamic limit. To this aim we consider a non-negative initial data $(f_0^\varepsilon)_{\varepsilon>0}$ and suppose that there exists a constant $C > 0$, such that for all $\varepsilon > 0$,

$$\|\rho_0^\varepsilon\|_{L^\infty} \leq C \tag{2.13}$$

and

$$\int \left(1 + |\mathbf{x}|^4 + |v|^4 + |w|^4\right) f_0^\varepsilon(\mathbf{x}, v, w) \mathrm{d}w \mathrm{d}v \mathrm{d}\mathbf{x} \leq C. \tag{2.14}$$

Theorem 2.5. *Let $T > 0$, Ψ be a non-negative, symmetric kernel verifying (2.1) and $(f_0^\varepsilon)_\varepsilon$, a sequence of initial data such that for all $\varepsilon > 0$, (2.4), (2.5), (2.13) and (2.14) are satisfied. Assume that (ρ_0, V_0, W_0) verifies (2.7), and*

$$\|\rho_0^\varepsilon - \rho_0\|_{L^2}^2 + \int \rho_0^\varepsilon(\mathbf{x}) \left[|V_0^\varepsilon(\mathbf{x}) - V_0(\mathbf{x})|^2 + |W_0^\varepsilon(\mathbf{x}) - W_0(\mathbf{x})|^2\right] \mathrm{d}\mathbf{x} \leq C\, \varepsilon^{1/(d+6)}. \tag{2.15}$$

Then, the macroscopic quantities $(\rho_0^\varepsilon, V^\varepsilon, W^\varepsilon)$, computed from the solution f^ε to (1.7), verify that for all $t \in [0, T]$,

$$\int_{\mathbb{R}^d} \left[|V^\varepsilon(t, \mathbf{x}) - V(t, \mathbf{x})|^2 + |W^\varepsilon(t, \mathbf{x}) - W(t, \mathbf{x})|^2\right] \rho_0^\varepsilon(\mathbf{x}) \mathrm{d}\mathbf{x} \leq C_T\, \varepsilon^{1/(d+6)},$$

where (V, W) is the unique solution to (1.14).

Let further assume that (ρ_0, V_0, W_0) are such that,

$$\rho_0(\mathbf{x}) = \int_{\mathbb{R}} F_0(\mathbf{x}, w)\, \mathrm{d}w, \quad W_0(\mathbf{x}) = \begin{cases} \dfrac{1}{\rho_0(\mathbf{x})} \displaystyle\int_{\mathbb{R}} F_0(\mathbf{x}, w)\, w\, \mathrm{d}w, & \text{if } \rho_0(\mathbf{x}) > 0, \\ \\ 0, & \text{else,} \end{cases}$$

for $F_0 \in \mathcal{M}(\mathbb{R}^{d+1})$, and $V_0 = 0$ on $\mathbb{R}^d \setminus \operatorname{Supp}_{ess}(\rho_0)$. Moreover, consider the function F_0^ε such that for all $\mathbf{x} \in \mathbb{R}^d$ and all $w \in \mathbb{R}$,

$$F_0^\varepsilon(\mathbf{x}, w) = \int f_0^\varepsilon(\mathbf{x}, v, w)\, \mathrm{d}v.$$

If $F_0^\varepsilon \rightharpoonup F_0$ weakly-$\star$ in $\mathcal{M}\left(\mathbb{R}^{d+1}\right)$ then we have for all $\varphi \in \mathscr{C}_b^0(\mathbb{R}^{d+2})$:

$$\int \varphi(\mathbf{x}, v, w)\, f^\varepsilon(t, \mathbf{x}, v, w)\, \mathrm{d}v\, \mathrm{d}w\, \mathrm{d}\mathbf{x} \to \int \varphi(\mathbf{x}, V(t, \mathbf{x}), w)\, F(t, \mathrm{d}\mathbf{x}, \mathrm{d}w),$$

strongly in $L^1_{loc}(0, T)$ as $\varepsilon \to 0$, where (F, V) is the unique solution to (2.8).

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Sketch of the Proof of Theorem 2.5. Our approach is based on the application of relative entropy method [18, 19] and more recently [29], and will be explained in section 4. The first step of the proof is to introduce a relative entropy which allows us to compare solutions of (1.12) with those of (1.8), and then derive an estimate which will enable us to prove that this relative entropy tends to 0 as ε vanishes. Deriving such an estimate is the most difficult part of the analysis. More precisely, the difficulties come from: (i) the reaction term $\partial_v(f^\varepsilon N(v))$ which introduces moments of order 4 that we will need to control, and (ii) the fact that V^ε and W^ε are not a priori bounded in $L^\infty(\mathbb{R}^d)$. We will overcome this problem using two entropy inequalities that will be proved in Section 3.

The rest of the paper is devoted to the proof of Theorem 2.5, where we get *a priori* estimates uniformly with respect to $\varepsilon > 0$ on the solution $(f^\varepsilon)_{\varepsilon>0}$ constructed in Proposition 2.2 and study the behavior of a relative entropy [18, 19].

3 A priori estimates

In this section, we prove some a priori estimates of the moments of a solution to (1.7) which will be crucial for the proof of Theorem 2.5. For all $i \in \mathbb{N}$ and $z \in \{\mathbf{x}, v, w\}$, we denote by μ_i^z the moment of order i in z of f^ε, defined as

$$\mu_i^z(t) := \int_{\mathbb{R}^{d+2}} |z|^i f^\varepsilon(t, \mathbf{x}, v, w) \, \mathrm{d}\mathbf{x}\mathrm{d}v\mathrm{d}w.$$

whereas μ_i is given by

$$\mu_i(t) := \mu_i^{\mathbf{x}}(t) + \mu_i^v(t) + \mu_i^w(t).$$

Throughout this sequel, let $T > 0$ and $\varepsilon > 0$, and suppose that f^ε is a well-defined solution to (1.7) for all $t \in (0, T]$ obtained in Proposition 2.2. For any $p \in \mathbb{N}^*$, we first establish some a priori estimates on μ_{2p}^v and μ_{2p}^w.

Lemma 3.1 (Moment estimates)**.** *Consider the solution f^ε to* (1.7) *given by Proposition 2.2 and $p^* \in \mathbb{N}^*$*

$$\mu_{2p^*}^v(0) + \mu_{2p^*}^w(0) < \infty.$$

Then there exists $C > 0$, only depending on p^* and τ, such that for all $p \in [1, p^*]$,

$$\frac{1}{2p} \frac{\mathrm{d}}{\mathrm{d}t} \left[\mu_{2p}^v + \mu_{2p}^w\right](t) + \mu_{2p+2}^v(t) + \frac{1}{\varepsilon} \mathcal{D}_p(t) \leq C \left(\left[\mu_{2p}^v + \mu_{2p}^w\right](t) + 1\right), \tag{3.1}$$

where $\mathcal{D}_p(t)$ is non-negative and defined as

$$\mathcal{D}_p(t) := \int_{\mathbb{R}^{d+2}} f^\varepsilon \, v^{2p-1} \left(v - V^\varepsilon\right) \rho_0^\varepsilon \, \mathrm{d}\mathbf{x}\mathrm{d}v\mathrm{d}w.$$

Proof. Consider f^ε a well-defined solution to (1.7) and $p \in \mathbb{N}^*$, we compute the time evolution of moments in $|v|^{2p}$ and $|w|^{2p}$, hence we have

$$\frac{1}{2p} \frac{\mathrm{d}}{\mathrm{d}t} \left[\mu_{2p}^v + \mu_{2p}^w\right](t) = I_1 + I_2 + I_3 + I_4 + I_5,$$

where

$$
\begin{cases}
I_1 &:= \ + \displaystyle\int_{\mathbb{R}^{d+2}} v^{2p-1}\, N(v)\, f^\varepsilon \, \mathrm{d}\mathbf{x}\mathrm{d}v\mathrm{d}w, \\[2.2em]
I_2 &:= \ -\dfrac{1}{\varepsilon} \displaystyle\int_{\mathbb{R}^{d+2}} f^\varepsilon\, \rho^\varepsilon\, v^{2p-1}\, (v - V^\varepsilon)\, \mathrm{d}\mathbf{x}\mathrm{d}v\mathrm{d}w, \\[2.2em]
I_3 &:= \ - \displaystyle\int_{\mathbb{R}^{d+2}} v^{2p-1}\, \mathcal{K}_\Psi[f^\varepsilon]\, f^\varepsilon \, \mathrm{d}\mathbf{x}\mathrm{d}v\mathrm{d}w, \\[2.2em]
I_4 &:= \ - \displaystyle\int_{\mathbb{R}^{d+2}} v^{2p-1}\, w\, f^\varepsilon \, \mathrm{d}\mathbf{x}\mathrm{d}v\mathrm{d}w, \\[2.2em]
I_5 &:= \ + \displaystyle\int_{\mathbb{R}^{d+2}} w^{2p-1}\, A(v,w)\, f^\varepsilon \, \mathrm{d}\mathbf{x}\mathrm{d}v\mathrm{d}w.
\end{cases}
$$

First of all we treat the term I_1, since $v^{2p-1}\, N(v) = |v|^{2p} - |v|^{2p+2}$, we get

$$
I_1 = \int_{\mathbb{R}^{d+2}} \left(|v|^{2p} - |v|^{2p+2} \right) f^\varepsilon \, \mathrm{d}\mathbf{x}\mathrm{d}v\mathrm{d}w = \mu^v_{2p}(t) - \mu^v_{2p+2}(t).
$$

Furthermore, we estimate the second term I_2 and show that it is non-positive. Indeed, from the definition of V^ε, we observe that

$$
\int \rho^\varepsilon\, (V^\varepsilon)^{2p-1} \left(\int f^\varepsilon\, (v - V^\varepsilon)\, \mathrm{d}v\, \mathrm{d}w \right) \mathrm{d}\mathbf{x} = 0.
$$

Therefore, since $p \geq 1$, the function $v \mapsto v^{2p-1}$ is non-decreasing, which yields

$$
\begin{aligned}
\varepsilon\, I_2 &= \ - \int f^\varepsilon \left[\left(v^{2p-1} - (V^\varepsilon)^{2p-1} \right) + (V^\varepsilon)^{2p-1} \right] (v - V^\varepsilon)\, \rho^\varepsilon\, \mathrm{d}v\mathrm{d}w\mathrm{d}\mathbf{x} \\
&= \ - \int f^\varepsilon \left(v^{2p-1} - (V^\varepsilon)^{2p-1} \right) (v - V^\varepsilon)\, \rho^\varepsilon\, \mathrm{d}v\mathrm{d}w\mathrm{d}\mathbf{x} \leq 0.
\end{aligned}
$$

As a consequence, we obtain that $\varepsilon\, I_2 = -\mathcal{D}_p \leq 0$.

Then, we deal with I_3 and prove that it contributes to the dissipation of moments. Indeed, by symmetry of Ψ we may reformulate I_3, using the shorthand notation $f^{\varepsilon\prime} = f^\varepsilon(\mathbf{x}', v', w')$ and $f^\varepsilon = f^\varepsilon(\mathbf{x}, v, w)$, as

$$
\begin{aligned}
I_3 &= \ -\frac{1}{2} \iint \Psi(\mathbf{x} - \mathbf{x}')\, v^{2p-1} \left(v - v' \right) f^{\varepsilon\prime}(t)\, f^\varepsilon(t)\, \mathrm{d}\mathbf{x}\mathrm{d}v\mathrm{d}w\mathrm{d}\mathbf{x}'\mathrm{d}v'\mathrm{d}w' \\
&\quad -\frac{1}{2} \iint \Psi(\mathbf{x} - \mathbf{x}')\, v'^{2p-1} \left(v' - v \right) f^{\varepsilon\prime}(t)\, f^\varepsilon(t)\, \mathrm{d}\mathbf{x}\mathrm{d}v\mathrm{d}w\mathrm{d}\mathbf{x}'\mathrm{d}v'\mathrm{d}w' \\
&\leq \ 0.
\end{aligned}
$$

Finally, we can easily compute I_4, which yields

$$
\begin{aligned}
I_4 &\leq \ \int f^\varepsilon \left(\frac{2p-1}{2p} |v|^{2p} + \frac{1}{2p} |w|^{2p} \right) \mathrm{d}\mathbf{x}\mathrm{d}v\mathrm{d}w \\
&\leq \ \frac{2p-1}{2p}\, \mu_{2p}(t),
\end{aligned}
$$

whereas the last term I_5 gives

$$
\begin{aligned}
I_5 &= \tau \int \left[w^{2p-1} v + a\, w^{2p-1} - b\, |w|^{2p} \right] f^\varepsilon \, \mathrm{d}\mathbf{x} \mathrm{d}v \mathrm{d}w \\
&\leq \tau \int \left[\frac{1}{2p} |v|^{2p} + \frac{2\,p-1}{2\,p} |w|^{2p} + \frac{2\,p-1}{2\,p} |w|^{2p} + \frac{|a|^{2p}}{2p} \right] f^\varepsilon \, \mathrm{d}\mathbf{x} \mathrm{d}v \mathrm{d}w \\
&\leq \tau \frac{2p-1}{p} \mu_{2p}(t) + \frac{\tau |a|^{2p}}{2p}.
\end{aligned}
$$

Therefore, gathering the previous results, we get the entropy inequality (3.1) with $C > 0$, only depending on p, τ and a. ∎

This Lemma will be helpful to pass to the limit $\varepsilon \to 0$ in (1.7). The first consequence is some moment estimates in $\mathbf{u} = (v, w)$ and also $\mathbf{x}$.

Corollary 3.2. *Under the assumptions of Lemma 3.1 with $p^* = 2$, we choose an initial datum f_0^ε such that (2.14) is satisfied. Then, there exists a constant $C_T > 0$, which does not depend on $\varepsilon > 0$, such that for any $k \in [0, 4]$,*

$$
\begin{cases}
\mu_k(t) \leq C_T, & t \in [0, T], \\[2mm]
\displaystyle\int_0^T \mu_{k+2}^v(t) \, \mathrm{d}t \leq C_T.
\end{cases}
\tag{3.2}
$$

Proof. First we observe that

$$
\mu_0(t) = 3 \int_{\mathbb{R}^{d+2}} f^\varepsilon(t) \, \mathrm{d}v \mathrm{d}w \mathrm{d}\mathbf{x} = 3 \, \|\rho_0^\varepsilon\|_{L^1},
\tag{3.3}
$$

which gives the result with $k = 0$.

Then for $k = 4$, we apply Lemma 3.1 with $p = 2$ and integrate with respect to $t \in [0, T]$ and by the Grönwall's lemma, it yields the estimates on the second order moment in $\mathbf{u} = (v, w)$, there exists a constant $C_T > 0$ such that for any $t \in [0, T]$

$$
\mu_4^v(t) + \mu_4^w(t) \leq C_T.
$$

Furthermore, since ρ^ε does not depend on time, we also have for all $t \in [0, T]$,

$$
\mu_4^{\mathbf{x}}(t) = \int_{\mathbb{R}^d} |\mathbf{x}|^4 \, \rho_0^\varepsilon \, \mathrm{d}\mathbf{x} < \infty,
$$

hence from hypothesis (2.14) which gives us the uniform control of the moment of order 4 in $\mathbf{x}$ of f_0^ε, there exists a constant $C_T > 0$, independent of $\varepsilon > 0$, such that for all $t \in [0, T]$,

$$
\mu_4(t) \leq C_T.
\tag{3.4}
$$

On the other hand, from the latter result and the dissipative terms obtained in Lemma 3.1, there exists a constant $C_T > 0$ such that

$$
\int_0^T \mu_6^v(t) \, \mathrm{d}t \leq C_T.
\tag{3.5}
$$

Interpolating (3.3) and (3.4)-(3.5), it yields the result with $0 \leq k \leq 4$. ∎

Another consequence of Lemma 3.1 is the control of the dissipation $\mathcal{D}_1(.)$, which is a crucial step to characterize the limit of the sequence $(f^\varepsilon)_{\varepsilon>0}$ when $\varepsilon \to 0$.

Corollary 3.3. *Under the assumptions of Lemma 3.1 with $p^* = 1$, we choose an initial datum f_0^ε such that (2.14) is satisfied. Then, there exists a constant $C_T > 0$, such that*

$$\int_0^T \int_{\mathbb{R}^{d+2}} f^\varepsilon(t)\,|v - V^\varepsilon(t,\mathbf{x})|^2\,\rho_0^\varepsilon(\mathbf{x})\,\mathrm{d}\mathbf{x}\,\mathrm{d}v\,\mathrm{d}w\,\mathrm{d}t \;\le\; C_T\,\varepsilon. \qquad (3.6)$$

Proof. We first apply Corollary 3.2 to obtain a uniform estimate on the moments $\mu_2(.)$,

$$\mu_2(t) \;\le\; C_T, \quad t \in [0,T].$$

Then, integrating the entropy inequality (3.1) between 0 and T and using the positivity of $\mathcal{D}_1(.)$, we find

$$\frac{1}{\varepsilon}\int_0^T \int_{\mathbb{R}^{d+2}} f^\varepsilon(t,\mathbf{x},v,w)\,|v - V^\varepsilon(t,\mathbf{x})|^2\,\rho_0^\varepsilon(\mathbf{x})\mathrm{d}\mathbf{x}\,\mathrm{d}v\,\mathrm{d}w\,\mathrm{d}t$$

$$\le \mu_2(0) \,+\, 2\,C \int_0^T (1 + \mu_2(t))\,\mathrm{d}t \;\le\; C_T,$$

from which we easily deduce (3.6). ∎

This last result is not enough to justify the asymptotic limit. Hopefully, it can be improved by removing the weight ρ_0^ε in the previous estimate.

Lemma 3.4. *Consider the solution f^ε to (1.7) given by Proposition 2.2, where the initial datum f_0^ε satisfies (2.14). Then, there exists a constant $C_T > 0$, independent of $\varepsilon > 0$, such that*

$$\int_0^T \int_{\mathbb{R}^{d+2}} f^\varepsilon(t,\mathbf{x},v,w)|v - V^\varepsilon(t,\mathbf{x})|^2\,\mathrm{d}\mathbf{x}\,\mathrm{d}v\,\mathrm{d}w\,\mathrm{d}t \le C_T\,\varepsilon^{2/(d+6)}. \qquad (3.7)$$

Proof. Let us fix $\varepsilon > 0$ and $T > 0$. First, we set

$$I^\varepsilon \;=\; \int_0^T \int f^\varepsilon |v - V^\varepsilon|^2\,\mathrm{d}w\,\mathrm{d}v\,\mathrm{d}\mathbf{x}\,\mathrm{d}t$$

and notice that according to the definition of V^ε,

$$I^\varepsilon \;=\; \int_0^T \int f^\varepsilon \left(|v|^2 + |V^\varepsilon|^2 - 2vV^\varepsilon\right)\mathrm{d}w\,\mathrm{d}v\,\mathrm{d}\mathbf{x}\,\mathrm{d}t$$

$$\le \int_0^T \int f^\varepsilon |v|^2 \mathrm{d}w\,\mathrm{d}v\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \;<\; +\infty.$$

So it gives that $f^\varepsilon |v - V^\varepsilon|^2 \in L^1([0,T] \times \mathbb{R}^{d+2})$. Our strategy to prove (3.7) is to divide $\mathbb{R}^d$ into several subsets where $f^\varepsilon |v - V^\varepsilon|^2$ is easier to control.

We consider any $\eta > 0$ and define the set $\mathcal{A}_\varepsilon$

$$\mathcal{A}_\varepsilon := \left\{ \mathbf{x} \in \mathbb{R}^d \;\mid\; \rho_0^\varepsilon(\mathbf{x}) = 0 \right\},$$

and $\mathcal{B}_\varepsilon^\eta$ given by

$$\mathcal{B}_\varepsilon^\eta := \left\{ \mathbf{x} \in \mathbb{R}^d \;\mid\; \rho_0^\varepsilon(\mathbf{x}) > \eta \right\},$$

whereas $\mathcal{C}_\varepsilon^\eta = \mathbb{R}^d \backslash (A_\varepsilon \cup B_\varepsilon^\eta)$, that is,

$$\mathcal{C}_\varepsilon^\eta := \left\{ \mathbf{x} \in \mathbb{R}^d \mid 0 < \rho_0^\varepsilon(\mathbf{x}) \leq \eta \right\}.$$

Thus, we have $I^\varepsilon = I_1^\varepsilon + I_2^\varepsilon + I_3^\varepsilon$, where

$$\begin{cases} I_1^\varepsilon := \displaystyle\int_0^T \int_{\mathcal{A}_\varepsilon} \int f^\varepsilon |v - V^\varepsilon|^2 \mathrm{d}w \, \mathrm{d}v \, \mathrm{d}\mathbf{x} \, \mathrm{d}t, \\[2mm] I_2^\varepsilon := \displaystyle\int_0^T \int_{\mathcal{B}_\varepsilon^\eta} \int f^\varepsilon |v - V^\varepsilon|^2 \mathrm{d}w \, \mathrm{d}v \, \mathrm{d}\mathbf{x} \, \mathrm{d}t, \\[2mm] I_3^\varepsilon := \displaystyle\int_0^T \int_{\mathcal{C}_\varepsilon^\eta} \int f^\varepsilon |v - V^\varepsilon|^2 \mathrm{d}w \, \mathrm{d}v \, \mathrm{d}\mathbf{x} \, \mathrm{d}t. \end{cases}$$

On the one hand, since $f^\varepsilon \geq 0$, we directly have that $f^\varepsilon = 0$ when $\rho_0^\varepsilon = 0$, that is when $\mathbf{x} \in \mathcal{A}_\varepsilon$, thus we have

$$I_1^\varepsilon = \int_0^T \int_{\mathcal{A}_\varepsilon} \int f^\varepsilon |v - V^\varepsilon|^2 \mathrm{d}\mathbf{x}\mathrm{d}v\mathrm{d}w\mathrm{d}t = 0. \tag{3.8}$$

On the other hand, for $\mathbf{x} \in \mathcal{B}_\varepsilon^\eta$, we know that $\rho_0^\varepsilon(\mathbf{x}) > \eta$, therefore it yields that

$$\begin{aligned} I_2^\varepsilon &\leq \int_0^T \int_{\mathcal{B}_\varepsilon^\eta} \int f^\varepsilon |v - V^\varepsilon|^2 \frac{\rho_0^\varepsilon}{\eta} \, \mathrm{d}\mathbf{x} \, \mathrm{d}v \, \mathrm{d}w \, \mathrm{d}t \\[2mm] &\leq \frac{1}{\eta} \int_0^T \int f^\varepsilon |v - V^\varepsilon|^2 \rho_0^\varepsilon \, \mathrm{d}\mathbf{x} \, \mathrm{d}v \, \mathrm{d}w \, \mathrm{d}t. \end{aligned}$$

Hence, by application of Corollary 3.3, we get the following estimate:

$$I_2^\varepsilon = \mathcal{O}\left(\frac{\varepsilon}{\eta}\right). \tag{3.9}$$

It remains to control the last term I_3^ε. To this aim we bound it by the sum of three terms : for any $R > 0$, we have

$$I_3^\varepsilon \leq \int_0^T \int_{\mathcal{C}_\varepsilon^\eta} \int f^\varepsilon |v|^2 \mathrm{d}v \, \mathrm{d}\mathbf{x} \, \mathrm{d}w \, \mathrm{d}t =: I_{3,1}^\varepsilon + I_{3,2}^\varepsilon + I_{3,3}^\varepsilon,$$

where

$$\begin{cases} I_{3,1}^\varepsilon := \displaystyle\int_0^T \int_{\mathcal{C}_\varepsilon^\eta} \int_{\{|v|>R\}} f^\varepsilon |v|^2 \mathrm{d}v \, \mathrm{d}\mathbf{x} \, \mathrm{d}w \, \mathrm{d}t, \\[2mm] I_{3,2}^\varepsilon := \displaystyle\int_0^T \int_{\mathcal{C}_\varepsilon^\eta \cap B^c(0,R)} \int_{\{|v|\leq R\}} f^\varepsilon |v|^2 \mathrm{d}v \, \mathrm{d}\mathbf{x} \, \mathrm{d}w \, \mathrm{d}t, \\[2mm] I_{3,3}^\varepsilon := \displaystyle\int_0^T \int_{\mathcal{C}_\varepsilon^\eta \cap B(0,R)} \int_{\{|v|\leq R\}} f^\varepsilon |v|^2 \mathrm{d}v \, \mathrm{d}\mathbf{x} \, \mathrm{d}w \, \mathrm{d}t. \end{cases}$$

For $k > 2$, the estimates on moments in velocity gives that

$$I_{3,1}^\varepsilon \leq \frac{1}{R^{k-2}} \int_0^T \int f^\varepsilon |v|^k \, \mathrm{d}v \, \mathrm{d}\mathbf{x} \, \mathrm{d}w \, \mathrm{d}t = \frac{1}{R^{k-2}} \int_0^T \mu_k^v(t) \, \mathrm{d}t, \tag{3.10}$$

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where the last term is uniformly bounded according to Corollary 3.2 for $k \in [0, 6]$.

Furthermore, we estimate $I_{3,2}^\varepsilon$ using a similar argument as for $I_{3,1}^\varepsilon$ but now using moments in space, that is, for $p \in [2, 4]$,

$$I_{3,2}^\varepsilon \leq \int_0^T \int \frac{|\mathbf{x}|^p}{R^p} f^\varepsilon(t) R^2 \, \mathrm{d}w \, \mathrm{d}v \, \mathrm{d}\mathbf{x} \, \mathrm{d}t \leq \frac{1}{R^{p-2}} \int_0^T \mu_p^{\mathbf{x}}(t) \, \mathrm{d}t, \qquad (3.11)$$

where $\mu_p^{\mathbf{x}}(.)$ is uniformly bounded according to Corollary 3.2 for $p \in [0, 4]$. Finally, the last term $I_{3,3}^\varepsilon$ can be computed as

$$I_{3,3}^\varepsilon \leq R^2 \int_0^T \int_{\mathcal{C}_\varepsilon^\eta \cap B(0,R)} \rho_0^\varepsilon \, \mathrm{d}\mathbf{x} \, \mathrm{d}t \leq \tilde{C} \, T \, R^{d+2} \, \eta, \qquad (3.12)$$

where $\tilde{C}$ is the positive constant such that $|B(0,R)| = \tilde{C} R^d$. By summing (3.10), (3.11) and (3.12), we can conclude that there exists a constant $C = C(T)$ independent on R, η and ε such that

$$I_3^\varepsilon \leq C \left(R^{d+2} \eta + \frac{1}{R^{p-2}} + \frac{1}{R^{k-2}} \right).$$

For simplicity we choose $p = k = 4$ and optimize the value of R, which leads to

$$I_3^\varepsilon \leq C_T \, \eta^{2/(d+4)}. \qquad (3.13)$$

Finally by summing (3.8), (3.9) and (3.13), we get that there exists a positive constant $C_T > 0$ independent on η and ε, such that,

$$I^\varepsilon \leq C_T \left(\eta^{2/(d+4)} + \frac{\varepsilon}{\eta} \right).$$

Hence, we can choose $\eta = \varepsilon^{(d+4)/(d+6)}$ and

$$\int_0^T \int f^\varepsilon |v - V^\varepsilon|^2 \, \mathrm{d}\mathbf{x} \, \mathrm{d}v \, \mathrm{d}w \, \mathrm{d}t \leq C_T \, \varepsilon^{2/(d+6)}.$$

■

4   Relative entropy estimate and proof of Theorem 2.5

Following ideas from [21, 29], our proof of Theorem 2.5 relies on a relative entropy argument to estimate the distance between a solution of (1.8) and a solution of (1.12) on $[0, T]$ for some finite $T > 0$ as $\varepsilon \to 0$, with well-prepared initial conditions. Throughout this section, by a solution $(\rho_0, \rho_0 V, \rho_0 W)$ of (1.12), we refer to the solution constructed in Proposition 2.3. We start this section with the definition of the relative entropy that we will be using. Then, we present an equality satisfied by the relative entropy, which will be useful to estimate it. This estimate will finally be the key argument to conclude the proof.

4.1   Definition of relative entropy

We want to use a relative entropy argument which enables us to compare solutions of (1.12) with the solutions of (1.7) seen on its hydrodynamic form (1.8). We first introduce the notion of entropy.

Definition 4.1 (Entropy). *For all functions V and $W : \mathbb{R}^d \longrightarrow \mathbb{R}$, and for any non-negative function $\rho : \mathbb{R}^d \to \mathbb{R}$, we define for $\mathcal{Z} = (\rho, \rho V, \rho W)$, the entropy $\eta(\mathcal{Z})$ by*

$$\eta(\mathcal{Z}) := \rho \, \frac{|V|^2 + |W|^2}{2}. \tag{4.1}$$

Note that if we define $P = \rho V$ and $Q = \rho W$, then we have

$$\eta(\mathcal{Z}) = \frac{P^2 + Q^2}{2\,\rho}.$$

As a consequence, the differential of η with respect to its variables $\mathcal{Z}$ is given by

$$D\eta(\mathcal{Z}) = \begin{pmatrix} D_\rho\eta \\ D_P\eta \\ D_Q\eta \end{pmatrix} = \begin{pmatrix} -\dfrac{|V|^2 + |W|^2}{2} \\ V \\ W \end{pmatrix}. \tag{4.2}$$

Using the definition of entropy (4.1), we can now introduce the notion of relative entropy or Bregman divergence.

Definition 4.2 (Relative Entropy). *For all functions V_1, W_1, V_2 and $W_2 : \mathbb{R}^d \to \mathbb{R}$ and for all non-negative functions ρ_1 and $\rho_2 : \mathbb{R}^d \to \mathbb{R}$, we define for $\mathcal{Z}_i = (\rho_i, \rho_i V_i, \rho_i W_i)$, with $i = 1, 2$, the relative entropy*

$$\eta(\mathcal{Z}_1|\mathcal{Z}_2) := \eta(\mathcal{Z}_1) - \eta(\mathcal{Z}_2) - D\eta(\mathcal{Z}_2) \cdot (\mathcal{Z}_1 - \mathcal{Z}_2)$$

which gives us after computation

$$\eta(\mathcal{Z}_1|\mathcal{Z}_2) = \rho_1 \, \frac{|V_2 - V_1|^2 + |W_2 - W_1|^2}{2}. \tag{4.3}$$

This relative entropy will be useful to "compare" the weak solution $(\rho_0^\varepsilon, \rho_0^\varepsilon V^\varepsilon, \rho_0^\varepsilon W^\varepsilon)$ to (1.8) with the classical solution $(\rho_0, \rho_0 V, \rho_0 W)$ to (1.12). Let us finally remark that in the simplified case where we assume $\rho_0 = \rho_0^\varepsilon$, then η reduces to the Euclidean norm $\|\cdot\|_2$.

4.2 Relative entropy equality

In this subsection, we prove an equality satisfied by the relative entropy defined by (4.3) between an arbitrary smooth function and a solution of the hydrodynamic equations (1.12). The purpose of this result is to split the relative entropy dissipation into one part due to the macroscopic solution (1.12) and another part which estimates the difference between the two solutions.

Lemma 4.3. *Under the assumption that Ψ is non-negative, symmetric and satisfies (2.1), we consider (V, W) the solution to the hydrodynamic equation (2.6) given by Proposition 2.3. Then, for any $\widetilde{\mathcal{Z}} = (\widetilde{\rho}, \widetilde{\rho}\widetilde{V}, \widetilde{\rho}\widetilde{W})$ such that $\widetilde{\rho}$ is non-negative, $\widetilde{\rho} \in L^1 \cap L^\infty(\mathbb{R}^d)$, whereas $\widetilde{V}$ and $\widetilde{W}$ are both differentiable in time and such that for any $t \in [0, T]$,*

$$\widetilde{\rho}\left(|\widetilde{V}(t)|^4 + |\widetilde{W}(t)|^4\right) \in L^1(\mathbb{R}^d),$$

the following equality holds:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \int \eta(\widetilde{\mathcal{Z}}|\mathcal{Z})\,\mathrm{d}\mathbf{x} \;=\; & \frac{\mathrm{d}}{\mathrm{d}t} \int \eta(\widetilde{\mathcal{Z}})\mathrm{d}\mathbf{x} \\
& - \int D\eta(\mathcal{Z})\left[\partial_t\widetilde{\mathcal{Z}} - \mathcal{F}(\widetilde{\mathcal{Z}})\right]\mathrm{d}\mathbf{x} + \mathcal{R}(\widetilde{\mathcal{Z}}|\mathcal{Z}) + \mathcal{S}(\widetilde{\mathcal{Z}}),
\end{aligned} \tag{4.4}$$

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with $\mathcal{R} = \mathcal{R}_l + \mathcal{R}_{nl}$ and $\mathcal{S} = \mathcal{S}_l + \mathcal{S}_{nl}$, where $\mathcal{R}_l$ and $\mathcal{S}_l$ contain local terms

$$
\begin{cases}
\mathcal{R}_l(\widetilde{\mathcal{Z}}|\mathcal{Z}) \;:=\; \displaystyle\int \widetilde{\rho}\left(V - \widetilde{V}\right)\left(N(V) - W - N(\widetilde{V}) + \widetilde{W}\right)\mathrm{d}\mathbf{x} \\[2mm]
\qquad\qquad + \displaystyle\int \widetilde{\rho}\left(W - \widetilde{W}\right)\left(A(V,W) - A(\widetilde{V},\widetilde{W})\right)\mathrm{d}\mathbf{x}, \\[3mm]
\mathcal{S}_l(\widetilde{\mathcal{Z}}) \;:=\; -\displaystyle\int \widetilde{\rho}\left[\widetilde{V}\,N(\widetilde{V}) - \widetilde{V}\,\widetilde{W} + \widetilde{W}\,A(\widetilde{V},\widetilde{W})\right]\mathrm{d}\mathbf{x}.
\end{cases}
\tag{4.5}
$$

whereas $\mathcal{R}_{nl}(\widetilde{\mathcal{Z}}|\mathcal{Z})$ and $\mathcal{S}_{nl}(\widetilde{\mathcal{Z}})$ gather nonlocal terms

$$
\begin{cases}
\mathcal{R}_{nl}(\widetilde{\mathcal{Z}}|\mathcal{Z}) \;:=\; \displaystyle\int \widetilde{\rho}\left[(V - \widetilde{V})\left(\mathcal{L}_{\rho_0}(V) - \mathcal{L}_{\widetilde{\rho}}(\widetilde{V})\right)\right]\mathrm{d}\mathbf{x}, \\[3mm]
\mathcal{S}_{nl}(\widetilde{\mathcal{Z}}) \;:=\; -\dfrac{1}{2}\displaystyle\iint \Psi(\mathbf{x} - \mathbf{y})\,\widetilde{\rho}(\mathbf{x})\,\widetilde{\rho}(\mathbf{y})\left|\widetilde{V}(t,\mathbf{x}) - \widetilde{V}(t,\mathbf{y})\right|^2 \mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}
\end{cases}
\tag{4.6}
$$

Proof. First of all, it is worth noticing that for all $t \in [0;T]$, since $V(t)$ and $W(t) \in L^\infty(\mathbb{R}^d)$ and since $\rho_0 \in L^1(\mathbb{R}^d)$, then for all $i \in \mathbb{N}$, $\rho_0\left(|V(t)|^i + |W(t)|^i\right) \in L^1(\mathbb{R}^d)$. From the definition (4.2) of $\eta(\widetilde{\mathcal{Z}}|\mathcal{Z})$, we have

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int \eta(\widetilde{\mathcal{Z}}|\mathcal{Z})\,\mathrm{d}\mathbf{x} &= \int\left[\partial_t\eta(\widetilde{\mathcal{Z}}) - \partial_t\eta(\mathcal{Z}) - \partial_t D\eta(\mathcal{Z})\cdot\left(\widetilde{\mathcal{Z}} - \mathcal{Z}\right) - D\eta(\mathcal{Z})\cdot\partial_t\left(\widetilde{\mathcal{Z}} - \mathcal{Z}\right)\right]\mathrm{d}\mathbf{x} \\
&= I_1 + I_2,
\end{aligned}
$$

with

$$
\begin{cases}
I_1 \;:=\; \displaystyle\int \partial_t\eta(\widetilde{\mathcal{Z}}) - D\eta(\mathcal{Z})\cdot\left[\partial_t\widetilde{\mathcal{Z}} - \mathcal{F}(\widetilde{\mathcal{Z}})\right]\mathrm{d}\mathbf{x}, \\[3mm]
I_2 \;:=\; -\displaystyle\int\left[\partial_t D\eta(\mathcal{Z})\cdot\left(\widetilde{\mathcal{Z}} - \mathcal{Z}\right) + D\eta(\mathcal{Z})\cdot\mathcal{F}(\widetilde{\mathcal{Z}})\right]\mathrm{d}\mathbf{x},
\end{cases}
$$

where $\mathcal{F}$ is defined in (1.12)-(1.13).

On the one hand, the term I_1 corresponds to the variation of entropy which simply gives

$$
I_1 = \frac{\mathrm{d}}{\mathrm{d}t}\int \eta(\widetilde{\mathcal{Z}})\,\mathrm{d}\mathbf{x} - \int D\eta(\mathcal{Z})\cdot\left[\partial_t\widetilde{\mathcal{Z}} - \mathcal{F}(\widetilde{\mathcal{Z}})\right]\mathrm{d}\mathbf{x}.
\tag{4.7}
$$

On the other hand, we decompose I_2 as $I_2 = I_{21} + I_{22}$ with

$$
\begin{cases}
I_{21} \;:=\; \displaystyle\int \partial_t D\eta(\mathcal{Z})\cdot\left(\widetilde{\mathcal{Z}} - \mathcal{Z}\right)\mathrm{d}\mathbf{x}, \\[3mm]
I_{22} \;:=\; -\displaystyle\int D\eta(\mathcal{Z})\cdot\mathcal{F}(\widetilde{\mathcal{Z}})\mathrm{d}\mathbf{x}.
\end{cases}
$$

Using the definition of $D\eta(\mathcal{Z})$ in (4.2) and since $\mathcal{Z}$ is solution to (1.12), we have

$$I_{21} := -\int \partial_t \begin{pmatrix} -\dfrac{|V|^2 + |W|^2}{2} \\ V \\ W \end{pmatrix} \cdot \begin{pmatrix} \widetilde{\rho} - \rho_0 \\ \widetilde{\rho}\widetilde{V} - \rho_0\, V \\ \widetilde{\rho}\widetilde{W} - \rho_0\, W \end{pmatrix} \mathrm{d}\mathbf{x}$$

$$= \int (\widetilde{\rho} - \rho_0)\, [V\, (\mathcal{L}_{\rho_0}(V) + N(V) - W) + W\, A(V,W)]\, \mathrm{d}\mathbf{x}$$

$$- \int \left(\widetilde{\rho}\widetilde{V} - \rho_0 V\right) (\mathcal{L}_{\rho_0}(V) + N(V) - W)\, \mathrm{d}\mathbf{x}$$

$$- \int \left(\widetilde{\rho}\widetilde{W} - \rho_0\, W\right) A(V,W)\, \mathrm{d}\mathbf{x},$$

hence it yields

$$I_{21} = \int \widetilde{\rho}\left[\left(V - \widetilde{V}\right)[\mathcal{L}_{\rho_0}(V) + N(V) - W] + A(V,W)\left(W - \widetilde{W}\right)\right] \mathrm{d}\mathbf{x}.$$

Furthermore, from the definition of $\mathcal{F}(\widetilde{\mathcal{Z}})$ in (1.12)-(1.13) and $D\eta(\mathcal{Z})$ in (4.2), we obtain

$$I_{22} = -\int \widetilde{\rho}\left(V\left[\mathcal{L}_{\widetilde{\rho}}(\widetilde{V}) + N(\widetilde{V}) - \widetilde{W}\right] + W\, A(\widetilde{V}, \widetilde{W})\right) \mathrm{d}\mathbf{x}.$$

Then, gathering the latter two equalities and after reordering, we have

$$I_2 = \int \widetilde{\rho}\left[(V - \widetilde{V})\left(\mathcal{L}_{\rho_0}(V) - \mathcal{L}_{\widetilde{\rho}}(\widetilde{V})\right) - \widetilde{V}\,\mathcal{L}_{\widetilde{\rho}}(\widetilde{V})\right] \mathrm{d}\mathbf{x}$$

$$+ \int \widetilde{\rho}\left[\left(V - \widetilde{V}\right)\left(N(V) - N(\widetilde{V}) - W + \widetilde{W}\right) - \widetilde{V}\, N(\widetilde{V}) + \widetilde{V}\,\widetilde{W}\right] \mathrm{d}\mathbf{x}$$

$$+ \int \widetilde{\rho}\left[\left(W - \widetilde{W}\right)\left(A(V,W) - A(\widetilde{V},\widetilde{W})\right) - \widetilde{W}\, A(\widetilde{V},\widetilde{W})\right] \mathrm{d}\mathbf{x},$$

which can be also written as

$$I_2 = \mathcal{I}_\psi + \mathcal{R}_l(\widetilde{\mathcal{Z}}|\mathcal{Z}) + \mathcal{S}_l(\widetilde{\mathcal{Z}}), \tag{4.8}$$

where $\mathcal{R}_l$ and $\mathcal{S}_l$ are given in (4.5) whereas the first term $\mathcal{I}_\psi$ is given by

$$\mathcal{I}_\psi := \int \widetilde{\rho}\left[(V - \widetilde{V})\left(\mathcal{L}_{\rho_0}(V) - \mathcal{L}_{\widetilde{\rho}}(\widetilde{V})\right) - \widetilde{V}\,\mathcal{L}_{\widetilde{\rho}}(\widetilde{V})\right] \mathrm{d}\mathbf{x}.$$

Thus, we set

$$\begin{cases} \mathcal{R}_{nl} = \displaystyle\int \widetilde{\rho}\left[(V - \widetilde{V})\left(\mathcal{L}_{\rho_0}(V) - \mathcal{L}_{\widetilde{\rho}}(\widetilde{V})\right)\right] \mathrm{d}\mathbf{x}, \\[2ex] \mathcal{S}_{nl} = -\displaystyle\int \widetilde{\rho}\,\widetilde{V}\,\mathcal{L}_{\widetilde{\rho}}(\widetilde{V})\,\mathrm{d}\mathbf{x} \end{cases}$$

and a direct computation gives

$$\mathcal{S}_{nl} = -\frac{1}{2}\iint \Psi(\mathbf{x} - \mathbf{y})\,\widetilde{\rho}(\mathbf{x})\,\widetilde{\rho}(\mathbf{y})\left|\widetilde{V}(t,\mathbf{x}) - \widetilde{V}(t,\mathbf{y})\right|^2 \mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}.$$

We conclude the proof by gathering the last equality together with (4.7) and (4.8). ∎

4.3 Relative entropy estimate

Now that we have established the relative entropy equality (4.4), we apply it with $\widetilde{\rho} = \rho_0^\varepsilon$, $\widetilde{V} = V^\varepsilon$ and $\widetilde{W} = W^\varepsilon$ to estimate the relative entropy between the weak solution $(V^\varepsilon, W^\varepsilon)$ and the classical solution (V, W). More precisely, we prove the following result.

Proposition 4.4. *Under the assumptions of Theorem 2.5, there exists $C_T > 0$ such that we have for all $t \in (0, T]$:*

$$\int_{\mathbb{R}^d} \rho_0^\varepsilon(\mathbf{x}) \left(|V^\varepsilon(t, \mathbf{x}) - V(t, \mathbf{x})|^2 + |W^\varepsilon(t, \mathbf{x}) - W(t, \mathbf{x})|^2 \right) \mathrm{d}\mathbf{x} \leq C_T \, \varepsilon^{1/(d+6)}, \qquad (4.9)$$

where (V, W) is the solution to (2.6) and $(\rho_0^\varepsilon, \rho_0^\varepsilon V^\varepsilon, \rho_0^\varepsilon W^\varepsilon)$ are the macroscopic quantities computed from f^ε the solution to (1.7) on $[0, T]$.

Proof. Consider f^ε a solution to (1.7) on $[0, T]$ given in Proposition 2.2. From Corollary 3.2, we get that for any $t \in [0, T]$, the moment $\mu_4(t)$ is uniformly bounded with respect to $\varepsilon > 0$. Therefore applying the Hölder inequality, we obtain that for all $\mathbf{x} \in \mathbb{R}^d$ such that $\rho_0^\varepsilon(\mathbf{x}) > 0$ and for all $t \in [0; T]$,

$$\rho_0^\varepsilon(\mathbf{x}) |V^\varepsilon(t, \mathbf{x})|^4 = \frac{1}{|\rho_0^\varepsilon(\mathbf{x})|^3} \left(\int v f^\varepsilon(t, \mathbf{x}, v, w) \mathrm{d}v \, \mathrm{d}w \right)^4,$$

$$\leq \int |v|^4 f^\varepsilon(t, \mathbf{x}, v, w) \, \mathrm{d}v \, \mathrm{d}w. \qquad (4.10)$$

Note that the last inequality remains true when $\rho_0^\varepsilon(\mathbf{x}) = 0$ and the same argument applies when we replace V^ε by W^ε.

Consequently, since $\rho^\varepsilon \in L^1(\mathbb{R}^d)$, we get for any $0 \leq p \leq 4$ and $t \in [0, T]$,

$$\rho_0^\varepsilon \left(|V^\varepsilon(t)|^p + |W^\varepsilon(t)|^p \right) \in L^1(\mathbb{R}^d).$$

Thus, we can compute the time evolution of the entropy $\eta(\mathcal{Z}^\varepsilon)$ where $\mathcal{Z}^\varepsilon = (\rho_0^\varepsilon, \rho_0^\varepsilon V^\varepsilon, \rho_0^\varepsilon W^\varepsilon)$ corresponds to the moments of f^ε with respect to $(1, v, w)$, that is $\mathcal{Z}^\varepsilon$ is solution to (1.8). It yields that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int \eta(\mathcal{Z}^\varepsilon(t)) \, \mathrm{d}\mathbf{x} + \mathcal{S}(\mathcal{Z}^\varepsilon(t)) = \int V^\varepsilon(t) \, \mathcal{E}(f^\varepsilon(t)) \, \mathrm{d}\mathbf{x}, \qquad (4.11)$$

where $\mathcal{S}(\mathcal{Z}^\varepsilon)$ is given by (4.5)-(4.6) when $\widetilde{\mathcal{Z}} = \mathcal{Z}^\varepsilon$ and the error term $\mathcal{E}(f^\varepsilon)$ is defined in (1.10).

Then we consider (V, W) the solution to (2.6) given in Proposition 2.3, hence we have for $\mathcal{Z} = (\rho_0, \rho_0 V, \rho_0 W)$,

$$\int D\eta(\mathcal{Z}(t)) \left[\partial_t \mathcal{Z}^\varepsilon(t) - \mathcal{F}(\mathcal{Z}^\varepsilon(t)) \right] \mathrm{d}\mathbf{x} = \int V(t) \, \mathcal{E}(f^\varepsilon(t)) \mathrm{d}\mathbf{x}. \qquad (4.12)$$

Therefore, applying Lemma 4.3 with $\widetilde{\mathcal{Z}} = (\rho_0^\varepsilon, \rho_0^\varepsilon V^\varepsilon, \rho_0^\varepsilon W^\varepsilon)$ and using (4.11) and (4.12), we simply get the following equality

$$\frac{d}{dt} \int \eta(\mathcal{Z}^\varepsilon(t) | \mathcal{Z}(t)) \, \mathrm{d}\mathbf{x} = \int (V^\varepsilon - V) \, \mathcal{E}(f^\varepsilon)(t, \mathbf{x}) \mathrm{d}\mathbf{x} + \mathcal{R}(\mathcal{Z}^\varepsilon | \mathcal{Z}), \qquad (4.13)$$

where $\mathcal{R} = \mathcal{R}_l + \mathcal{R}_{nl}$ is given in (4.5)-(4.6). On the one hand, we estimate the term $\mathcal{R}_l$ by

$$
\begin{aligned}
|\mathcal{R}_l(\mathcal{Z}^\varepsilon|\mathcal{Z})| &\leq \left| \int \rho_0^\varepsilon \, (V - V^\varepsilon) \, (N(V) - N(V^\varepsilon) - (W - W^\varepsilon)) \, \mathrm{d}\mathbf{x} \right| \\
&\quad + \left| \int \rho_0^\varepsilon \, (W - W^\varepsilon) \, (A(V, W) - A(V^\varepsilon, W^\varepsilon)) \, \mathrm{d}\mathbf{x} \right| \\
&\leq \int \left(|V - V^\varepsilon|^2 + \frac{1}{2} \left(|V - V^\varepsilon|^2 + |W - W^\varepsilon|^2 \right) \right) \rho_0^\varepsilon \, \mathrm{d}\mathbf{x} \\
&\quad + \frac{\tau}{2} \int \left(|V - V^\varepsilon|^2 + |W - W^\varepsilon|^2 \right) \rho_0^\varepsilon \, \mathrm{d}\mathbf{x} \\
&\leq \frac{3 + \tau}{2} \int \eta(\mathcal{Z}^\varepsilon|\mathcal{Z}) \mathrm{d}\mathbf{x}.
\end{aligned}
$$

On the other hand, we estimate the second term $\mathcal{R}_{nl}(\mathcal{Z}^\varepsilon|\mathcal{Z})$ as

$$
\begin{aligned}
|\mathcal{R}_{nl}(\mathcal{Z}^\varepsilon|\mathcal{Z})| &\leq \iint \Psi(\mathbf{x} - \mathbf{y}) \, \rho_0^\varepsilon(\mathbf{x}) \, |\rho_0(\mathbf{y}) - \rho_0^\varepsilon(\mathbf{y})| \, |V(t, \mathbf{y}) - V(t, \mathbf{x})| \, |V(t, \mathbf{x}) - V^\varepsilon(t, \mathbf{x})| \, \mathrm{d}\mathbf{x} \, \mathrm{d}\mathbf{y} \\
&\leq \|V\|_{L^\infty} \|\Psi\|_{L^1} \left(\|\rho_0^\varepsilon\|_{L^\infty} \|\rho_0 - \rho_0^\varepsilon\|_{L^2}^2 + \int \eta(\mathcal{Z}^\varepsilon|\mathcal{Z}) \mathrm{d}\mathbf{x} \right).
\end{aligned}
$$

Gathering these last inequalities, using the uniform control of $\|\rho_0^\varepsilon\|_{L^\infty}$ given by hypothesis (2.13), we have shown that there exists a constant $C_T > 0$, which does not depend on $\varepsilon > 0$ such that for all $t \in [0, T]$,

$$
\int_0^t |\mathcal{R}(\mathcal{Z}^\varepsilon(s)|\mathcal{Z}(s))| \mathrm{d}s \leq C_T \left[\|\rho_0 - \rho_0^\varepsilon\|_{L^2}^2 + \int_0^t \int \eta(\mathcal{Z}^\varepsilon(s)|\mathcal{Z}(s)) \, \mathrm{d}\mathbf{x} \, \mathrm{d}s \right]. \tag{4.14}
$$

It remains to estimate the error term

$$
\begin{aligned}
\left| \int (V^\varepsilon(t) - V(t)) \, \mathcal{E}(f^\varepsilon)(t) \mathrm{d}\mathbf{x} \right| &\leq \frac{3}{2} \int |V^\varepsilon(t) - V(t)| \, |V^\varepsilon(t) - v| \left[(V^\varepsilon(t))^2 + v^2 \right] f^\varepsilon(t) \mathrm{d}v \mathrm{d}w \mathrm{d}\mathbf{x} \\
&\leq \alpha(t) \left(\int |V^\varepsilon(t) - v|^2 \, f^\varepsilon(t) \, \mathrm{d}v \, \mathrm{d}w \, \mathrm{d}\mathbf{x} \right)^{1/2},
\end{aligned}
$$

where $\alpha(t)$ is given by

$$
\alpha(t) := \frac{3}{2} \left(\int \left[(V^\varepsilon(t))^2 + v^2 \right]^2 \left[V^\varepsilon(t) - V(t) \right]^2 \, f^\varepsilon(t) \, \mathrm{d}v \, \mathrm{d}w \, \mathrm{d}\mathbf{x} \right)^{1/2}.
$$

Using that V is uniformly bounded in L^∞ according to Proposition 2.3 and since

$$
\rho_0^\varepsilon \, |V^\varepsilon(t, \mathbf{x})|^6 \leq \int |v|^6 f^\varepsilon(t, \mathbf{x}, v, w) \mathrm{d}v \mathrm{d}w
$$

we deduce from Corollary 3.2 that there exists a constant $C_T > 0$, which does not depend on ε, such that

$$
\int_0^T \alpha^2(s) \, \mathrm{d}s \leq C_T.
$$

It yields from Lemma 3.7 and the Cauchy-Schwarz inequality that

$$
\int_0^T \left| \int (V^\varepsilon(t) - V(t)) \, \mathcal{E}(f^\varepsilon)(t) \mathrm{d}\mathbf{x} \right| \mathrm{d}t \leq C_T \, \varepsilon^{1/(d+6)}. \tag{4.15}
$$

Finally integrating (4.13) on the time interval $[0, t]$ we get from the previous estimates (4.14) and (4.15) and using the Grönwall's lemma that

$$\int \eta(\mathcal{Z}^\varepsilon(t)|\mathcal{Z}(t))\,\mathrm{d}\mathbf{x} \;\leq\; C_T \left[\int \eta(\mathcal{Z}^\varepsilon(0)|\mathcal{Z}(0))\,\mathrm{d}\mathbf{x} + \|\rho_0 - \rho_0^\varepsilon\|_{L^2}^2 + \varepsilon^{1/(d+6)} \right]$$

From the assumption (2.15), we get that for all $t \in [0, T]$,

$$\int \eta(\mathcal{Z}^\varepsilon(t)|\mathcal{Z}(t))\,\mathrm{d}\mathbf{x} \;\leq\; C_T\, \varepsilon^{1/(d+6)},$$

which concludes the proof. ∎

4.4 Conclusion − Proof of Theorem 2.5

In this section, we complete the proof of Theorem 2.5 using the entropy estimates previously established to show the convergence of f^ε in the limit $\varepsilon \to 0$.

First, we set

$$F^\varepsilon(t, \mathbf{x}, w) := \int f^\varepsilon(t, \mathbf{x}, v, w)\,\mathrm{d}v,$$

with an initial datum F_0^ε given by

$$F_0^\varepsilon = \int_{\mathbb{R}} f_0^\varepsilon\,\mathrm{d}v.$$

Noticing that since f^ε is compactly supported in v for any $\varepsilon > 0$, we can choose a test function in (2.2) independent of $v \in \mathbb{R}$, hence the distribution F^ε satisfies the following equation, (1.7)

$$\int_0^T \int_{\mathbb{R}^{d+1}} F^\varepsilon\, \partial_t \varphi \,+\, \tau \left[\int_{\mathbb{R}} v f^\varepsilon \mathrm{d}v \,+\, (a - b\,w)\, F^\varepsilon \right] \partial_w \varphi \,\mathrm{d}\mathbf{x}\mathrm{d}w\mathrm{d}t$$
$$+ \int_{\mathbb{R}^{d+1}} F_0^\varepsilon\, \varphi(0)\,\mathrm{d}\mathbf{x}\mathrm{d}w \,=\, 0, \quad \forall \varphi \in \mathscr{C}_c^\infty([0,T] \times \mathbb{R}^{d+1}),$$

or after reordering

$$\int_0^T \int_{\mathbb{R}^{d+1}} F^\varepsilon\, [\partial_t \varphi \,+\, A(V(t,\mathbf{x}), w)\, \partial_w \varphi]\, \mathrm{d}\mathbf{x}\mathrm{d}w\mathrm{d}t \,+\, \int_{\mathbb{R}^{d+1}} F_0^\varepsilon\, \varphi(0)\,\mathrm{d}\mathbf{x}\mathrm{d}w$$
$$= \tau \int_0^T \int_{\mathbb{R}^{d+2}} (V(t,\mathbf{x}) - v)\, f^\varepsilon\, \partial_w \varphi\, \mathrm{d}v\mathrm{d}\mathbf{x}\mathrm{d}w\mathrm{d}t, \quad \forall \varphi \in \mathscr{C}_c^1([0,T] \times \mathbb{R}^{d+1}),$$

where V is solution to (1.12).

On the one hand, using that up to a subsequence F^ε converges weakly-$\star$ in $\mathcal{M}((0,T) \times \mathbb{R}^{d+1})$ to a limit $F \in \mathcal{M}((0,T) \times \mathbb{R}^{d+1})$, we can pass to the limit on the left hand side by linearity. On the other hand, from Lemma 3.4 and Proposition 4.4, we get when $\varepsilon \to 0$,

$$\int_0^T \int f^\varepsilon |v - V(t,\mathbf{x})|^2 \mathrm{d}\mathbf{x}\mathrm{d}v\mathrm{d}w\mathrm{d}s \;\leq\; \int_0^T \int f^\varepsilon \left(|v - V^\varepsilon(t,\mathbf{x})|^2 + |V^\varepsilon(t,\mathbf{x}) - V(t,\mathbf{x})|^2 \right) \mathrm{d}\mathbf{z}\mathrm{d}s$$
$$\leq\; C_T\, \varepsilon^{1/(d+6)},$$

hence it yields that since ρ^ε does not depend on time,

$$\left| \int (V(t,\mathbf{x}) - v)\, f^\varepsilon\, \partial_w \varphi\, \mathrm{d}v\mathrm{d}\mathbf{x}\mathrm{d}w\mathrm{d}t \right| \leq C_T\, \|\partial_w \varphi\|_{L^\infty}\, \|\rho_0^\varepsilon\|_{L^1}^{1/2}\, \varepsilon^{1/(2d+12)}.$$

Thus, passing to the limit $\varepsilon \to 0$, it proves that F is a measure solution (2.8). Furthermore, by uniqueness of the solution to (2.8), we get the convergence for the sequence $(F^\varepsilon)_{\varepsilon > 0}$.

Finally let us show that for any $\varphi \in \mathscr{C}_b^0(\mathbb{R}^{d+2})$,

$$\int \varphi(\mathbf{x}, v, w)\, f^\varepsilon(t, \mathbf{x}, v, w)\, \mathrm{d}v\, \mathrm{d}w\, \mathrm{d}\mathbf{x} \to \int \varphi(\mathbf{x}, V(t, \mathbf{x}), w)\, F(t, \mathrm{d}\mathbf{x}, \mathrm{d}w),$$

strongly in $L_{\mathrm{loc}}^1(0, T)$ as $\varepsilon \to 0$. Consider $0 < t < t' \leq T$. We start with showing the convergence for any $\varphi \in \mathscr{C}_c^1(\mathbb{R}^{d+2})$, and then we will conclude using a density argument. Consider $\varphi \in \mathscr{C}_c^1(\mathbb{R}^{d+2})$, we have:

$$\begin{aligned}
\mathcal{I} &:= \int_t^{t'} \left| \int \varphi(\mathbf{x}, v, w)\, f^\varepsilon(s, \mathbf{x}, v, w)\, \mathrm{d}v\, \mathrm{d}w\, \mathrm{d}\mathbf{x} - \int \varphi(\mathbf{x}, V(s, \mathbf{x}), w)\, F(s, \mathrm{d}\mathbf{x}, \mathrm{d}w) \right| \mathrm{d}s \\
&\leq \mathcal{I}_1 + \mathcal{I}_2,
\end{aligned}$$

where

$$\begin{cases}
\mathcal{I}_1 &:= \displaystyle\int_t^{t'} \left| \int (\varphi(\mathbf{x}, v, w) - \varphi(\mathbf{x}, V(s, \mathbf{x}), w))\, f^\varepsilon(s, \mathbf{x}, v, w)\, \mathrm{d}v\, \mathrm{d}w\, \mathrm{d}\mathbf{x} \right| \mathrm{d}s, \\
\mathcal{I}_2 &:= \displaystyle\int_t^{t'} \left| \int \varphi(\mathbf{x}, V(s, \mathbf{x}), w)\, (F^\varepsilon(s, \mathrm{d}\mathbf{x}, \mathrm{d}w) - F(s, \mathrm{d}\mathbf{x}, \mathrm{d}w)) \right| \mathrm{d}s.
\end{cases}$$

We estimate the first term $\mathcal{I}_1$ using the regularity of φ and the Cauchy-Schwarz inequality:

$$\begin{aligned}
\mathcal{I}_1 &\leq \|\partial_v \varphi\|_{L^\infty} \int_0^T \int |v - V(s, \mathbf{x})|\, f^\varepsilon(s, \mathbf{x}, v, w)\, \mathrm{d}v\, \mathrm{d}w\, \mathrm{d}\mathbf{x}\, \mathrm{d}s \\
&\leq \|\partial_v \varphi\|_{L^\infty} \left(\int_0^T \int f^\varepsilon(s, \mathbf{x}, v, w)\, \mathrm{d}v\, \mathrm{d}w\, \mathrm{d}\mathbf{x}\, \mathrm{d}s \right)^{1/2} \\
&\quad \times \left(\int_0^T \int f^\varepsilon(s, \mathbf{x}, v, w)\, |v - V(s, \mathbf{x})|^2\, \mathrm{d}v\, \mathrm{d}w\, \mathrm{d}\mathbf{x}\, \mathrm{d}s \right)^{1/2} \\
&\leq \|\partial_v \varphi\|_{L^\infty}\, (T\, \|f_0^\varepsilon\|_{L^1}\, C_T)^{1/2}\, \varepsilon^{1/(2d+12)},
\end{aligned}$$

whereas the second term $\mathcal{I}_2$ also converges to zero when ε goes to zero since F^ε converges weakly-$\star$ in $\mathcal{M}((0, T) \times \mathbb{R}^{d+1})$ to F. Using a density argument, this shows the convergence of f^ε in $L_{\mathrm{loc}}^1((0, T), \mathcal{M}(\mathbb{R}^{d+2}))$, so this concludes the proof of Theorem 2.5.

A   Proof of Proposition 2.2

This appendix is devoted to the proof of the existence and uniqueness of a solution f^ε to (1.7). Let $T > 0$ and $\varepsilon > 0$ be fixed. The main difficulty is that we cannot use a compactness argument based on an average lemma as in [30] for example, since there is no transport term of the form $v \nabla_x f^\varepsilon$ in (1.7). Thus, our strategy is to linearize the equation (1.7) in order to construct a Cauchy sequence which converges towards a solution to (1.7). First of all, we need to prove some extra a priori estimates on f^ε. In the rest of this section, since ε is fixed, for the sake of clarity, we will note f instead of f^ε, but all the following estimates are not uniform in ε.

A.1 A priori estimates

We recall that for any solution f of (1.7) and for all $t \in [0, T]$, $\|f(t)\|_{L^1} = \|f(0)\|_{L^1}$. This subsection is devoted to the proof of some a priori estimates.

First of all, we set the system of characteristic equations associated to (1.7) for all $(\mathbf{x}, v, w) \in \mathbb{R}^{d+2}$ and all $t \in [0, T]$:

$$
\begin{cases}
\dfrac{\mathrm{d}\Sigma^v(s)}{\mathrm{d}s} = N(\Sigma^v(s)) - \Sigma^w(s) - [\Phi_\varepsilon * \rho_0](\mathbf{x})\Sigma^v(s) + [\Phi_\varepsilon * j_f(s)](\mathbf{x}), \\[2mm]
\dfrac{\mathrm{d}\Sigma^w(s)}{\mathrm{d}s} = A(\Sigma^v(s), \Sigma^w(s)), \\[2mm]
\Sigma^v(t) = v, \quad \Sigma^w(t) = w,
\end{cases}
\tag{A.1}
$$

where f is a solution of (1.7) and j_f is defined with:

$$
j_f := \int f\, v \,\mathrm{d}v \,\mathrm{d}w.
$$

We define the flow of (A.1) for all $\mathbf{z} = (\mathbf{x}, v, w) \in \mathbb{R}^{d+2}$ for all s, t in $[0, T]$:

$$
\Sigma(s, t, \mathbf{z}) = (\Sigma^v(s, t, \mathbf{z}), \Sigma^w(s, t, \mathbf{z})) \quad , \quad \Sigma(t, t, \mathbf{z}) = (v, w).
$$

We start with proving the well-posedness of the flow of the characteristic equation (A.1) and an estimate of the support of a solution to (1.7).

Lemma 1.1 (Well-posedness of the characteristic system and estimate of the support). *Consider an initial data f_0 satisfying (2.4) and (2.5), and suppose that there exists f a smooth non-negative solution to (1.7) such that for all $t \in [0; T]$, $\|f(t)\|_{L^1} = \|f(0)\|_{L^1}$. Then, the characteristic system (A.1) is well-posed. Furthermore, there exists a positive constant $R_{T,\varepsilon}$ such that:*

$$
\sup_{\substack{t \in [0,T] \\ \mathbf{x} \in \mathbb{R}^d}} \operatorname{Supp}(f(t, \mathbf{x}, \cdot, \cdot)) \subset B(0, R_{T,\varepsilon}),
\tag{A.2}
$$

and there exist two positive constants C_1 and C_2 such that:

$$
R_{T,\varepsilon} \leq C_2 e^{C_1(1+\frac{1}{\varepsilon})T}.
\tag{A.3}
$$

Proof. The Cauchy-Lipschitz Theorem yields the local existence and uniqueness of the flow of the characteristic equation (A.1). Define for all $s \in [0, T]$:

$$
R_\varepsilon(s) := \sup \left\{ \|\Sigma(s', 0, \mathbf{z})\| \quad | \quad s' \in [0; s],\, \mathbf{z} \in \mathbb{R}^d \times B(0, R_0^\varepsilon) \right\}.
$$

Our purpose is to estimate R_ε using the following energy estimate. Let $s \in [0; T]$, $\mathbf{z} = (\mathbf{x}, v, w) \in \mathbb{R}^d \times B(0, R_0^\varepsilon)$. We have:

$$\|\Sigma(s,0,\mathbf{z})\|^2 = \|(v,w)\|^2 + 2\int_0^s \Sigma(s',0,\mathbf{z})\,\partial_s\Sigma(s',0,\mathbf{z})\,\mathrm{d}s'$$

$$= \|(v,w)\|^2 + 2\int_0^s \big[\Sigma^v\,N(\Sigma^v) - \Sigma^v\,\Sigma^w$$

$$-\,[\Phi_\varepsilon * \rho_0](\mathbf{x})\,|\Sigma^v|^2 + [\Phi_\varepsilon * j_f(s')](\mathbf{x})\,\Sigma^v + \Sigma^w\,A(\Sigma^v,\Sigma^w)\big]\,\mathrm{d}s'$$

$$\le |R_0^\varepsilon|^2 + 2\int_0^s \Big[|\Sigma^v|^2 + \frac{1}{2}\,(|\Sigma^v|^2 + |\Sigma^w|^2)$$

$$+\,R_\varepsilon(s)\,[\phi_\varepsilon * \rho_0](\mathbf{x})\,\Sigma^v + \frac{\tau}{2}\,(|\Sigma^v|^2 + |\Sigma^w|^2) + \frac{\tau}{2}\,(a^2 + |\Sigma^w|^2)\Big]\,\mathrm{d}s'$$

$$\le |R_0^\varepsilon|^2 + T\,\tau\,a^2 + \int_0^s \Big(3 + 2\,\tau + 2\,\|\rho_0\|_{L^\infty}\,\|\Psi\|_{L^1} + \frac{2}{\varepsilon}\,\|\rho_0\|_{L^\infty}\Big)\,|R_\varepsilon(s')|^2\,\mathrm{d}s'.$$

Thus, passing to the supremum in $\mathbf{x} \in \mathbb{R}^d$ and in $(v,w) \in B(0, R_0^\varepsilon)$, we get that for all $s \in [0,T]$:

$$|R_\varepsilon(s)|^2 \le |R_0^\varepsilon|^2 + T\,\tau\,a^2 + \int_0^s \Big(3 + 2\,\tau + 2\,\|\rho_0\|_{L^\infty}\,\|\Psi\|_{L^1} + \frac{2}{\varepsilon}\,\|\rho_0\|_{L^\infty}\Big)\,|R_\varepsilon(s')|^2\,\mathrm{d}s'.$$

Using the Grönwall's inequality, we get that there exist two positive constants C_1 and C_2 which depend only on T, R_0^ε, $\|\Psi\|_{L^1}$, ρ_0, τ and a such that for all $s \in [0,T]$:

$$|R_\varepsilon(s)|^2 \le C_2\,e^{C_1\,(1+\frac{1}{\varepsilon})\,T}.$$

Therefore, the function $\Sigma(\cdot,0,\cdot)$ is well-posed in $\mathscr{C}^1([0,T]^2 \times B(0,R_0^\varepsilon),\,L^\infty(\mathbb{R}^d_{\mathbf{x}}))$. Similarly, for all $t \in [0,T]$, we show that $\Sigma(\cdot,t,\cdot)$ is well-posed in $\mathscr{C}^1([0,T]^2 \times B(0,R_\varepsilon(t)),\,L^\infty(\mathbb{R}^d_{\mathbf{x}}))$. Furthermore, we can conclude that

$$\sup_{\substack{t \in [0,T] \\ \mathbf{x} \in \mathbb{R}^d}} \mathrm{Supp}(f(t,\mathbf{x},\cdot,\cdot)) \subset B(0, R_\varepsilon(T)).$$

∎

Now, using this estimate on the propagation of the support in $\mathbf{u} = (v,w)$ of any solution f to (1.7), we can estimate f and $\nabla_\mathbf{u} f$ in L^∞.

Lemma 1.2 (Estimates in L^∞)**.** *Consider an initial data f_0 satisfying (2.4) and (2.5), and suppose that there exists f a smooth non-negative solution to (1.7) such that for all $t \in [0,T]$, $\|f(t)\|_{L^1} = \|f_0\|_{L^1}$. Then, there exists a positive constant $C_{T,\varepsilon}$ such that for all $t \in [0,T]$,*

$$\|f(t)\|_{L^\infty} \le C_{T,\varepsilon} \quad,\ and \quad \|\nabla_\mathbf{u} f(t)\|_{L^\infty} \le C_{T,\varepsilon}.$$

Proof. We write (1.7) in a non-conservative form:

$$\partial_t f + \mathbf{A} \cdot \nabla_\mathbf{u} f = -\,\mathrm{div}_\mathbf{u}(\mathbf{A})\,f, \tag{A.4}$$

where $\mathbf{A}$ is the advection field of (1.7) given for all $t \in [0,T]$ and all $\mathbf{z} = (\mathbf{x},v,w) \in \mathbb{R}^{d+2}$ by:

$$\mathbf{A}(t,\mathbf{z}) := \begin{pmatrix} N(v) - w - [\Phi_\varepsilon * \rho_0](\mathbf{x})\,v + [\Phi_\varepsilon * j_f(t)](\mathbf{x}) \\[2mm] A(v,w) \end{pmatrix}.$$

Thus, for all $t \in [0, T]$ and all $\mathbf{z} = (\mathbf{x}, v, w) \in \mathbb{R}^{d+2}$, we note $\mathbf{u} = (v, w)$ and we have:

$$
\begin{aligned}
-\operatorname{div}_{\mathbf{u}} (\mathbf{A}(t, \mathbf{z})) &= -N'(v) + [\Phi_\varepsilon * \rho_0](\mathbf{x}) + \tau\, b \\
&= 3\,v^2 - 1 + [\Phi_\varepsilon * \rho_0](\mathbf{x}) + \tau\, b.
\end{aligned}
$$

Then, we get that for all $t \in [0, T]$ and $\mathbf{z} = (\mathbf{x}, v, w) \in \mathbb{R}^{d+2}$:

$$
f(t, \mathbf{z}) = f_0(\mathbf{x}, \Sigma(0, t, \mathbf{z})) - \int_0^t (\operatorname{div}_{\mathbf{u}} (\mathbf{A})\ f)\,(t, \mathbf{x}, \Sigma(s, t, \mathbf{z}))\, \mathrm{d}s.
$$

Consequently, we have for all $t \in [0, T]$:

$$
\|f(t)\|_{L^\infty} \leq \|f_0\|_{L^\infty} + \int_0^t \| \operatorname{div}_{\mathbf{u}} (\mathbf{A}(s, \cdot))\, f(s)\|_{L^\infty}\, \mathrm{d}s.
$$

Moreover, according to Lemma 1.1, there exists a positive constant $R_{T,\varepsilon}$ satisfying (A.2), and therefore, for all $s,\, t \in [0, T]$ and $\mathbf{z} = (\mathbf{x}, \mathbf{u}) \in \mathbb{R}^d \times B(0, R_{T,\varepsilon})$,

$$
|\operatorname{div}_{\mathbf{u}} (\mathbf{A}(s, \mathbf{z}))| \leq 3\,|R_{T,\varepsilon}|^2 + 1 + \|\Phi_\varepsilon\|_{L^1}\, \|\rho_0\|_{L^\infty} + \tau\, b.
$$

Therefore, the Grönwall's inequality gives us that there exists a positive constant C_1 such that for all $t \in [0, T]$:

$$
\|f(t)\|_{L^\infty} \leq \|f_0\|_{L^\infty} e^{C_1\, t}.
$$

Then, by differentiating (A.4) with respect to v and w, we have:

$$
\begin{cases}
\partial_t (\partial_v f) + \mathbf{A} \cdot \nabla_{\mathbf{u}} (\partial_v f) &= S^v(t, \mathbf{z})\, \partial_v f - \tau\, \partial_w f - N''(v)\, f, \\
\partial_t (\partial_w f) + \mathbf{A} \cdot \nabla_{\mathbf{u}} (\partial_w f) &= S^w(t, \mathbf{z})\, \partial_w f + \partial_v f,
\end{cases}
$$

where S^v and S^w are given for all $t \in [0, T]$ and all $\mathbf{z} = (\mathbf{x}, v, w) \in \mathbb{R}^{d+2}$ by:

$$
\begin{cases}
S^v(t, \mathbf{z}) := -2\,N'(v) + 2\,[\Phi_\varepsilon * \rho_0]\,(\mathbf{x}) + \tau\, b, \\
S^w(t, \mathbf{z}) := -N'(v) + [\Phi_\varepsilon * \rho_0]\,(\mathbf{x}) + 2\,\tau\, b.
\end{cases}
$$

Therefore, using the estimate of the support of f (A.2), the Grönwall's inequality gives us that there exist two positive constants C_2 and C_3 such that for all $t \in [0, T]$:

$$
\|\nabla_{\mathbf{u}} f(t)\|_{L^\infty} \leq C_3\, e^{C_2\, t}.
$$

■

A.2 Proof of existence and uniqueness

We proceed with a linearization of the equation (1.7) in order to construct a Cauchy sequence $\{f_n^\varepsilon\}_{n \in \mathbb{N}}$ of non-negative functions in the Banach space $\mathscr{C}^0([0, T] \times \mathbb{R}_{\mathbf{u}}^2,\, L^\infty(\mathbb{R}_{\mathbf{x}}^d))$, such that for all $n \in \mathbb{N}$, the Lemmas 1.1 and 1.2 give that there exist two positive constants $R_{T,\varepsilon}$ and $C_{T,\varepsilon}$ independent on n such that for all $n \in \mathbb{N}$:

$$
\sup_{\substack{t \in [0, T] \\ \mathbf{x} \in \mathbb{R}^d}} \operatorname{Supp}(f_n^\varepsilon(t, \mathbf{x}, \cdot, \cdot)) \subset B(0, R_{T,\varepsilon}),
$$

$$
\|f_n^\varepsilon(t)\|_{L^\infty} \leq C_{T,\varepsilon} \quad \text{and} \quad \|\nabla_{\mathbf{u}} f_n^\varepsilon(t)\|_{L^\infty} \leq C_{T,\varepsilon}.
$$

As $\mathscr{C}^0([0,T]\times\mathbb{R}^2_{\mathbf{u}}, L^\infty(\mathbb{R}^d_{\mathbf{x}}))$ is a Banach space, the Cauchy sequence $\{f^\varepsilon_n\}_{n\in\mathbb{N}}$ strongly converges in $\mathscr{C}^0([0,T]\times\mathbb{R}^2_{\mathbf{u}}, L^\infty(\mathbb{R}^d_{\mathbf{x}}))$ towards a function f^ε, which is non-negative and

$$f^\varepsilon \in \mathscr{C}^0([0,T]\times\mathbb{R}^2_{\mathbf{u}}, L^\infty(\mathbb{R}^d_{\mathbf{x}})).$$

Hence, passing to the limit $n\to+\infty$ in the linearized equation, f^ε is a weak solution to (1.7) in the sense of (2.2) which satisfies the support estimate (A.2). Consequently, we deduce

$$f^\varepsilon \in \mathscr{C}^0\left([0,T], L^1(\mathbb{R}^{d+2})\right),$$

hence we can apply Lemma 1.1 with f^ε, which yields that the characteristics $\Sigma \in \mathscr{C}^1([0,T]^2 \times \mathbb{R}^2_{\mathbf{u}}, L^\infty(\mathbb{R}^d_{\mathbf{x}}))$ are well-defined. As previously, we note $\mathbf{A}$ the advection field of (1.7). Therefore, for all $t \in [0,T]$, $\mathbf{u}=(v,w) \in \mathbb{R}^2$ and $\mathbf{x} \in \mathbb{R}^d$,

$$f^\varepsilon(t,\mathbf{x},\mathbf{u}) \,=\, f^\varepsilon_0(\mathbf{x},\Sigma(0,t,\mathbf{x},\mathbf{u}))\,\exp\left(-\int_0^t \operatorname{div}_{\mathbf{u}}\mathbf{A}(s,\mathbf{x},\Sigma(s,t,\mathbf{x},\mathbf{u}))\mathrm{d}s\right).$$

Finally, using the regularity of $\operatorname{div}_{\mathbf{u}}\mathbf{A}$, Σ and f^ε_0 which satisfies (2.4), we get that

$$f^\varepsilon, \nabla_{\mathbf{u}}f^\varepsilon \in L^\infty((0,T)\times\mathbb{R}^{d+2}).$$

B   Proof of Proposition 2.3

This appendix is devoted to the proof of Proposition 2.3. We apply a fixed point argument to get the existence and uniqueness of a classical solution (V,W) of (2.6). We first set

$$\mathscr{E} \,:=\, \mathscr{C}^0([0,T], L^\infty(\mathbb{R}^d))$$

and for a fixed $M > 0$ we define

$$\mathscr{K}_T \,:=\, \left\{(V,W) \in \mathscr{E}^2 \,;\, \|V(t)-V_0\|_{L^\infty(\mathbb{R}^d)} + \|W(t)-W_0\|_{L^\infty(\mathbb{R}^d)} \le M, \,\forall t \in [0,T]\right\},$$

equipped with the norm for $U = (V,W) \in \mathscr{K}_T$,

$$\|\|U\|\| \,:=\, \sup_{t\in[0,T]}\left(\|V(t)\|_{L^\infty(\mathbb{R}^d)} + \|W(t)\|_{L^\infty(\mathbb{R}^d)}\right).$$

Consider the application Γ such that for all $U = (V,W) \in \mathscr{K}_T$, for all $t \in [0,T]$ and almost every $\mathbf{x} \in \mathbb{R}^d$,

$$\Gamma[U](t,\mathbf{x}) \,:=\, U_0(\mathbf{x}) \,+\, \int_0^t \left[\begin{array}{l} \mathcal{L}_{\rho_0}(V)(s,\mathbf{x})\mathrm{d}s + N(V(s,\mathbf{x})) - W(s,\mathbf{x}) \\[2mm] A(V,W)(s,\mathbf{x}) \end{array}\right]\mathrm{d}s.$$

On the one hand, since $\Psi \in L^1(\mathbb{R}^d)$ and $N \in \mathscr{C}^1(\mathbb{R})$, we prove that Γ is well-defined

$$\|\|\Gamma[U]-U_0\|\| \,\le\, C\,T$$

for some constant $C > 0$ independent of time. As a consequence, we can choose T small enough so that $\Gamma[U] \in \mathscr{K}_T$. On the other hand, Γ is contractive : for U_1 $U_2 \in \mathscr{K}_T$ and $(t,\mathbf{x})$ be in $[0,T]\times\mathbb{R}^d$, we have

$$|\Gamma[U_1] \,-\, \Gamma[U_2]|\,(t,\mathbf{x}) \le C\,t\,\|\|U_1 - U_2\|\|,$$

where $C > 0$ does not depend of time $t > 0$, hence for T small enough, Γ is a contraction from $\mathscr{K}_T$ to $\mathscr{K}_T$, which is a complete for the distance associated to the norm $\|\|.\|\|$ previously defined. Therefore, using the Banach fixed point theorem, there exists a unique $U \in \mathscr{K}_T$ such that $\Gamma[U] = U$, that is, $U = (V, W)$ is solution to (2.6) for the initial condition $U_0 = (V_0, W_0)$. Moreover since V and $W \in \mathscr{C}^0([0, T], L^\infty(\mathbb{R}^d))$ and using the Duhamel's formula, it yields that

$$V\,,\;W \in \mathscr{C}^1([0, T], L^\infty(\mathbb{R}^d)).$$

Finally, an energy estimate gives that

$$
\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(|V(t,\mathbf{x})|^2 + |W(t,\mathbf{x})|^2\right) \;\leq\;\; & \left(\frac{3+\tau}{2} + \|\Psi\|_{L^1}\,\|\rho_0\|_{L^\infty}\right)\|V(t)\|_{L^\infty(\mathbb{R}^d)}^2 \\
& + \;\; \frac{1+2\,\tau}{2}\,\|W(t)\|_{L^\infty(\mathbb{R}^d)}^2 + \frac{\tau\,a^2}{2}.
\end{aligned}
$$

Using Grönwall's lemma, we can conclude that there exists a constant $C > 0$, only depending on a, τ and Ψ, such that

$$\|\|(V, W)\|\| \;\leq\; (\|\|(V_0, W_0)\|\| + C)\; e^{C\,T}.$$

It allows to prove that this unique solution is global in time.

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