MULTIDIMENSIONAL STABILITY OF PLANAR TRAVELING WAVES FOR THE SCALAR NONLOCAL ALLEN-CAHN EQUATION

Grégory Faye

CAMS - Ecole des Hautes Etudes en Sciences Sociales
190-198 avenue de France, 75013 Paris, France
and
CNRS, UMR 5219, Institut de Mathématiques de Toulouse
31062 Toulouse Cedex, France

(Communicated by Chongchun Zeng)

Abstract. We prove the multidimensional stability of planar traveling waves for scalar nonlocal Allen-Cahn equations using semigroup estimates. We show that if the traveling wave is spectrally stable in one space dimension, then it is stable in n-space dimension, $n \geq 2$, with perturbations of the traveling wave decaying like $t^{-(n-1)/4}$ as $t \to +\infty$ in $H^k(\mathbb{R}^n)$ for $k \geq \left\lceil \frac{n+1}{2} \right\rceil$.

1. **Introduction.** We consider the scalar nonlocal Allen-Cahn equation

$$\partial_t u(\mathbf{x}, t) = -u(\mathbf{x}, t) + \int_{\mathbb{R}^n} \mathcal{K}(\mathbf{x} - \mathbf{y}) u(\mathbf{y}, t) \mathrm{d}\mathbf{y} + f(u(\mathbf{x}, t)) :$$
$$= -u(\mathbf{x}, t) + \mathcal{K} *_{\mathbf{x}} u(\mathbf{x}, t) + f(u(\mathbf{x}, t)) \qquad (1)$$

where $u \in \mathbb{R}$, $(\mathbf{x}, t) \in \mathbb{R}^n \times \mathbb{R}^+$ and f is a smooth function of bistable type with three zeros, 0, 1 and $a \in (0, 1)$. A prototypical example for f is the cubic nonlinearity of form $f_{\mathrm{cu}}(u) := u(1 - u)(u - a)$. Here $\mathcal{K} \in L^1(\mathbb{R})$ is a nonnegative function with $\int_{\mathbb{R}^n} \mathcal{K}(\mathbf{x}) \mathrm{d}\mathbf{x} = 1$ and that is even with respect to each variable. A traveling wave $\varphi(\xi)$ is a smooth function of the variable $\xi = \mathbf{e} \cdot \mathbf{x} - ct$, for $\mathbf{e} \in \mathbb{S}^{n-1}$ and some $c \in \mathbb{R}$, which is a solution of (1) satisfying the limits $\lim_{\xi \to -\infty} \varphi(\xi) = 1$ and $\lim_{\xi \to +\infty} \varphi(\xi) = 0$. Without loss of generality, we suppose that $\mathbf{e} = (1, 0, \ldots, 0)$. In the moving frame $\mathbf{x} = (\xi, \mathbf{z}) \in \mathbb{R} \times \mathbb{R}^{n-1}$, equation (1) can be written as

$$\partial_t u(\mathbf{x}, t) - c \partial_\xi u(\mathbf{x}, t) = -u(\mathbf{x}, t) + \int_{\mathbb{R}^n} \mathcal{K}(\mathbf{x} - \mathbf{y}) u(\mathbf{y}, t) \mathrm{d}\mathbf{y} + f(u(\mathbf{x}, t)) \qquad (2)$$

such that the traveling wave $\varphi(\xi)$ is a stationary solution of (2). If we define $\mathcal{K}_0 : \mathbb{R} \to \mathbb{R}$ as

$$\mathcal{K}_0(\xi) = \int_{\mathbb{R}^{n-1}} \mathcal{K}(\xi, \mathbf{z}) \mathrm{d}\mathbf{z} \qquad (3)$$

then (φ, c) satisfies

$$-c\dot{\varphi}(\xi) = -\varphi(\xi) + \int_{\mathbb{R}} \mathcal{K}_0(\xi-\zeta)\varphi(\zeta)\mathrm{d}\zeta + f(\varphi(\xi)), \quad \lim_{\xi \to -\infty} \varphi(\xi) = 1 \text{ and } \lim_{\xi \to +\infty} \varphi(\xi) = 0, \ (4)$$

where $\dot{}$ stands for $\frac{\mathrm{d}}{\mathrm{d}\xi}$ and φ is decreasing.

Main assumptions. Throughout the paper, we will assume the following hypotheses for f and $\mathcal{K}$ which ensure the existence and uniqueness (modulo translation) of a solution (φ, c) to (4), see [3].

Hypothesis (H1) *We suppose that the nonlinearity f satisfies the following properties:*

(i) $f \in \mathcal{C}^\infty(\mathbb{R})$;
(ii) $f(u) = 0$ *precisely when* $u \in \{0, a, 1\}$;
(iii) $f'(0) < 0$, $f'(1) < 0$ *and* $f'(a) > 0$.

Note that we only need $f \in \mathcal{C}^2(\mathbb{R})$ to obtain the existence result of [3] and here we require more regularity to obtain uniform bounds on the nonlinear terms in our stability analysis.

Hypothesis (H2) *We suppose that the kernel $\mathcal{K}$ satisfies the following properties:*

(i) $\mathcal{K} \geq 0$, *is even with respect to each variable;*
(ii) $\mathcal{K} \in W^{1,1}(\mathbb{R}^n)$;
(iii) $\int_{\mathbb{R}^n} \mathcal{K}(\mathbf{x})\mathrm{d}\mathbf{x} = 1$, $\int_{\mathbb{R}^n} \|\mathbf{x}\|\mathcal{K}(\mathbf{x})\mathrm{d}\mathbf{x} < \infty$ *and* $\int_{\mathbb{R}^n} \|\mathbf{x}\|^2\mathcal{K}(\mathbf{x})\mathrm{d}\mathbf{x} < \infty$;
(iv) $\widehat{\mathcal{K}}(\mathbf{k}) = 1 - d_0\|\mathbf{k}\|^2 + o(\|\mathbf{k}\|^2)$ *as* $\mathbf{k} \to 0$ *with* $d_0 > 0$.

Here, $W^{k,p}(\mathbb{R}^n)$ denotes the Sobolev space with its usual norm and we use the notation $H^k(\mathbb{R}^n) := W^{k,2}(\mathbb{R}^n)$. The symbol $\widehat{\mathcal{K}}$ denotes the Fourier transform of $\mathcal{K}$ defined as

$$\widehat{\mathcal{K}}(\mathbf{k}) = \int_{\mathbb{R}^n} \mathcal{K}(\mathbf{x})e^{-i\mathbf{k}\cdot\mathbf{x}}\mathrm{d}\mathbf{x}, \quad \mathbf{k} \in \mathbb{R}^n.$$

The first assumption is natural from a modeling point of view while the second and third assumptions are required to ensure the existence of traveling wave solution φ to equation (4). The third and forth assumptions also imply that

$$\forall j \in [\![1, n]\!] \quad \int_{\mathbb{R}^n} x_j\mathcal{K}(\mathbf{x})\mathrm{d}\mathbf{x} = 0 \text{ and } d_0 = \frac{1}{2n}\int_{\mathbb{R}^n} \|\mathbf{x}\|^2\mathcal{K}(\mathbf{x})\mathrm{d}\mathbf{x} > 0.$$

Furthermore, as $-1 + \widehat{\mathcal{K}}(\mathbf{k}) \sim -d_0\|\mathbf{k}\|^2$ for $\mathbf{k} \to 0$, in the long wavelength limit, the linear operator $u \mapsto -u + \mathcal{K} * u$ approaches the Laplacian $d_0\Delta_{\mathbb{R}^n}$ and we recover the classical Allen-Cahn equation. Remark that in the short wavelength limit we have $-1 + \widehat{\mathcal{K}}(\mathbf{k}) \sim -1$ for $\|\mathbf{k}\| \to \infty$ such that $u \mapsto -u + \mathcal{K} * u$ is a bounded operator which is a very different feature from the Laplacian. Note that with Hypothesis (H2) for the kernel $\mathcal{K}$ we recover all the hypotheses of [3] for $\mathcal{K}_0$.

In this paper, we are concerned with determining the stability of the traveling wave φ. We are thus let to study the spectral properties of the linear operator $\mathcal{L}$

$$\begin{array}{rrcl} \mathcal{L}: & H^1(\mathbb{R}^n) & \longrightarrow & L^2(\mathbb{R}^n) \\ & u & \longmapsto & -u + \mathcal{K} *_{\mathbf{x}} u + c\frac{\mathrm{d}}{\mathrm{d}\xi} + f'(\varphi)u. \end{array} \quad (5)$$

It is natural to assume that the wave φ is linearly stable in one space dimension to get stability in higher in space dimensions. In fact, it is consequence of Hypotheses (H1) and (H2) on f and $\mathcal{K}$. First, define the linear operator $\mathcal{L}_0$ associated to

equation (4)

$$\mathcal{L}_0 : \quad \begin{aligned} H^1(\mathbb{R}) &\longrightarrow L^2(\mathbb{R}) \\ u &\longmapsto -u + \mathcal{K}_0 *_\xi u + c\frac{\mathrm{d}}{\mathrm{d}\xi} + f'(\varphi)u. \end{aligned} \tag{6}$$

and its adjoint operator $\mathcal{L}_0^*$

$$\mathcal{L}_0^* : \quad \begin{aligned} \mathcal{D}(\mathcal{L}_0^*) \subset L^2(\mathbb{R}) &\longrightarrow L^2(\mathbb{R}) \\ u &\longmapsto -u + \mathcal{K}_0 *_\xi u - c\frac{\mathrm{d}}{\mathrm{d}\xi} + f'(\varphi)u. \end{aligned} \tag{7}$$

Lemma 1.1 ([2, 6])**.** *Suppose that Hypotheses* $(H1)$ *and* $(H2)$ *are satisfied, then*

(i) 0 *is an algebraic simple eigenvalue of* $\mathcal{L}_0$ *with negative eigenfunction* φ';
(ii) *there exists* $\gamma_0 > 0$ *such that* $\sigma_{ess}(\mathcal{L}_0) \subset \{\lambda \mid |\Re(\lambda)| < -\gamma_0\}$;
(iii) *there exists a unique negative solution* $\psi \in H^1(\mathbb{R})$ *which solves* $\mathcal{L}_0^*\psi = 0$ *with* $\int_\mathbb{R} \varphi'(\xi)\psi(\xi)\mathrm{d}\xi = 1$.

Since the eigenvalue zero is isolated, there exists a spectral projection operator, $\mathcal{P}$, onto the null space of $\mathcal{L}_0$ given by

$$\mathcal{P}u = \frac{1}{2\pi i} \int_\Gamma (\mathcal{L}_0 - \lambda)^{-1} u \mathrm{d}\lambda, \tag{8}$$

where Γ is a simple closed curve in the complex plane enclosing the zero eigenvalue. If $\langle \cdot, \cdot \rangle$ denotes the scalar product on $L^2(\mathbb{R})$ then we can write $\mathcal{P}$ as

$$\mathcal{P}u(\xi, \mathbf{z}) = \langle \psi, u \rangle(\mathbf{z})\varphi'(\xi) := \left(\int_\mathbb{R} \psi(\xi)u(\xi, \mathbf{z})\mathrm{d}\xi \right) \varphi'(\xi). \tag{9}$$

We define the operator $\mathcal{Q}$ as $\mathcal{Q}u := u - \mathcal{P}u$.

Main result. We can now state our main result. The perturbation of the wave will be written as

$$u(\mathbf{x}, t) := \varphi(\xi - \rho(\mathbf{z}, t)) + v(\xi - \rho(\mathbf{z}, t), \mathbf{z}, t) \tag{10}$$

where $\rho : \mathbb{R}^{n-1} \to \mathbb{R} \in H^k(\mathbb{R}^{n-1})$ and $v : \mathbb{R}^n \to \mathbb{R} \in H^k(\mathbb{R}^n)$ is in the range of the operator $\mathcal{L}_0$ that is $\mathcal{P}v = 0$. And we set

$$\mathcal{E}_0 := \|v_0\|_{W^{1,1}(\mathbb{R}^n)} + \|v_0\|_{H^k(\mathbb{R}^n)} + \|\rho_0\|_{W^{1,1}(\mathbb{R}^{n-1})} + \|\rho_0\|_{H^{k+1}(\mathbb{R}^{n-1})}.$$

Theorem 1.2. *Let* $n \geq 2$ *and* $k \geq \left[\frac{n+1}{2}\right]$. *Suppose that Hypotheses* $(H1)$ *and* $(H2)$ *are satisfied. There exists* $C > 0$ *such that if* $\mathcal{E}_0$ *is sufficiently small, then the traveling wave solution* φ *of equation* (2) *is stable in the sense that the perturbation* (ρ, v) *given in* (10) *satisfies the decay estimates for all* $t \geq 0$

$$\|v(t)\|_{H^k(\mathbb{R}^n)} \leq C(1+t)^{-\frac{n-1}{4}-1}\mathcal{E}_0, \tag{11a}$$

$$\|\rho(t)\|_{H^k(\mathbb{R}^{n-1})} \leq C(1+t)^{-\frac{n-1}{4}}\mathcal{E}_0, \tag{11b}$$

$$\|\nabla_\mathbf{z} \cdot \rho(t)\|_{H^k(\mathbb{R}^{n-1})} \leq C(1+t)^{-\frac{n+1}{4}}\mathcal{E}_0, \tag{11c}$$

where $\nabla_\mathbf{z} = (\partial_{x_2}, \cdots, \partial_{x_n})$.

Note that Theorem 1.2 is well known in the case of local diffusion, namely when the nonlocal term $-u + \mathcal{K} *_\mathbf{x} u$ in equation (1) is replaced by the standard Laplacian $\Delta = \sum_{i=1}^n \partial_{x_i}^2$ on $\mathbb{R}^n$. Xin [18] was the first to prove these results in dimension $n \geq 4$ in the local case. His results were then extended to the remaining dimensions $n = 2, 3$ in [14] and generalized to systems of bistable reaction-diffusion equations by Kapitula [13]. Our strategy of proof will be similar to as [18, 13] where semigroup

estimates for the associated linearized operator are used to prove the multidimensional stability of the traveling wave φ. It is important to remark that in dimension $n \geq 4$, these semigroup estimates are sufficient to prove Theorem 1.2. For the remaining dimensions $n = 2, 3$, the proof essentially relies on the decomposition of the perturbation as written in (10) which basically allows one to split the problem into two parts. One part controls the drift of the perturbations along the translates of the wave and another part which controls the remaining part of the perturbations and will decay faster in time. Although our proof will follow the strategy developed in [18, 13], we still have to deal with the nonlocal nature of our equations. In our case, we use point-wise Green's functions estimates to obtain sharp decay estimates of the linear part of our linearized operator. These types of estimates are reminiscent of the ones obtained by Hoffman and coworkers [11] in the study of multi-dimensional stability of planar traveling of lattice differential equations, which are discrete version of equation (1). In the nonlocal setting, using super- and sub- solution technique, Chen [5] has been able to prove the uniform multidimensional stability of the traveling wave φ of equation (1). As a direct consequence, our Theorem 1.2 generalizes Chen's result.

An application. This present work was initially motivated by the study of Bates and Chen [1] where they prove a multidimensional stability result for a slightly different multidimensional nonlocal Allen-Cahn equation. Their idea was to consider a generalization of the Laplacian in n-dimension for which, each component $\partial_{x_i}^2$ of Δ is approximated by the convolution operator $-u + \mathcal{J} *_{x_i} u$. They obtain an equation of form

$$\partial_t u = \sum_{i=1}^{n} \left(-u + \mathcal{J} *_{x_i} u \right) + f(u), \tag{12}$$

with

$$\mathcal{J} *_{x_i} u(\mathbf{x}) := \int_{\mathbb{R}} \mathcal{J}(y) u(x_1, \cdots, x_i - y, \cdots, x_n) \mathrm{d}y.$$

The kernel $\mathcal{J}$ satisfies the following Hypothesis.

Hypothesis (H3) *We suppose that the kernel $\mathcal{J}$ satisfies the following properties:*

(i) $\mathcal{J} \geq 0$, *is even;*
(ii) $\mathcal{J} \in W_{\eta}^{1,1}(\mathbb{R})$ *for $\eta > 0$.*

Here $W_{\eta}^{1,1}(\mathbb{R})$ denotes the exponentially weighted function space defined as

$$W_{\eta}^{1,1}(\mathbb{R}) := \left\{ u \in W^{1,1}(\mathbb{R}) \mid e^{\eta|\cdot|} u \in L^1(\mathbb{R}) \text{ and } e^{\eta|\cdot|} \partial_x u \in L^1(\mathbb{R}) \right\}.$$

A direct consequence of Hypothesis (H3) is that $\widehat{\mathcal{J}}(k) = 1 - d_0 k^2 + o(k^2)$ as $k \to 0$ for $d_0 > 0$. In this setting, the traveling wave φ is solution of (4) with $\mathcal{K}_0 = \mathcal{J}$ and the linearized operator $\mathcal{L}_0$ has the same expression as in equation (6) and thus Lemma 1.1 is also verified provided that f satisfies Hypothesis (H1).

To study the stability of the traveling wave φ, we work with the same decomposition

$$u(\mathbf{x}, t) = \varphi(\xi - \rho(\mathbf{z}, t)) + v(\xi - \rho(\mathbf{z}, t), \mathbf{x}, t), \quad t \geq 0, \tag{13}$$

with $\rho : \mathbb{R}^{n-1} \to \mathbb{R} \in H^k(\mathbb{R}^{n-1})$ and $v : \mathbb{R}^n \to \mathbb{R} \in H^k(\mathbb{R}^n)$ is in the range of the operator $\mathcal{L}_0$ that is $\mathcal{P}v = 0$. We also set

$$\widetilde{\mathcal{E}}_0 := \|v_0\|_{L^1(\mathbb{R}^n)} + \|v_0\|_{H^k(\mathbb{R}^n)} + \|\rho_0\|_{W^{1,1}(\mathbb{R}^{n-1})} + \|\rho_0\|_{H^{k+1}(\mathbb{R}^{n-1})}.$$

As a bi-product of our proof, we obtain the following result.

Theorem 1.3. *Let $n \geq 2$ and $k \geq \left[\frac{n+1}{2}\right]$. Suppose that Hypotheses (H1) and (H3) are satisfied. There exists $C > 0$ such that if $\widetilde{\mathcal{E}}_0$ is sufficiently small, then the traveling wave solution φ of equation* (12) *is stable in the sense that the perturbation (ρ, v) given in* (13) *satisfies the decay estimates for all $t \geq 0$*

$$\|v(t)\|_{H^k(\mathbb{R}^n)} \leq C(1+t)^{-\frac{n+1}{2}} \widetilde{\mathcal{E}}_0, \tag{14a}$$

$$\|\rho(t)\|_{H^k(\mathbb{R}^{n-1})} \leq C(1+t)^{-\frac{n-1}{4}} \widetilde{\mathcal{E}}_0, \tag{14b}$$

$$\|\nabla_{\mathbf{z}} \cdot \rho(t)\|_{H^k(\mathbb{R}^{n-1})} \leq C(1+t)^{-\frac{n+1}{4}} \widetilde{\mathcal{E}}_0. \tag{14c}$$

Note that Bates and Chen [1] only proved Theorem 1.3 in dimension $n \geq 4$ and thus our result generalizes their analysis to the remaining dimensions $n = 2, 3$. Compared to Theorem 1.2, we obtain a sharper decay of v component of the perturbation. This is a consequence of the fact that the projection $\mathcal{P}$ commutes with each linear operator $-u + \mathcal{J} *_{x_i} u$ for $i = 2 \cdots n$.

Outline. The paper is organized in three parts. In section 2, we prove Theorem 1.2 in dimension $n \geq 4$. In the following section we study the semigroup associated to the linear operator $\mathcal{L}$ and derive estimates crucial for our nonlinear stability analysis. Finally, in section 4, we prove Theorem 1.2 for the remaining dimensions $n = 2, 3$ and Theorem 1.3 is proved in section 5.

2. **Stability in high dimension.** In this section, we give a simple proof of Theorem 1.2 in the high-dimensional case $n \geq 4$, following ideas that have been developed for the multidimensional local Allen-Cahn equations [18, 13] and then, generalized to reaction-diffusion systems for example [10]. The main ingredient of the proof is an estimate (see (18)) for the linearized evolution operator $\mathcal{L}$ which will be established in the following section 3.

We consider a solution $u(\mathbf{x}, t) = \varphi(\xi) + v(\mathbf{x}, t)$ of (1) which satisfies the equation

$$\partial_t v(\mathbf{x}, t) = \mathcal{L}v(\mathbf{x}, t) + \mathcal{H}(v(\mathbf{x}, t)), \tag{15}$$

where

$$\mathcal{H}(v) := f(\varphi + v) - f(\varphi) - f'(\varphi)v. \tag{16}$$

The Cauchy problem associated to equation (15) with initial condition $v_0 \in H^k(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$, with $k \geq n + 1$ and $n \geq 4$ is locally well-posed in $H^k(\mathbb{R}^n)$. This is equivalent to say that for any $v_0 \in H^k(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ there exists a time $T > 0$ such that (15) as a unique mild solution in $H^k(\mathbb{R}^n)$ defined on $[0, T]$ satisfying $v(0) = v_0$. The integral formulation of (15) is given by

$$v(t) = \mathcal{S}_{\mathcal{L}}(t)v_0 + \int_0^t \mathcal{S}_{\mathcal{L}}(t-s)\mathcal{H}(v(s))ds, \tag{17}$$

where $\mathcal{S}_{\mathcal{L}}$ is the semigroup associated to the linear operator $\mathcal{L}$. Anticipating the estimates derived in the following sections (see (39)), there exist positive constants C and θ such that

$$\|\mathcal{S}_{\mathcal{L}}(t)v\|_{H^k(\mathbb{R}^n)} \leq C \left((1+t)^{-\frac{n-1}{4}} \|v\|_{L^1(\mathbb{R}^n)} + e^{-\theta t} \|v\|_{H^k(\mathbb{R}^n)} \right). \tag{18}$$

The nonlinear contribution $\mathcal{H}(v)$ is at least quadratic in v close to the origin. As a consequence, we can find a positive nondecreasing function $\kappa : \mathbb{R}_+ \to \mathbb{R}_+$ such that, for all $t \in [0, T]$,

$$|\mathcal{H}(v)| \leq \kappa(R)|v|^2, \quad \text{for } |v| \leq R.$$

Let $T_* > 0$ be the maximal time of existence of a solution $v \in H^k(\mathbb{R}^n)$ with initial condition $v_0 \in H^k(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$. For $t \in [0, T*)$ we define

$$\Phi(t) = \sup_{0 \leq s \leq t} (1+s)^{\frac{n-1}{4}} \|v(s)\|_{H^k(\mathbb{R}^n)}.$$

Using estimate (18) directly into the integral formulation (17) yields

$$\|v(t)\|_{H^k(\mathbb{R}^n)} \leq \|\mathcal{S}_{\mathcal{L}}(t)v_0\|_{H^k(\mathbb{R}^n)} + \int_0^t \|\mathcal{S}_{\mathcal{L}}(t-s)\mathcal{H}(v(s))\|_{H^k(\mathbb{R}^n)}ds$$

$$\lesssim (1+t)^{-\frac{n-1}{4}} \|v_0\|_{L^1(\mathbb{R}^n)} + e^{-\theta t}\|v_0\|_{H^k(\mathbb{R}^n)}$$

$$+ \kappa(\Phi(t)) \int_0^t (1+t-s)^{-\frac{n-1}{4}} \|v(s)\|_{H^k(\mathbb{R}^n)}^2 ds$$

$$+ \kappa(\Phi(t)) \int_0^t e^{-\theta(t-s)} \|v(s)\|_{H^k(\mathbb{R}^n)}^2 ds.$$

Here, and throughout the paper we use the notation $A \lesssim B$ whenever $A \leq \kappa B$ for $\kappa > 0$ a constant independent of time t. From Lemma A.1, there exist constants $C_1 > 0$ and $C_2 > 0$ so that

$$\int_0^t (1+t-s)^{-\frac{n-1}{4}}(1+s)^{-\frac{n-1}{2}} ds \leq C_1(1+t)^{-\frac{n-1}{4}},$$

$$\int_0^t e^{-\theta(t-s)}(1+s)^{-\frac{n-1}{2}} ds \leq C_2(1+t)^{-\frac{n-1}{4}}.$$

Note that the first inequality is a consequence of our careful choice of n. Indeed, this inequality is only true for $\frac{n-1}{4} > \frac{1}{2}$ ($n \geq 4$). Then, for all $t \in [0, T_*)$ we have

$$\Phi(t) \leq C_0 \left(\|v_0\|_{L^1(\mathbb{R}^n)} + \|v_0\|_{H^k(\mathbb{R}^n)} \right) + \widetilde{C}_0 \kappa(\Phi(t))\Phi(t)^2,$$

for some positive constants C_0 and $\widetilde{C}_0$. Suppose that the initial condition v_0 is small enough so that

$$2C_0 \left(\|v_0\|_{L^1(\mathbb{R}^n)} + \|v_0\|_{H^k(\mathbb{R}^n)} \right) < 1 \quad \text{and} \quad 4C_0\widetilde{C}_0\kappa(1) \left(\|v_0\|_{L^1(\mathbb{R}^n)} + \|v_0\|_{H^k(\mathbb{R}^n)} \right) < 1,$$

then

$$\Phi(t) \leq 2C_0 \left(\|v_0\|_{L^1(\mathbb{R}^n)} + \|v_0\|_{H^k(\mathbb{R}^n)} \right) < 1,$$

for all $t \in [0, T_*)$. This implies that the maximal time of existence is $T_* = +\infty$ and the solution v of (15) satisfies:

$$\sup_{t \geq 0} (1+t)^{\frac{n-1}{4}} \|v(t)\|_{H^k(\mathbb{R}^n)} \leq 2C_0 \left(\|v_0\|_{L^1(\mathbb{R}^n)} + \|v_0\|_{H^k(\mathbb{R}^n)} \right).$$

3. **Linear estimates.** We first start this section by deriving the nonlinear problem that we will be solving in the next section and then derive estimates of the corresponding linear parts.

3.1. **Setup of the problem.** We represent each solution $u(\mathbf{x}, t)$ of (2) through the decomposition

$$u(\mathbf{x}, t) := \varphi(\xi - \rho(\mathbf{z}, t)) + v(\xi - \rho(\mathbf{z}, t), \mathbf{z}, t) \tag{19}$$

where $\rho : \mathbb{R}^{n-1} \to \mathbb{R} \in H^1(\mathbb{R}^{n-1})$ and $v : \mathbb{R}^n \to \mathbb{R} \in H^1(\mathbb{R}^n)$. Note that the perturbation v can be written as

$$v(\mathbf{x}, t) = u(\xi + \rho(\mathbf{z}, t), \mathbf{z}, t) - \varphi(\xi),$$

and v is in range of $\mathcal{L}_0$ such that $\mathcal{P}v = 0$. Note that such a decomposition (10) is always possible. Indeed, suppose that $w \in H^k(\mathbb{R}^n)$ is given and small enough. In

order to use the decomposition (10), we need to find a unique pair $(\rho(w), v(w)) \in H^k(\mathbb{R}^{n-1}) \times H^k(\mathbb{R}^n)$ with $\mathcal{P}v = 0$ that satisfies

$$\varphi + w = \mathcal{T}_\rho \cdot (\varphi + v),$$

where $\mathcal{T}_\rho \cdot \varphi(\xi) = \varphi(\xi - \rho)$ and $\mathcal{T}_\rho \cdot v(\xi, \mathbf{z}, t) = v(\xi - \rho, \mathbf{z}, t)$. Reproducing the standard argument of Kapitula in [13], we use Taylor's theorem to write

$$\varphi - \mathcal{T}_{-\rho} \cdot \varphi = -\rho \int_0^1 \mathcal{T}_{s\rho} \cdot \varphi' \mathrm{d}s,$$

so that we obtain the equivalent equation

$$\mathcal{T}_{-\rho} \cdot w = v - \rho \int_0^1 \mathcal{T}_{s\rho} \cdot \varphi' \mathrm{d}s.$$

Taking the inner product with ψ the eigenfunction of the adjoint operator $\mathcal{L}_0^*$ associated to the zero eigenvalue yields

$$\langle \mathcal{T}_{-\rho} \cdot w, \psi \rangle = -\rho \left\langle \int_0^1 \mathcal{T}_{s\rho} \cdot \varphi' \mathrm{d}s, \psi \right\rangle.$$

Here, we have used the fact that we look for solution v so such that $\mathcal{P}v = 0$. We can then define the functional $F : H^k(\mathbb{R}^n) \times H^k(\mathbb{R}^{n-1}) \to H^k(\mathbb{R}^{n-1})$ by

$$F(w, \rho) = \langle \mathcal{T}_{-\rho} \cdot w, \psi \rangle + \rho \left\langle \int_0^1 \mathcal{T}_{s\rho} \cdot \varphi' \mathrm{d}s, \psi \right\rangle,$$

with $F(0, 0) = 0$ and Fréchet derivative $D_\rho F(0, 0) = \mathrm{id}$ where id is the identity operator. By the implicit function theorem with have on a neighborhood of $(0, 0)$ the existence of $\rho(w)$ such that $F(w, \rho(w)) = 0$. We then apply the projection $\mathcal{Q}$ to the equation $\mathcal{T}_{-\rho} \cdot w = v - \rho \int_0^1 \mathcal{T}_{s\rho} \cdot \varphi' \mathrm{d}s$ and obtain

$$v = \mathcal{Q}\mathcal{T}_{-\rho(w)} \cdot w + \mathcal{Q}\left(\rho(w) \int_0^1 \mathcal{T}_{s\rho(w)} \cdot \varphi' \mathrm{d}s\right),$$

which clearly admits a solution $v(w)$ with $\mathcal{P}v(w) = 0$. As a conclusion, all sufficiently small perturbation can be written as in equation (10).

We can now substitute the Ansatz (19) into (2) to get the evolution equation:

$$-\partial_t \rho \dot\varphi_\rho - c\varphi_\rho + \partial_t v_\rho - c\partial_\xi v_\rho - \partial_t \rho \partial_\xi v_\rho = -\varphi_\rho - v_\rho + \mathcal{K} *_{\mathbf{x}} \varphi_\rho + \mathcal{K} *_{\mathbf{x}} v_\rho + f(\varphi_\rho + v_\rho),$$

with $\varphi_\rho = \varphi(\cdot - \rho)$ and $v_\rho = v(\cdot - \rho, \cdot, \cdot)$. As φ is solution of (4), we obtain:

$$(\partial_t - \mathcal{L}) v = (\partial_t - \mathcal{L})(\rho\dot\varphi) + \mathcal{H}(v) + \mathcal{N}(\rho, v) + \mathcal{R}(\rho, v) \tag{20}$$

where the nonlinear term $\mathcal{H}$ has been defined in (16), $\mathcal{R}(\rho, v) := \partial_t \rho \partial_\xi v$ and the remaining term $\mathcal{N}$ is split into two different parts

$$\mathcal{N}(\rho, v) := \mathcal{N}_1(\rho) + \mathcal{N}_2(\rho, v), \tag{21}$$

where

$$\mathcal{N}_1(\rho)(\mathbf{x}, t) :$$
$$= \int_{\mathbb{R}^n} \mathcal{K}(\mathbf{x} - \mathbf{x}')\varphi\left(\xi' + \rho(\mathbf{z}, t) - \rho(\mathbf{z}', t)\right) \mathrm{d}\xi' \mathrm{d}\mathbf{z}' - \int_{\mathbb{R}^n} \mathcal{K}(\mathbf{x} - \mathbf{x}')\varphi(\xi')\mathrm{d}\xi' \mathrm{d}\mathbf{z}'$$
$$- \int_{\mathbb{R}^n} \mathcal{K}(\mathbf{x} - \mathbf{x}')\dot\varphi(\xi')\left(\rho(\mathbf{z}, t) - \rho(\mathbf{z}', t)\right) \mathrm{d}\xi' \mathrm{d}\mathbf{z}', \tag{22}$$

and

$$\mathcal{N}_2(\rho,v)(\mathbf{x},t) := \int_{\mathbb{R}^n} \mathcal{K}(\mathbf{x}-\mathbf{x}')v\left(\xi'+\rho(\mathbf{z},t)-\rho(\mathbf{z}',t),\mathbf{z}',t\right)\mathrm{d}\xi'\mathrm{d}\mathbf{z}'$$
$$-\int_{\mathbb{R}^n} \mathcal{K}(\mathbf{x}-\mathbf{x}')v(\xi',\mathbf{z}',t)\mathrm{d}\xi'\mathrm{d}\mathbf{z}'. \tag{23}$$

One can check that the third term of $\mathcal{N}_1(\varphi,\rho)$ is actually $\mathcal{L}(\rho\dot\varphi)$ as

$$\mathcal{L}(\rho\dot\varphi) = -\rho\dot\varphi + \mathcal{K}*_{\mathbf{x}}(\rho\dot\varphi) + c\rho\ddot\varphi + f'(\varphi)\rho\dot\varphi$$
$$= \mathcal{K}*_{\mathbf{x}}(\rho\dot\varphi) - \rho\mathcal{K}_0*\dot\varphi$$
$$= -\int_{\mathbb{R}^n} \mathcal{K}(\mathbf{x}-\mathbf{x}')\dot\varphi(\xi')\left(\rho(\mathbf{z},t)-\rho(\mathbf{z}',t)\right)\mathrm{d}\xi'\mathrm{d}\mathbf{z}'.$$

Finally, if we denote $\mathcal{S}_{\mathcal{L}}(t)$ the semigroup generated by the linear operator $\mathcal{L}$, applying Duhamel's formula to (20), we obtain

$$v(t) = \mathcal{S}_{\mathcal{L}}(t)v_0 + \rho(t)\dot\varphi - \mathcal{S}_{\mathcal{L}}(t)(\rho_0\dot\varphi)$$
$$+ \int_0^t \mathcal{S}_{\mathcal{L}}(t-s)\left(\mathcal{H}(v(s)) + \mathcal{N}(\rho(s),v(s)) + \mathcal{R}(\rho(s),v(s))\right)\mathrm{d}s.$$

As v is in the range of $\mathcal{L}_0$, we must have

$$v(t) = \mathcal{Q}\mathcal{S}_{\mathcal{L}}(t)v_0 - \mathcal{Q}\mathcal{S}_{\mathcal{L}}(t)(\rho_0\dot\varphi)$$
$$+ \int_0^t \mathcal{Q}\mathcal{S}_{\mathcal{L}}(t-s)\left(\mathcal{H}(v(s)) + \mathcal{N}(\rho(s),v(s)) + \mathcal{R}(\rho(s),v(s))\right)\mathrm{d}s, \tag{24a}$$
$$\rho(t) = \langle \mathcal{S}_{\mathcal{L}}(t)(\rho_0\dot\varphi - v_0), \psi\rangle$$
$$- \int_0^t \langle \mathcal{S}_{\mathcal{L}}(t-s)\left(\mathcal{H}(v(s)) + \mathcal{N}(\rho(s),v(s)) + \mathcal{R}(\rho(s),v(s))\right), \psi\rangle\mathrm{d}s. \tag{24b}$$

In the following sections, we will derive estimates on $\mathcal{S}_{\mathcal{L}}(t)$.

3.2. **Study of $\mathcal{S}_{\mathcal{L}}(t)$.** We consider the initial value problem

$$\partial_t u = \mathcal{L}u, \quad u(\cdot,0) = u_0 \in H^k(\mathbb{R}^n), \tag{25}$$

which has the solution $u(\xi,\mathbf{z},t) = \mathcal{S}_{\mathcal{L}}(t)u_0(\xi,\mathbf{z})$. From its definition, $\mathcal{L}$ can be written as

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{A},$$

where the operator $\mathcal{A}$ is defined as

$$\mathcal{A}u := -\mathcal{K}_0 *_\xi u + \mathcal{K} *_{\mathbf{x}} u, \quad u \in L^2(\mathbb{R}^n). \tag{26}$$

Let $\hat u$ represent the Fourier transform of u in $\mathbf{z}$:

$$\hat u(\xi,\widetilde{\mathbf{k}},t) = \int_{\mathbb{R}^{n-1}} u(\xi,\mathbf{z},t)e^{-i\mathbf{z}\cdot\widetilde{\mathbf{k}}}\mathrm{d}\mathbf{z}$$

so that (25) is transformed into

$$\partial_t \hat u(\xi,\widetilde{\mathbf{k}},t) = \mathcal{L}_0\hat u(\xi,\widetilde{\mathbf{k}},t) + \mathcal{B}(\widetilde{\mathbf{k}})\hat u(\xi,\widetilde{\mathbf{k}},t),$$

where

$$\mathcal{B}(\widetilde{\mathbf{k}})\hat u(\xi,\widetilde{\mathbf{k}}) := -\widehat{\mathcal{K}}_{n-1}(\mathbf{0}_{n-1}) *_\xi \hat u(\xi,\widetilde{\mathbf{k}}) + \widehat{\mathcal{K}}_{n-1}(\widetilde{\mathbf{k}}) *_\xi \hat u(\xi,\widetilde{\mathbf{k}}),$$

with

$$\widehat{\mathcal{K}}_{n-1}(\xi,\widetilde{\mathbf{k}}) := \int_{\mathbb{R}^{n-1}} \mathcal{K}(\mathbf{x})e^{-i\mathbf{z}\cdot\widetilde{\mathbf{k}}}\mathrm{d}\mathbf{z}.$$

We have used the fact that $\widehat{\mathcal{K}}_{n-1}(\xi, \mathbf{0}_{n-1}) = \mathcal{K}_0$ by definition. We readily note that for each $\widetilde{\mathbf{k}} \in \mathbb{R}^{n-1}$, the operator

$$\mathcal{L}(\widetilde{\mathbf{k}}) : \quad H^1(\mathbb{R}) \quad \longrightarrow \quad L^2(\mathbb{R})$$
$$u \quad \longmapsto \quad \mathcal{L}_0 u + \mathcal{B}(\widetilde{\mathbf{k}})u$$

defines a $\mathcal{C}^0$ semigroup with $\mathcal{L}(\mathbf{0}_{n-1}) = \mathcal{L}_0$.

Lemma 3.1. *The family of operators $\mathcal{L}(\widetilde{\mathbf{k}})$ satisfies the following properties.*

(i) *Near $\widetilde{\mathbf{k}} = \mathbf{0}_{n-1}$, the only eigenvalue λ is a smooth function of $\widetilde{\mathbf{k}}$ and the expression of $\lambda(\widetilde{\mathbf{k}})$ reads:*

$$\lambda(\widetilde{\mathbf{k}}) = -A\|\widetilde{\mathbf{k}}\|^2 + o\left(\|\widetilde{\mathbf{k}}\|^3\right), \tag{27}$$

with $A := d_0 \langle \mathcal{K}_0 *_\xi \varphi', \psi \rangle$.

(ii) *$\sigma\left(\mathcal{L}(\widetilde{\mathbf{k}})\right) \subset \{\Re(\lambda) < 0\}$, for $\widetilde{\mathbf{k}} \neq \mathbf{0}_{n-1}$.*

Proof. For the first property (i), we apply perturbation theory to the linear operator $\mathcal{L}(\widetilde{\mathbf{k}})$ for $\widetilde{\mathbf{k}}$ near zero. To this end, we define

$$\mathcal{F} : \quad \mathbb{R}^{n-1} \times \mathcal{C} \times H_\perp^1(\mathbb{R}) \quad \longrightarrow \quad L^2(\mathbb{R})$$
$$(\widetilde{\mathbf{k}}, \lambda, w) \quad \longmapsto \quad \left(\mathcal{L}(\widetilde{\mathbf{k}}) - \lambda\right)(\dot\varphi + w),$$

where $H_\perp^1(\mathbb{R}) = \left\{u \in H^1(\mathbb{R}) \mid \langle u, \dot\varphi \rangle = 0\right\}$. Applying the implicit function theorem, we see that there exist a small neighborhood of the origin and smooth functions $\lambda(\widetilde{\mathbf{k}})$ and $w(\widetilde{\mathbf{k}})$ such that $\mathcal{F}(\widetilde{\mathbf{k}}, \lambda(\widetilde{\mathbf{k}}), w(\widetilde{\mathbf{k}})) = 0$ on that neighborhood. We denote $q(\widetilde{\mathbf{k}}) = \dot\varphi + w(\widetilde{\mathbf{k}})$. Similarly for the adjoint operator $\mathcal{L}^*(\widetilde{\mathbf{k}})$, we have a smooth continuation of ψ given by $q^*(\widetilde{\mathbf{k}})$ so that

$$\langle q(\widetilde{\mathbf{k}}), q^*(\widetilde{\mathbf{k}}) \rangle = 1.$$

Differentiating $\mathcal{F}(\widetilde{\mathbf{k}}, \lambda(\widetilde{\mathbf{k}}), w(\widetilde{\mathbf{k}})) = 0$ with respect to $\widetilde{k}_j$, for any j, we find

$$\partial_{\widetilde{k}_j}\lambda(\mathbf{0}) = \partial_{\widetilde{k}_j}\left(\langle \mathcal{B}(\widetilde{\mathbf{k}})\varphi', \psi \rangle\right)_{\widetilde{\mathbf{k}} = \mathbf{0}_{n-1}} = 0.$$

Indeed, if $\ell \in \mathbb{R}$, we have that

$$\widehat{\mathcal{B}(\widetilde{\mathbf{k}})u}(\ell) = \int_{\mathbb{R}} \mathcal{B}(\widetilde{\mathbf{k}})u(\xi)e^{-\xi\ell}d\xi$$
$$= \left(-\widehat{\mathcal{K}}(\ell, \mathbf{0}_{n-1}) + \widehat{\mathcal{K}}(\ell, \widetilde{\mathbf{k}})\right)\hat{u}(\ell)$$
$$\sim -d_0\|\widetilde{\mathbf{k}}\|^2\hat{u}(\ell)$$

as $\|\widetilde{\mathbf{k}}\| \to 0$. Similarly, we find that for any j and l,

$$\partial^2_{\widetilde{k}_j\widetilde{k}_l}\lambda(\mathbf{0}) = 0.$$

Finally, for any j, we have that

$$\partial^2_{\widetilde{k}_j\widetilde{k}_j}\lambda(\mathbf{0}) = -2d_0 \langle \mathcal{K}_0 *_\xi \varphi', \psi \rangle,$$

which gives the desired expansion.

For the second property (ii), we have just seen that $\lambda(\widetilde{\mathbf{k}}) \neq 0$ for small values of $\widetilde{\mathbf{k}}$. For any $u \in H^1(\mathbb{R})$ we have that

$$\langle \mathcal{L}(\widetilde{\mathbf{k}})u, u \rangle = -\langle u, u \rangle + \langle \widehat{\mathcal{K}}_{n-1}(\widetilde{\mathbf{k}}) *_\xi u, u \rangle + \langle f'(\varphi)u, u \rangle$$

$$\leq \left(-1 + \widehat{\mathcal{K}}_{n-1}(0, \widetilde{\mathbf{k}}) + \sup_{\varphi \in [0,1]} f'(\varphi) \right) \langle u, u \rangle.$$

As $\widehat{\mathcal{K}}_{n-1}(0, \widetilde{\mathbf{k}}) \to 0$ as $\|\widetilde{\mathbf{k}}\| \to \infty$, there exist $M > 0$ and $c_M > 0$ so that for all $\|\widetilde{\mathbf{k}}\| \geq M$,

$$-1 + \widehat{\mathcal{K}}_{n-1}(0, \widetilde{\mathbf{k}}) + \sup_{\varphi \in [0,1]} f'(\varphi) < -c_M.$$

This implies that $\Re(\lambda) < -c_M < 0$ for all $\lambda \in \sigma(\mathcal{L}(\widetilde{\mathbf{k}}))$ with $\|\widetilde{\mathbf{k}}\| \geq M$. For the region in-between, compactness and local robustness of the spectrum ensure that $\sigma\left(\mathcal{L}(\widetilde{\mathbf{k}})\right) \subset \{\Re(\lambda) < 0\}$. $\square$

Based on Lemma 3.1, there exists $\epsilon > 0$, so that $\lambda(\widetilde{\mathbf{k}})$ is a simple eigenvalue of $\mathcal{L}(\widetilde{\mathbf{k}})$ in $\|\widetilde{\mathbf{k}}\| \leq 2\epsilon$. As a consequence, there exists a smooth spectral projection operator, $\mathcal{P}(\widetilde{\mathbf{k}})$, given by

$$\mathcal{P}(\widetilde{\mathbf{k}})u = \frac{1}{2\pi i} \int_\Gamma \left(\mathcal{L}(\widetilde{\mathbf{k}}) - \lambda \right)^{-1} u d\lambda, \tag{28}$$

where Γ is a simple closed curve in the complex plane enclosing the zero eigenvalue. More conveniently, we write $\mathcal{P}(\widetilde{\mathbf{k}})$ as

$$\mathcal{P}(\widetilde{\mathbf{k}})u(\xi) = \left(\int_\mathbb{R} q^*(\widetilde{\mathbf{k}}, \xi)u(\xi)d\xi \right) q(\widetilde{\mathbf{k}}, \xi) := \langle q^*(\widetilde{\mathbf{k}}), u \rangle q(\widetilde{\mathbf{k}}), \quad \|\widetilde{\mathbf{k}}\| \leq 2\epsilon. \tag{29}$$

Following some ideas developed in [17, 12] for viscous conservation laws, we introduce a smooth cutoff function $\chi(\widetilde{\mathbf{k}})$ that is identically one for $\|\widetilde{\mathbf{k}}\| \leq \epsilon$ and identically zero for $\|\widetilde{\mathbf{k}}\| \geq 2\epsilon$. We can then split the solution operator $\mathcal{S}_\mathcal{L}(t)u_0$ into a low-frequency part

$$\mathcal{S}_\mathcal{L}^I(t)u_0(\xi, \mathbf{z}) := \frac{1}{(2\pi)^{n-1}} \int_{\mathbb{R}^{n-1}} e^{i\widetilde{\mathbf{k}} \cdot \mathbf{z}} e^{\mathcal{L}(\widetilde{\mathbf{k}})t} \left[\chi(\widetilde{\mathbf{k}})\mathcal{P}(\widetilde{\mathbf{k}})\hat{u}_0(\xi, \widetilde{\mathbf{k}}) \right] \mathrm{d}\widetilde{\mathbf{k}} \tag{30}$$

and the associated high-frequency part

$$\mathcal{S}_\mathcal{L}^{II}(t)u_0(\xi, \mathbf{z}) := \frac{1}{(2\pi)^{n-1}} \int_{\mathbb{R}^{n-1}} e^{i\widetilde{\mathbf{k}} \cdot \mathbf{z}} e^{\mathcal{L}(\widetilde{\mathbf{k}})t} \left[\left(\mathrm{id} - \chi(\widetilde{\mathbf{k}})\mathcal{P}(\widetilde{\mathbf{k}}) \right) \hat{u}_0(\xi, \widetilde{\mathbf{k}}) \right] \mathrm{d}\widetilde{\mathbf{k}}, \tag{31}$$

where id denotes the identity. One can easily check that we have $\mathcal{S}_\mathcal{L}(t) = \mathcal{S}_\mathcal{L}^I(t) + \mathcal{S}_\mathcal{L}^{II}(t)$.

3.2.1. *Low-frequency bounds.* We introduce the Green kernel associated with $\mathcal{S}_\mathcal{L}^I(t)$ as

$$\mathbf{G}^I(\mathbf{x}, t; \mathbf{x}') := \mathcal{S}_\mathcal{L}^I(t)\delta_{\mathbf{x}'}(\mathbf{x}), \tag{32}$$

where $\mathbf{x} = (\xi, \mathbf{z})$ and $\mathbf{x}' = (\xi', \mathbf{z}')$.

Proposition 1. *The Green kernel* $\mathbf{G}^I$ *satisfies*

$$\mathbf{G}^I(\mathbf{x}, t; \mathbf{x}') = \frac{1}{(2\pi)^{n-1}} \int_{\mathbb{R}^{n-1}} e^{i\widetilde{\mathbf{k}} \cdot (\mathbf{z} - \mathbf{z}')} \chi(\widetilde{\mathbf{k}}) e^{\lambda(\widetilde{\mathbf{k}})t} q(\widetilde{\mathbf{k}}, \xi) q^*(\widetilde{\mathbf{k}}, \xi') d\widetilde{\mathbf{k}}. \tag{33}$$

Proof. First of all, through a direct computation, we have

$$\widehat{\delta_{\mathbf{x}'}}(\xi, \widetilde{\mathbf{k}}) = \int_{\mathbb{R}^{n-1}} e^{-i\widetilde{\mathbf{k}}\cdot\mathbf{z}} \delta_{(\xi', \mathbf{z}')}(\xi, \mathbf{z}) d\mathbf{z} = e^{-i\widetilde{\mathbf{k}}\cdot\mathbf{z}'} \delta_{\xi'}(\xi).$$

Using the properties of the spectral projection $\mathcal{P}(\widetilde{\mathbf{k}})$, we further have

$$\mathcal{P}(\widetilde{\mathbf{k}})\widehat{\delta_{\mathbf{x}'}}(\xi, \widetilde{\mathbf{k}}) = q^*(\widetilde{\mathbf{k}}, \xi')q(\widetilde{\mathbf{k}}, \xi).$$

Finally, noticing that $e^{\mathcal{L}(\widetilde{\mathbf{k}})t}q(\widetilde{\mathbf{k}}, \xi) = e^{\lambda(\widetilde{\mathbf{k}})t}q(\widetilde{\mathbf{k}}, \xi)$, we obtain the desired formula. $\square$

Proposition 2. *The low-frequency Green function $\mathbf{G}^I(\mathbf{x}, t; \mathbf{x}')$ of* (32) *can be decomposed as $\mathbf{G}^I(\mathbf{x}, t; \mathbf{x}') := \dot{\varphi}(\xi)\Psi(\mathbf{z} - \mathbf{z}', t; \xi') + \widetilde{\mathbf{G}}^I(\mathbf{x}, t; \mathbf{x}')$, for which the following estimates hold:*

$$\sup_{\xi'} \|\Psi(\cdot, t; \xi')\|_{L^2(\mathbb{R}^{n-1})} \lesssim (1+t)^{-\frac{n-1}{4}}, \tag{34a}$$

$$\sup_{\xi'} \|\Psi(\cdot, t)\|_{H^k(\mathbb{R}^{n-1})} \lesssim (1+t)^{-\frac{n-1}{4} - \frac{|\alpha|}{2}}, \tag{34b}$$

$$\sup_{\mathbf{x}'} \|\widetilde{\mathbf{G}}^I(\cdot, t; \mathbf{x}')\|_{L^2(\mathbb{R}^n)} \lesssim (1+t)^{-\frac{n-1}{4} - 1}, \tag{34c}$$

$$\sup_{\mathbf{x}'} \|\widetilde{\mathbf{G}}^I(\cdot, t; \mathbf{x}')\|_{H^k(\mathbb{R}^n)} \lesssim (1+t)^{-\frac{n-1}{4} - \frac{|\alpha|+2}{2}}. \tag{34d}$$

for $\alpha \in \mathbb{Z}_+^{n-1}$ with $|\alpha| \le k$. Moreover, $\mathcal{P}\widetilde{\mathbf{G}}^I(\mathbf{x}, t; \mathbf{x}') = 0$.

Proof. The idea of the proof is based on the remark that for $\|\widetilde{\mathbf{k}}\| \le 2\epsilon$, the smooth eigenfunctions $q(\widetilde{\mathbf{k}}, \xi)$ and $q^*(\widetilde{\mathbf{k}}, \xi)$ have an expansion of the form

$$q(\widetilde{\mathbf{k}}, \xi) = \dot{\varphi}(\xi) + \mathcal{O}\left(\|\widetilde{\mathbf{k}}\|^2\right),$$

$$q^*(\widetilde{\mathbf{k}}, \xi') = \psi(\xi') + \mathcal{O}\left(\|\widetilde{\mathbf{k}}\|^2\right).$$

This leads us to introduce an auxiliary function $\widetilde{\Psi}(\mathbf{z}, t)$ of the form

$$\widetilde{\Psi}(\mathbf{z}, t) := \frac{1}{(2\pi)^{n-1}} \int_{\mathbb{R}^{n-1}} e^{i\widetilde{\mathbf{k}}\cdot\mathbf{z}} \chi(\widetilde{\mathbf{k}}) e^{\lambda(\widetilde{\mathbf{k}})t} d\widetilde{\mathbf{k}},$$

so that we formally have

$$\mathbf{G}^I(\mathbf{x}, t; \mathbf{x}') - \dot{\varphi}(\xi)\psi(\xi')\widetilde{\Psi}(\mathbf{z} - \mathbf{z}', t) = \frac{1}{(2\pi)^{n-1}} \int_{\mathbb{R}^{n-1}} e^{i\widetilde{\mathbf{k}}\cdot(\mathbf{z}-\mathbf{z}')} \chi(\widetilde{\mathbf{k}}) e^{\lambda(\widetilde{\mathbf{k}})t} \mathcal{O}\left(\|\widetilde{\mathbf{k}}\|^2\right) d\widetilde{\mathbf{k}}. \tag{35}$$

On the one hand, a simple Fourier transform computation shows that

$$\frac{1}{(2\pi)^{n-1}} \int_{\mathbb{R}^{n-1}} e^{i\widetilde{\mathbf{k}}\cdot(\mathbf{z}-\mathbf{z}')} e^{-A\|\widetilde{\mathbf{k}}\|^2 t} d\widetilde{\mathbf{k}} = (4\pi At)^{-\frac{n-1}{2}} \exp\left(-\frac{\|\mathbf{z}-\mathbf{z}'\|^2}{4At}\right),$$

where $A = d_0 \langle \mathcal{K}_0 *_\xi \varphi', \psi \rangle$, which directly gives us bounds for $\widetilde{\Psi}(\cdot, t)$ that are similar to the standard diffusive bounds satisfied for the heat equations:

$$\|\widetilde{\Psi}(\cdot, t)\|_{L^2(\mathbb{R}^{n-1})} \lesssim (1+t)^{-\frac{n-1}{4}}, \tag{36a}$$

$$\|\widetilde{\Psi}(\cdot, t)\|_{H^k(\mathbb{R}^{n-1})} \lesssim (1+t)^{-\frac{n-1}{4} - \frac{|\alpha|}{2}}, \tag{36b}$$

for $\alpha \in \mathbb{Z}_+^{n-1}$ with $|\alpha| \le k$. On the other hand, because of the presence of terms of the form $\|\widetilde{\mathbf{k}}\|^2 e^{-A\|\widetilde{\mathbf{k}}\|^2 t}$ in the rest term of equation (35), the decay rate is improved by factor $(1+t)^{-1}$ so that we have the following estimates for $\widetilde{G}^I := \mathbf{G}^I - \dot{\varphi}\psi\widetilde{\Psi}$.

$$\sup_{\mathbf{x}'} \|\widetilde{G}^I(\cdot, t; \mathbf{x}')\|_{L^2(\mathbb{R}^n)} \lesssim (1+t)^{-\frac{n-1}{4}-1}, \tag{37a}$$

$$\sup_{\mathbf{x}'} \|\widetilde{G}^I(\cdot, t; \mathbf{x}')\|_{H^k(\mathbb{R}^n)} \lesssim (1+t)^{-\frac{n-1}{4}-\frac{|\alpha|}{2}-1}, \tag{37b}$$

for $\alpha \in \mathbb{Z}_+^{n-1}$ with $|\alpha| \le k$. Now, we can define $\Psi(\mathbf{z} - \mathbf{z}', t; \xi')$ as

$$\Psi(\mathbf{z} - \mathbf{z}', t; \xi') := \frac{1}{(2\pi)^{n-1}} \int_{\mathbb{R}^{n-1}} e^{i\widetilde{\mathbf{k}}\cdot\mathbf{z}} \chi(\widetilde{\mathbf{k}}) e^{\lambda(\widetilde{\mathbf{k}})t} q^*(\widetilde{\mathbf{k}}, \xi')\langle q(\widetilde{\mathbf{k}}, \cdot), \psi\rangle \widetilde{\mathbf{k}} \tag{38}$$

and set $\widetilde{\mathbf{G}}^I(\mathbf{x}, t; \mathbf{x}') := \mathbf{G}^I(\mathbf{x}, t; \mathbf{x}') - \dot{\varphi}(\xi)\Psi(\mathbf{z} - \mathbf{z}', t; \xi')$. And all the estimates (34) are readily obtained from (36) and (37). $\qquad\square$

Proposition 3. *The linear operator $\mathcal{S}_{\mathcal{L}}^I(t)$ satisfies the decay estimate*

$$\left\|\mathcal{S}_{\mathcal{L}}^I(t)u\right\|_{H^k(\mathbb{R}^n)} \lesssim (1+t)^{-\frac{n-1}{4}}\|u\|_{L^1(\mathbb{R}^n)}.$$

Furthermore, we have

$$\left\|\mathcal{Q}\mathcal{S}_{\mathcal{L}}^I(t)u\right\|_{H^k(\mathbb{R}^n)} \lesssim (1+t)^{-\frac{n-1}{4}-1}\|u\|_{L^1(\mathbb{R}^n)}.$$

Proof. The proof of the proposition easily follows from the estimates (34) by first noticing that

$$\mathcal{S}_{\mathcal{L}}^I(t)u(\mathbf{x}) = \int_{\mathbb{R}^n} \mathbf{G}^I(\mathbf{x}, t; \mathbf{x}')u(\mathbf{x}')\mathrm{d}\mathbf{x}', \quad \mathbf{x} \in \mathbb{R}^n,$$

and

$$\int_{\mathbb{R}^n} \left|\mathcal{S}_{\mathcal{L}}^I(t)u(\mathbf{x})\right|^2 \mathrm{d}\mathbf{x} \le \left(\sup_{\mathbf{x}'} \|\mathbf{G}^I(\cdot, t; \mathbf{x}')\|_{L^2(\mathbb{R}^n)}\right)^2 \left(\int_{\mathbb{R}^n} |u(\mathbf{x}')|\mathrm{d}\mathbf{x}'\right)^2.$$

The estimates in $H^k(\mathbb{R}^n)$ are obtained via similar computations. Finally, we recall the decomposition of $\mathbf{G}^I(\mathbf{x}, t; \mathbf{x}')$ implies that

$$\mathcal{Q}\mathbf{G}^I(\mathbf{x}, t; \mathbf{x}') = \widetilde{\mathbf{G}}^I(\mathbf{x}, t; \mathbf{x}'),$$

which completes the proof of the proposition. $\qquad\square$

3.2.2. *High-frequency bounds.* We now study the high-frequency bounds associated to $\mathcal{S}_{\mathcal{L}}^{II}(t)$. By definition, we have

$$\mathcal{S}_{\mathcal{L}}^{II}(t)u(\xi, \mathbf{z}) := \frac{1}{(2\pi)^{n-1}} \int_{\mathbb{R}^{n-1}} e^{i\widetilde{\mathbf{k}}\cdot\mathbf{z}} e^{\mathcal{L}(\widetilde{\mathbf{k}})t} \left[\left(\mathrm{id} - \chi(\widetilde{\mathbf{k}})\mathcal{P}(\widetilde{\mathbf{k}})\right)\hat{u}(\xi, \widetilde{\mathbf{k}})\right] \mathrm{d}\widetilde{\mathbf{k}},$$

where $\mathcal{P}(\widetilde{\mathbf{k}})$ is such that

$$\int_{\mathbb{R}} \left|e^{\mathcal{L}(\widetilde{\mathbf{k}})t}\left[\left(\mathrm{id} - \chi(\widetilde{\mathbf{k}})\mathcal{P}(\widetilde{\mathbf{k}})\right)\hat{u}(\xi, \widetilde{\mathbf{k}})\right]\right|^2 \mathrm{d}\xi \le e^{-2\theta t}\left\|u(\cdot, \widetilde{\mathbf{k}})\right\|_{L^2(\mathbb{R})}^2,$$

where $\theta > 0$ is a positive constant which depends only on ϵ. Then, using Parseval's inequality, one obtains

$$\left\|\mathcal{S}_{\mathcal{L}}^{II}(t)u\right\|_{L^2(\mathbb{R}^n)} \lesssim e^{-\theta t}\|u\|_{L^2(\mathbb{R}^n)},$$

from which one can deduce $H^k(\mathbb{R}^n)$-estimates.

Proposition 4. *The linear operator $\mathcal{S}_{\mathcal{L}}^{II}(t)$ satisfies the decay estimate*

$$\left\|\mathcal{S}_{\mathcal{L}}^{II}(t)u\right\|_{H^k(\mathbb{R}^n)} \lesssim e^{-\theta t}\|u\|_{H^k(\mathbb{R}^n)}.$$

As a conclusion, combining Proposition 3 and 4, we arrive at the linear estimate for semigroup $\mathcal{S}_{\mathcal{L}}(t)$ of $\mathcal{L}$. For $u \in H^k(\mathbb{R}^n)$, we have

$$\|\mathcal{S}_{\mathcal{L}}(t)u\|_{H^k(\mathbb{R}^n)} \lesssim (1+t)^{-\frac{n-1}{4}}\|u\|_{L^1(\mathbb{R}^n)} + e^{-\theta t}\|u\|_{H^k(\mathbb{R}^n)}. \tag{39}$$

As a consequence of our analysis, we also have

$$\|\mathcal{Q}\mathcal{S}_{\mathcal{L}}(t)u\|_{H^k(\mathbb{R}^n)} \lesssim (1+t)^{-\frac{n-1}{4}-1}\|u\|_{L^1(\mathbb{R}^n)} + e^{-\theta t}\|u\|_{H^k(\mathbb{R}^n)}, \tag{40}$$

and

$$\|\nabla_{\mathbf{z}} \cdot \mathcal{S}_{\mathcal{L}}(t)u\|_{H^k(\mathbb{R}^n)} \lesssim (1+t)^{-\frac{n-1}{4}-\frac{1}{2}}\|u\|_{L^1(\mathbb{R}^n)} + t^{-\frac{1}{2}}e^{-\theta t}\|u\|_{H^k(\mathbb{R}^n)}. \tag{41}$$

4. **Nonlinear stability in dimension** 2 **and** 3**.** In this section, we prove Theorem 1.2 for the remaining dimensions 2 and 3.

4.1. **Some nonlinear estimates.** We first give estimates on the nonlinear terms that appear in our system (24). More precisely, we will prove the following lemma.

Lemma 4.1. *Let $k \geq \left[\frac{n+1}{2}\right]$. There exists a $\delta > 0$ such that for any $v \in H^k(\mathbb{R}^n)$ and $\rho \in H^{k+1}(\mathbb{R}^{n-1})$ with $\|v\|_{H^k(\mathbb{R}^n)} \leq \delta$, $\|\rho\|_{H^k(\mathbb{R}^{n-1})} \leq \delta$ and $\|\nabla_{\mathbf{z}}\cdot\rho\|_{H^k(\mathbb{R}^{n-1})} \leq \delta$ we have*

$$\|\mathcal{H}(v)\|_{L^1(\mathbb{R}^n)}, \|\mathcal{H}(v)\|_{H^k(\mathbb{R}^n)} \leq C\|v\|_{H^k(\mathbb{R}^n)}^2, \tag{42a}$$

$$\|\mathcal{N}_1(\rho)\|_{L^1(\mathbb{R}^n)}, \|\mathcal{N}_1(\rho)\|_{H^k(\mathbb{R}^n)} \leq C\|\nabla_{\mathbf{z}}\cdot\rho\|_{H^k(\mathbb{R}^{n-1})}^2, \tag{42b}$$

$$\|\mathcal{N}_2(\rho,v)\|_{L^1(\mathbb{R}^n)}, \|\mathcal{N}_2(\rho,v)\|_{H^k(\mathbb{R}^n)} \leq C\|\rho\|_{H^k(\mathbb{R}^{n-1})}\|v\|_{H^k(\mathbb{R}^n)}, \tag{42c}$$

$$\|\mathcal{R}(\rho,v)\|_{L^1(\mathbb{R}^n)}, \|\mathcal{R}(\rho,v)\|_{H^k(\mathbb{R}^n)}$$
$$\leq C\left(\|v\|_{H^k(\mathbb{R}^n)}^2 + \|\rho\|_{H^k(\mathbb{R}^{n-1})}\|v\|_{H^k(\mathbb{R}^n)} + \|\nabla_{\mathbf{z}}\cdot\rho\|_{H^k(\mathbb{R}^{n-1})}^2\right). \tag{42d}$$

Proof. Throughout the proof we will use that from Sobolev embedding we have

$$\|uv\|_{H^k(\mathbb{R}^n)} \leq C\|u\|_{H^k(\mathbb{R}^n)}\|v\|_{H^k(\mathbb{R}^n)} \text{ and } \|u\|_{L^\infty(\mathbb{R}^n)} \leq C\|u\|_{H^k(\mathbb{R}^n)}.$$

Note that in order to obtain the nonlinear estimates (42), we will only use the above Sobolev embedding and Taylor's theorem together with the fact that both f and φ are smooth with the a priori bounds on v and ρ. As the proofs of each estimate are almost similar, we will present only the key points.

- For $\underline{\mathcal{H}(v)}$. We use Taylor's formula to write

$$\mathcal{H}(v) = f(\varphi+v) - f(\varphi) - f'(\varphi)v = v^2 \int_0^1 (1-s)f''(\varphi+sv)\mathrm{d}s. \tag{43}$$

As φ and v are both bounded, we have

$$\|\mathcal{H}(v)\|_{L^1(\mathbb{R}^n)} \leq C\|v\|_{H^k(\mathbb{R}^n)}^2 \text{ and } \|\mathcal{H}(v)\|_{L^2(\mathbb{R}^n)} \leq C\|v\|_{H^k(\mathbb{R}^n)}^2.$$

In order to obtain the $H^k(\mathbb{R}^n)$ bound, we take the successive derivatives of $\mathcal{H}(v)$ with respect to $\mathbf{x}$. To illustrate the computations, we present only the

computations for the derivative with respect to ξ. Taking the partial derivative of (43) yields

$$\partial_\xi \mathcal{H}(v) = 2v\partial_\xi v \int_0^1 (1-s)f''(\varphi + sv)\mathrm{d}s + v^3 \int_0^1 s(1-s)f'''(\varphi + sv)\mathrm{d}s$$
$$+ v^2\dot\varphi \int_0^1 (1-s)f'''(\varphi + sv)\mathrm{d}s.$$

Now, since $\dot\varphi$ is bounded, we have

$$\|\partial_\xi \mathcal{H}(v)\|_{L^2(\mathbb{R}^n)} \leq C\left(\|v\|_{H^k(\mathbb{R}^n)}^2 + \|v\|_{H^k(\mathbb{R}^n)}^3\right).$$

Finally, as f and φ are both smooth and as $\varphi^{(k)}$ is bounded for all k, we can continue the above procedure for as many spatial derivatives as necessary and thus obtain the desired estimate.

- For $\mathcal{N}_1(\rho)$. From the definition of $\mathcal{N}_1$ in (22), we have that

$$\mathcal{N}_1(\rho)(\mathbf{x}, t)$$
$$= \int_{\mathbb{R}^n} \mathcal{K}(\mathbf{x} - \mathbf{x}')\left(\rho(\mathbf{z}, t) - \rho(\mathbf{z}', t)\right)^2 \left(\int_0^1 (1-s)\ddot\varphi\left(\xi' + s\left(\rho(\mathbf{z}, t) - \rho(\mathbf{z}', t)\right)\right)\mathrm{d}s\right)\mathrm{d}\xi' d\mathbf{z}',$$

  from which we further note that

$$\rho(\mathbf{z}, t) - \rho(\mathbf{z}', t) = (\mathbf{z} - \mathbf{z}')\int_0^1 \nabla_\mathbf{z} \cdot \rho(\mathbf{z}' + \tau(\mathbf{z} - \mathbf{z}'), t)\mathrm{d}\tau.$$

  As a consequence,

$$\|\mathcal{N}_1(\rho)\|_{L^1(\mathbb{R}^n)} \leq \frac{1}{2}\|\ddot\varphi\|_{L^\infty(\mathbb{R})}\|\nabla_\mathbf{z} \cdot \rho\|_{L^2(\mathbb{R}^{n-1})}^2 \int_{\mathbb{R}^n} \mathbf{z}^2 \mathcal{K}(\mathbf{x})\mathrm{d}\xi\mathrm{d}\mathbf{z},$$
$$\leq C\|\nabla_\mathbf{z} \cdot \rho\|_{H^k(\mathbb{R}^{n-1})}^2.$$

  Here we have used the fact φ is a smooth function and that $\ddot\varphi(\xi) \to 0$ as $\xi \to \pm\infty$ to conclude that $\ddot\varphi \in L^\infty$ and that

$$\left|\int_0^1 (1-s)\ddot\varphi\left(\xi' + s\left(\rho(\mathbf{z}, t) - \rho(\mathbf{z}', t)\right)\right)\mathrm{d}s\right| \leq \frac{1}{2}\|\ddot\varphi\|_{L^\infty(\mathbb{R})}.$$

- For $\mathcal{N}_2(\rho, v)$. From the definition of $\mathcal{N}_2$ in (23), we have that

$$\mathcal{N}_2(\rho, v)(\mathbf{x}, t)$$
$$= \int_{\mathbb{R}^n} \mathcal{K}(\mathbf{x} - \mathbf{x}')\left(\rho(\mathbf{z}, t) - \rho(\mathbf{z}', t)\right)\int_0^1 \partial_\xi v(\xi + s\left(\rho(\mathbf{z}, t) - \rho(\mathbf{z}', t)\right), \mathbf{z}', t)\mathrm{d}\xi' d\mathbf{z}'.$$

  Using Cauchy-Schwarz inequality directly yields

$$\|\mathcal{N}_2(\rho, v)\|_{L^1(\mathbb{R}^n)} \leq C\|\rho\|_{H^k(\mathbb{R}^{n-1})}\|v\|_{H^k(\mathbb{R}^n)},$$

  and the other estimates follow easily.

- For $\mathcal{R}(\rho, v)$. For the last estimates on $\mathcal{R}(\rho, v)$, we project equation (20) along ψ so that we obtain

$$-\langle \mathcal{A}v, \psi\rangle = \partial_t\rho\left(1 + \langle\partial_\xi v, \psi\rangle\right) - \langle\mathcal{A}(\rho\dot\varphi), \psi\rangle + \langle\mathcal{H}(v) + \mathcal{N}(\rho, v), \psi\rangle.$$

  Provided that $\|v\|_{H^k(\mathbb{R}^n)}$ is small enough we can write

$$\partial_t\rho = \frac{1}{1 + \langle\partial_\xi v, \psi\rangle}\left(-\langle\mathcal{A}v, \psi\rangle + \langle\mathcal{A}(\rho\dot\varphi), \psi\rangle - \langle\mathcal{H}(v) + \mathcal{N}(\rho, v), \psi\rangle\right). \tag{44}$$

Then, multiplying the above equation by $\partial_\xi v$ and integrating over $\mathbb{R}^n$, one obtains the desired estimate

$$\|\mathcal{R}(\rho, v)\|_{L^1(\mathbb{R}^n)} \lesssim \|v\|_{H^k(\mathbb{R}^n)}^2 + \|\rho\|_{H^k(\mathbb{R}^{n-1})} \|v\|_{H^k(\mathbb{R}^n)} + \|\nabla_{\mathbf{z}} \cdot \rho\|_{H^k(\mathbb{R}^{n-1})}^2.$$

$\square$

Remark 1. Note that the evolution equation (44) that we obtained for ρ is equivalent to equation (24b) that was previously derived, provided that $\|v\|_{H^k(\mathbb{R}^n)}$ is small enough.

4.2. **Proof of Theorem 1.2.** We can now turn to the proof of our main Theorem 1.2. We first augment the system (24) with an additional equation for $\omega := \nabla_{\mathbf{z}} \cdot \rho$, so that we have the system of equations

$$v(t) = \mathcal{Q}\mathcal{S}_\mathcal{L}(t)v_0 - \mathcal{Q}\mathcal{S}_\mathcal{L}(t)(\rho_0 \dot\varphi)$$

$$+ \int_0^t \mathcal{Q}\mathcal{S}_\mathcal{L}(t - s)\left(\mathcal{H}(v(s)) + \mathcal{N}(\rho(s), v(s)) + \mathcal{R}(\rho(s), v(s))\right) \mathrm{d}s, \tag{45a}$$

$$\rho(t) = \langle \mathcal{S}_\mathcal{L}(t)(\rho_0 \dot\varphi - v_0), \psi \rangle$$

$$- \int_0^t \langle \mathcal{S}_\mathcal{L}(t - s)\left(\mathcal{H}(v(s)) + \mathcal{N}(\rho(s), v(s)) + \mathcal{R}(\rho(s), v(s))\right), \psi \rangle \mathrm{d}s, \tag{45b}$$

$$\omega(t) = \langle \mathcal{S}_\mathcal{L}(t)(\omega_0 \dot\varphi - \nabla_{\mathbf{z}} \cdot v_0), \psi \rangle$$

$$- \int_0^t \nabla_{\mathbf{z}} \cdot \langle \mathcal{S}_\mathcal{L}(t - s)\left(\mathcal{H}(v(s)) + \mathcal{N}(\rho(s), v(s)) + \mathcal{R}(\rho(s), v(s))\right), \psi \rangle \mathrm{d}s. \tag{45c}$$

In the above equation, we used the fact that $\nabla_{\mathbf{z}} \cdot \mathcal{S}_\mathcal{L}(t)f = \mathcal{S}_\mathcal{L}(t)\nabla_{\mathbf{z}} \cdot f$ and set $w_0 := \nabla_{\mathbf{z}} \cdot \rho_0$. We now define the Banach space $\mathcal{X} := H^k(\mathbb{R}^n) \times H^k(\mathbb{R}^{n-1}) \times H^k(\mathbb{R}^{n-1})$. Using standard semigroup theory, we obtain the following existence result for the system (45).

Lemma 4.2. *Suppose that the initial condition for* (45) *satisfies* $(v_0, \rho_0, \omega_0) \in \mathcal{X}$ *for* $k \geq \left\lceil \frac{n+1}{2} \right\rceil$. *Then, there exists* $T > 0$ *such that there exists a unique solution to* (45) *with* $(v(t), \rho(t), \omega(t)) \in \mathcal{X}$ *for all* $t \in [0, T]$.

Let $T_* > 0$ be the maximal time of existence of a solution $(v, \rho, \omega) \in \mathcal{X}$ with initial condition $v_0 \in H^k(\mathbb{R}^n) \cap W^{1,1}(\mathbb{R}^n)$, and $\rho, \omega_0 \in H^k(\mathbb{R}^{n-1}) \cap L^1(\mathbb{R}^{n-1})$. For $t \in [0, T*)$ we define

$$\Phi_v(t) := \sup_{0 \leq s \leq t} (1 + s)^{\frac{n-1}{4} + 1} \|v(s)\|_{H^k(\mathbb{R}^n)},$$

$$\Phi_\rho(t) := \sup_{0 \leq s \leq t} (1 + s)^{\frac{n-1}{4}} \|\rho(s)\|_{H^k(\mathbb{R}^{n-1})},$$

$$\Phi_\omega(t) := \sup_{0 \leq s \leq t} (1 + s)^{\frac{n+1}{4}} \|\omega(s)\|_{H^k(\mathbb{R}^{n-1})}$$

and

$$\mathcal{E}_0 := \|v_0\|_{W^{1,1}(\mathbb{R}^n)} + \|v_0\|_{H^k(\mathbb{R}^n)} + \|\rho_0\|_{W^{1,1}(\mathbb{R}^{n-1})} + \|\rho_0\|_{H^{k+1}(\mathbb{R}^{n-1})}.$$

Estimate for the v component. We apply our semigroup estimates to the first equation of system (45) to obtain

$$\|v(t)\|_{H^k(\mathbb{R}^n)}$$

$$\lesssim (1 + t)^{-\frac{n-1}{4} - 1}\mathcal{E}_0 + e^{-\theta t}\mathcal{E}_0$$

$$+ \int_0^t e^{-\theta(t-s)} \|\mathcal{H}(v(s)) + \mathcal{N}(\rho(s), v(s)) + \mathcal{R}(\rho(s), v(s))\|_{H^k(\mathbb{R}^n)} \, ds$$

$$+ \int_0^t (1+t-s)^{-\frac{n-1}{4}-1} \|\mathcal{H}(v(s)) + \mathcal{N}(\rho(s), v(s)) + \mathcal{R}(\rho(s), v(s))\|_{L^1(\mathbb{R}^n)} \, ds$$

$$\lesssim (1+t)^{-\frac{n-1}{4}-1} \mathcal{E}_0 + e^{-\theta t} \mathcal{E}_0$$

$$+ \int_0^t e^{-\theta(t-s)} \left(\|v(s)\|^2_{H^k(\mathbb{R}^n)} + \|\rho(s)\|_{H^k(\mathbb{R}^{n-1})} \|v(s)\|_{H^k(\mathbb{R}^n)} \right) ds$$

$$+ \int_0^t e^{-\theta(t-s)} \|\omega(s)\|^2_{H^k(\mathbb{R}^{n-1})} \, ds + \int_0^t (1+t-s)^{-\frac{n-1}{4}-1} \|v(s)\|^2_{H^k(\mathbb{R}^n)} \, ds$$

$$+ \int_0^t (1+t-s)^{-\frac{n-1}{4}-1} \left(\|\rho(s)\|_{H^k(\mathbb{R}^{n-1})} \|v(s)\|_{H^k(\mathbb{R}^n)} + \|\omega(s)\|^2_{H^k(\mathbb{R}^{n-1})} \right) ds.$$

We can now use the definition of Φ_v, Φ_ρ and Φ_ω to obtain the inequality

$$\|v(t)\|_{H^k(\mathbb{R}^n)}$$

$$\lesssim (1+t)^{-\frac{n-1}{4}-1} \mathcal{E}_0 + e^{-\theta t} \mathcal{E}_0 + \Phi_v^2(t) \int_0^t e^{-\theta(t-s)} (1+s)^{-\frac{n-1}{2}-2} ds$$

$$+ \Phi_v(t) \Phi_\rho(t) \int_0^t e^{-\theta(t-s)} (1+s)^{-\frac{n-1}{2}-1} ds + \Phi_\omega^2(t) \int_0^t e^{-\theta(t-s)} (1+s)^{-\frac{n+1}{2}} ds$$

$$+ \Phi_v^2(t) \int_0^t (1+t-s)^{-\frac{n-1}{4}-1} (1+s)^{-\frac{n-1}{2}-2} ds$$

$$+ \Phi_\omega^2(t) \int_0^t (1+t-s)^{-\frac{n-1}{4}-1} (1+s)^{-\frac{n+1}{2}} ds$$

$$+ \Phi_v(t) \Phi_\rho(t) \int_0^t (1+t-s)^{-\frac{n-1}{4}-1} (1+s)^{-\frac{n-1}{2}-1} ds.$$

We can now use the estimates of the Lemma A.1 to rewrite the above inequalities as

$$\|v(t)\|_{H^k(\mathbb{R}^n)}$$

$$\lesssim (1+t)^{-\frac{n-1}{4}-1} \mathcal{E}_0 + e^{-\theta t} \mathcal{E}_0 + \Phi_v^2(t)(1+t)^{-\frac{n-1}{2}-2} + \Phi_v(t) \Phi_\rho(t)(1+t)^{-\frac{n-1}{2}-1}$$

$$+ \Phi_\omega^2(t)(1+t)^{-\frac{n+1}{2}} + \Phi_v^2(t)(1+t)^{-\frac{n-1}{4}-1} + \Phi_v(t) \Phi_\rho(t)(1+t)^{-\frac{n-1}{4}-1}$$

$$+ \Phi_\omega^2(t)(1+t)^{-\frac{n-1}{4}-1}.$$

As a consequence, there exists a constant $C_v > 0$ such that for all $t \in [0, T_*)$ we have

$$\Phi_v(t) \leq C_v \left(\mathcal{E}_0 + \Phi_v^2(t) + \Phi_v(t) \Phi_\rho(t) + \Phi_\omega^2(t) \right). \tag{46}$$

Estimate for the ρ component. We repeat the procedure of the previous paragraph for the ρ component of system (45) to obtain

$$\|\rho(t)\|_{H^k(\mathbb{R}^{n-1})}$$

$$\lesssim (1+t)^{-\frac{n-1}{4}} \mathcal{E}_0 + e^{-\theta t} \mathcal{E}_0$$

$$+ \int_0^t e^{-\theta(t-s)} \left(\|v(s)\|^2_{H^k(\mathbb{R}^n)} + \|\rho(s)\|_{H^k(\mathbb{R}^{n-1})} \|v(s)\|_{H^k(\mathbb{R}^n)} \right) ds$$

$$+ \int_0^t e^{-\theta(t-s)} \|\omega(s)\|^2_{H^k(\mathbb{R}^{n-1})} \, ds + \int_0^t (1+t-s)^{-\frac{n-1}{4}} \|v(s)\|^2_{H^k(\mathbb{R}^n)} \, ds$$

$$+ \int_0^t (1+t-s)^{-\frac{n-1}{4}} \left(\|\rho(s)\|_{H^k(\mathbb{R}^{n-1})} \|v(s)\|_{H^k(\mathbb{R}^n)} + \|\omega(s)\|^2_{H^k(\mathbb{R}^{n-1})} \right) ds$$

$$\lesssim (1+t)^{-\frac{n-1}{4}} \mathcal{E}_0 + e^{-\theta t} \mathcal{E}_0 + \Phi_v^2(t)(1+t)^{-\frac{n-1}{2}-2} + \Phi_v(t)\Phi_\rho(t)(1+t)^{-\frac{n-1}{2}-1}$$

$$+ \Phi_\omega^2(t)(1+t)^{-\frac{n+1}{2}} + \Phi_v^2(t) \int_0^t (1+t-s)^{-\frac{n-1}{4}} (1+s)^{-\frac{n-1}{2}-2} ds$$

$$+ \Phi_v(t)\Phi_\rho(t) \int_0^t (1+t-s)^{-\frac{n-1}{4}} (1+s)^{-\frac{n-1}{2}-1} ds$$

$$+ \Phi_\omega^2(t) \int_0^t (1+t-s)^{-\frac{n-1}{4}} (1+s)^{-\frac{n+1}{2}} ds$$

$$\lesssim (1+t)^{-\frac{n-1}{4}} \mathcal{E}_0 + e^{-\theta t} \mathcal{E}_0 + \Phi_v^2(t)(1+t)^{-\frac{n-1}{2}-2} + \Phi_v(t)\Phi_\rho(t)(1+t)^{-\frac{n-1}{2}-1}$$

$$+ \Phi_\omega^2(t)(1+t)^{-\frac{n+1}{2}} + \Phi_v^2(t)(1+t)^{-\frac{n-1}{4}} + \Phi_v(t)\Phi_\rho(t)(1+t)^{-\frac{n-1}{4}}$$

$$+ \Phi_\omega^2(t)(1+t)^{-\frac{n-1}{4}}.$$

As a consequence, there exists a constant $C_\rho > 0$ such that for all $t \in [0, T_*)$ we have

$$\Phi_\rho(t) \leq C_\rho \left(\mathcal{E}_0 + \Phi_v^2(t) + \Phi_v(t)\Phi_\rho(t) + \Phi_\omega^2(t) \right). \tag{47}$$

Estimate for the ω component. Finally, for the ω component, we obtain using the same technique

$$\|\rho(t)\|_{H^k(\mathbb{R}^{n-1})}$$

$$\lesssim (1+t)^{-\frac{n+1}{4}} \mathcal{E}_0 + t^{-\frac{1}{2}} e^{-\theta t} \mathcal{E}_0 + \int_0^t (t-s)^{-\frac{1}{2}} e^{-\theta(t-s)} \|\rho(s)\|_{H^k(\mathbb{R}^{n-1})} \|v(s)\|_{H^k(\mathbb{R}^n)} \, ds$$

$$+ \int_0^t (t-s)^{-\frac{1}{2}} e^{-\theta(t-s)} \left(\|v(s)\|^2_{H^k(\mathbb{R}^n)} + \|\omega(s)\|^2_{H^k(\mathbb{R}^{n-1})} \right) ds$$

$$+ \int_0^t (1+t-s)^{-\frac{n+1}{4}} \left(\|v(s)\|^2_{H^k(\mathbb{R}^n)} + \|\rho(s)\|_{H^k(\mathbb{R}^{n-1})} \|v(s)\|_{H^k(\mathbb{R}^n)} + \|\omega(s)\|^2_{H^k(\mathbb{R}^{n-1})} \right) ds$$

$$\lesssim (1+t)^{-\frac{n+1}{4}} \mathcal{E}_0 + t^{-\frac{1}{2}} e^{-\theta t} \mathcal{E}_0 + \Phi_v^2(t)(1+t)^{-\frac{n-1}{2}-2-\frac{1}{2}} + \Phi_v(t)\Phi_\rho(t)(1+t)^{-\frac{n-1}{2}-1-\frac{1}{2}}$$

$$+ \Phi_\omega^2(t)(1+t)^{-\frac{n+1}{2}-\frac{1}{2}} + \Phi_v^2(t)(1+t)^{-\frac{n+1}{4}} + \Phi_v(t)\Phi_\rho(t)(1+t)^{-\frac{n+1}{4}} + \Phi_\omega^2(t)(1+t)^{-\frac{n+1}{4}}.$$

As a consequence, there exists a constant $C_\omega > 0$ such that for all $t \in [0, T_*)$ we have

$$\Phi_\omega(t) \leq C_\omega \left(\mathcal{E}_0 + \Phi_v^2(t) + \Phi_v(t)\Phi_\rho(t) + \Phi_\omega^2(t) \right). \tag{48}$$

Conclusion of the proof of Theorem 1.2. We can now define

$$\Phi(t) := \Phi_v(t) + \Phi_\rho(t) + \Phi_\omega(t).$$

From inequalities (46), (47) and (48), we have that there exists a constant $C > 0$ such that for all $t \in [0, T_*)$ we have

$$\Phi(t) \leq C \left(\mathcal{E}_0 + \Phi^2(t) \right),$$

from which it can be deduced that if $\mathcal{E}_0$ is small enough, then $\Phi(t) \leq C\mathcal{E}_0$ for all $t \in [0, T_*)$. This implies that the maximal time of existence $T_* = +\infty$ and that the

solution (v, ρ, ω) of system (45) satisfies:

$$\sup_{t \geq 0} (1 + t)^{\frac{n-1}{4} + 1} \|v(t)\|_{H^k(\mathbb{R}^n)} \leq C\mathcal{E}_0,$$

$$\sup_{t \geq 0} (1 + t)^{\frac{n-1}{4}} \|\rho(t)\|_{H^k(\mathbb{R}^{n-1})} \leq C\mathcal{E}_0,$$

$$\sup_{t \geq 0} (1 + t)^{\frac{n+1}{4}} \|\omega(t)\|_{H^k(\mathbb{R}^{n-1})} \leq C\mathcal{E}_0.$$

5. **Extension to the Bates and Chen model.** In this section, we modify our method to prove the multidimensional stability of traveling front solution for the Bates and Chen's model discussed in the introduction. In this case, the traveling wave φ is solution of (4) with $\mathcal{K}_0 = \mathcal{J}$. One of the key feature in that case is the fact that the projection $\mathcal{P}$ now commutes with linearized operator $\mathcal{L}_{bc}$

$$\mathcal{L}_{bc} : \quad H^1(\mathbb{R}^n) \quad \longrightarrow \quad L^2(\mathbb{R}^n)$$
$$u \qquad \longmapsto \quad -u + \mathcal{J} *_\xi u + c\frac{\mathrm{d}}{\mathrm{d}\xi} + f'(\varphi)u + \sum_{j=2}^n \left(-u + \mathcal{J} *_{x_j} u \right). \quad (49)$$

As a consequence, with our Ansatz of form

$$u(\mathbf{x}, t) := \varphi(\xi - \rho(\mathbf{z}, t)) + v(\xi - \rho(\mathbf{z}, t), \mathbf{z}, t)$$

where $\rho : \mathbb{R}^{n-1} \to \mathbb{R} \in H^1(\mathbb{R}^{n-1})$ and $v : \mathbb{R}^n \to \mathbb{R} \in H^1(\mathbb{R}^n)$ with $\mathcal{P}v = 0$, we obtain the equation

$$-\partial_t \rho \dot{\varphi} + \partial_t v = \mathcal{L}_{bc} v - \dot{\varphi} \mathcal{A}_{n-1}\rho + \mathcal{H}(v) + \widetilde{\mathcal{N}}(\rho, v) + \mathcal{R}(\rho, v),$$

where $\mathcal{H}$ and $\mathcal{R}$ were defined in the previous section and

$$\mathcal{A}_{n-1}v = \sum_{j=2}^n \left(-u + \mathcal{J} *_{x_j} u \right),$$

$$\widetilde{\mathcal{N}}(\rho, v) = \widetilde{\mathcal{N}}_1(\rho) + \widetilde{\mathcal{N}}_2(\rho, v),$$

with

$$\widetilde{\mathcal{N}}_1(\rho)(\mathbf{z}, t)$$
$$= \sum_{i=2}^n \left(\int_{\mathbb{R}} \mathcal{J}(y)\varphi(\xi + \rho(\mathbf{z}, t) - \rho(x_2, \cdots, x_i - y, \cdots, x_n, t))\mathrm{d}y \right)$$
$$+ \dot{\varphi}(\xi) \sum_{i=2}^n \left(\int_{\mathbb{R}} \mathcal{J}(y)\rho(x_2, \cdots, x_i - y, \cdots, x_n, t)\mathrm{d}y \right),$$
$$\widetilde{\mathcal{N}}_2(\rho, v)(\mathbf{x}, t)$$
$$= \sum_{i=2}^n \left(\int_{\mathbb{R}} \mathcal{J}(y)v(\xi + \rho(\mathbf{z}, t) - \rho(x_2, \cdots, x_i - y, \cdots, x_n, t), x_2, \cdots, x_i - y, \cdots, x_n, t)\mathrm{d}y \right)$$
$$+ \sum_{i=2}^n \left(\int_{\mathbb{R}} \mathcal{J}(y)v(\xi, x_2, \cdots, x_i - y, \cdots, x_n, t)\mathrm{d}y \right).$$

Using the projection $\mathcal{P}$, we obtain the system

$$\partial_t v = \mathcal{L}_{bc} v + \mathcal{Q}\left(\mathcal{H}(v) + \widetilde{\mathcal{N}}(\rho, v) + \mathcal{R}(\rho, v) \right), \quad (50a)$$

$$(1 + \langle \partial_\xi v, \psi \rangle)\partial_t \rho = \mathcal{A}_{n-1}\rho - \langle \mathcal{H}(v) + \widetilde{\mathcal{N}}(\rho, v), \psi \rangle. \quad (50b)$$

Note that $\mathcal{L}_{bc} = \mathcal{L}_0 + \mathcal{A}_{n-1}$ with

$$\mathcal{L}_0 v = -u + \mathcal{J} *_\xi u + c\frac{\mathrm{d}}{\mathrm{d}\xi} + f'(\varphi)u,$$

and that $\psi \in H^1(\mathbb{R})$ is such that $\mathcal{L}_0^* \psi = 0$ and $\int_{\mathbb{R}} \dot{\varphi}(\xi)\psi(\xi)\mathrm{d}\xi = 1$, $\mathcal{L}_0^*$ being the adjoint of $\mathcal{L}_0$.

As long as $\|v\|_{H^k(\mathbb{R}^n)}$ remains small, we can rewrite the second equation of system (50) as

$$\partial_t \rho = \mathcal{A}_{n-1}\rho - \frac{1}{1 + \langle \partial_\xi v, \psi \rangle} \left(\langle \partial_\xi v, \psi \rangle \mathcal{A}_{n-1}\rho + \langle \mathcal{H}(v) + \widetilde{\mathcal{N}}(\rho, v), \psi \rangle \right):$$
$$= \mathcal{A}_{n-1}\rho + \mathcal{M}(\rho, v).$$

Setting $\omega := \nabla_{\mathbf{z}} \cdot \rho$, we finally obtain the initial value problem

$$\partial_t v = \mathcal{L}_{bc}v + \mathcal{Q}\left(\mathcal{H}(v) + \widetilde{\mathcal{N}}(\rho, v) + \mathcal{R}(\rho, v) \right), \tag{51a}$$

$$\partial_t \rho = \mathcal{A}_{n-1}\rho + \mathcal{M}(\rho, v), \tag{51b}$$

$$\partial_t \omega = \mathcal{A}_{n-1}\omega + \nabla_{\mathbf{z}} \cdot \mathcal{M}(\rho, v), \tag{51c}$$

$$v(0) = v_0, \quad \rho(0) = \rho_0, \quad \omega(0) = \omega_0. \tag{51d}$$

5.1. Linear and nonlinear estimates.

In this section, we derive linear estimates for the semigroups generated by the linear operators $\mathcal{L}_{bc}$ and $\mathcal{A}_{n-1}$ together with estimates for the nonlinear terms.

5.1.1. Study of $\mathcal{S}_{\mathcal{L}_{bc}}(t)$.

We consider the initial value problem

$$\partial_t u = \mathcal{L}_{bc}u, \quad u(\cdot, 0) = u_0 \in H^k(\mathbb{R}^n), \tag{52}$$

which has the solution $u(\xi, \mathbf{z}, t) = \mathcal{S}_{\mathcal{L}_{bc}}(t)u_0(\xi, \mathbf{z})$. Taking the Fourier transform in $\mathbf{z}$ on both side of (52) yields

$$\partial_t \hat{u}(\xi, \widetilde{\mathbf{k}}, t) = \mathcal{L}_0 \hat{u}(\xi, \widetilde{\mathbf{k}}, t) + \sum_{j=1}^{n-1} \left(-1 + \widehat{\mathcal{J}}(\widetilde{k}_j) \right) \hat{u}(\xi, \widetilde{\mathbf{k}}, t), \quad \widetilde{\mathbf{k}} = \left(\widetilde{k}_1, \cdots, \widetilde{k}_{n-1} \right). \tag{53}$$

Setting $\mathcal{S}_{\mathcal{L}_0}(t)$ to represent the semigroup generated by $\mathcal{L}_0$, the solution of (52) are given by

$$u(\mathbf{x}, t) = \frac{1}{(2\pi)^{n-1}} \int_{\mathbb{R}^{n-1}} e^{i\widetilde{\mathbf{k}}\mathbf{z}} \mathcal{S}_{\mathcal{L}_0}(t) e^{\sum_{i=1}^{n-1}\left(-1+\widehat{\mathcal{J}}(\widetilde{k}_i)\right)t} \hat{u}_0(\xi, \widetilde{\mathbf{k}}, t)\mathrm{d}\widetilde{\mathbf{k}}.$$

By Lemma 1.1, there exists $\gamma_0 > 0$ such that

$$\|\mathcal{S}_{\mathcal{L}_0}(t)\mathcal{Q}u\|_{L^2(\mathbb{R}^n)} \lesssim e^{-\gamma_0 t}\|\mathcal{Q}u\|_{L^2(\mathbb{R}^n)}.$$

Note that the linear problem (52) is homogeneous in $\mathbf{z}$ so that $\mathcal{S}_{\mathcal{L}_{bc}}(t)$ can be differentiated with respect to x_j, $j = 2\ldots n$. Estimates for the ξ derivative follow from the regularity of solution (53). As a consequence, we have obtained the following Lemma.

Lemma 5.1. *The semigroup generated by the linear operator $\mathcal{L}_{bc}$ satisfies the decay estimate*

$$\|\mathcal{S}_{\mathcal{L}_{bc}}(t)\mathcal{Q}u\|_{H^k(\mathbb{R}^n)} \lesssim e^{-\gamma_0 t}\|\mathcal{Q}u\|_{H^k(\mathbb{R}^n)}.$$

5.1.2. *Study of $\mathcal{S}_{\mathcal{A}_{n-1}}(t)$.* The study of semigroup generated by $\mathcal{A}_{n-1}$ has already been done in [1, 4] and we only quote their results.

Lemma 5.2. *The semigroup generated by the linear operator $\mathcal{A}_{n-1}$ satisfies the decay estimate*

$$\|\mathcal{S}_{\mathcal{A}_{n-1}}(t)u\|_{H^k(\mathbb{R}^{n-1})} \lesssim (1+t)^{-\frac{n-1}{4}}\|u\|_{L^1(\mathbb{R}^{n-1})} + e^{-\theta t}\|u\|_{H^k(\mathbb{R}^{n-1})},$$

$$\|\nabla_{\mathbf{z}} \cdot \mathcal{S}_{\mathcal{A}_{n-1}}(t)u\|_{H^k(\mathbb{R}^{n-1})} \lesssim (1+t)^{-\frac{n-1}{4}-\frac{1}{2}}\|u\|_{L^1(\mathbb{R}^{n-1})} + t^{-\frac{1}{2}}e^{-\theta t}\|u\|_{H^k(\mathbb{R}^{n-1})}.$$

5.2. **Nonlinear estimates.** One can easily check that similar estimates as the one presented in Lemma 4.1 hold for the nonlinear terms $\widetilde{\mathcal{N}}_1(\rho)$ and $\widetilde{\mathcal{N}}_2(\rho, v)$. More precisely, we have the following Lemma.

Lemma 5.3. *Let $k \geq \left\lceil \frac{n+1}{2} \right\rceil$. There exists a $\delta > 0$ such that for any $v \in H^k(\mathbb{R}^n)$ and $\rho \in H^{k+1}(\mathbb{R}^{n-1})$ with $\|v\|_{H^k(\mathbb{R}^n)} \leq \delta$, $\|\rho\|_{H^k(\mathbb{R}^{n-1})} \leq \delta$ and $\|\nabla_{\mathbf{z}}\cdot\rho\|_{H^k(\mathbb{R}^{n-1})} \leq \delta$ we have*

$$\left\|\widetilde{\mathcal{N}}_1(\rho)\right\|_{L^1(\mathbb{R}^n)}, \left\|\widetilde{\mathcal{N}}_1(\rho)\right\|_{H^k(\mathbb{R}^n)} \lesssim \|\nabla_{\mathbf{z}} \cdot \rho\|^2_{H^k(\mathbb{R}^{n-1})}, \tag{54a}$$

$$\left\|\widetilde{\mathcal{N}}_2(\rho, v)\right\|_{L^1(\mathbb{R}^n)}, \left\|\widetilde{\mathcal{N}}_2(\rho, v)\right\|_{H^k(\mathbb{R}^n)} \lesssim \|\rho\|_{H^k(\mathbb{R}^{n-1})} \|v\|_{H^k(\mathbb{R}^n)}, \tag{54b}$$

$$\|\mathcal{M}(\rho, v)\|_{L^1(\mathbb{R}^{n-1})}, \|\mathcal{M}(\rho, v)\|_{H^k(\mathbb{R}^{n-1})} \lesssim \|v\|^2_{H^k(\mathbb{R}^n)} + \|\nabla_{\mathbf{z}} \cdot \rho\|^2_{H^k(\mathbb{R}^{n-1})}$$

$$+ \|\nabla_{\mathbf{z}} \cdot \rho\|^2_{H^k(\mathbb{R}^{n-1})}. \tag{54c}$$

Proof. Most of the proof is similar to that of Lemma 4.1 and is thus omitted. We only present, part of the computations for the estimate of the nonlinear term $\widetilde{\mathcal{N}}_1(\rho)$. For each element of the sum in $\widetilde{\mathcal{N}}_1(\rho)$, we use Taylor's formula and obtain

$$\widetilde{\mathcal{N}}_1(\rho)(\mathbf{z}, t)$$
$$= \sum_{j=2}^n \int_{\mathbb{R}} \mathcal{J}(y) \left(\rho(\mathbf{z}, t) - \mathcal{T}_y^j \cdot \rho(\mathbf{z}, t)\right)^2 \left(\int_0^1 (1-s)\ddot{\varphi}\left(\xi + \rho(\mathbf{z}, t) - \mathcal{T}_y^j \cdot \rho(\mathbf{z}, t)\right) \mathrm{d}s\right) \mathrm{d}y,$$

with $\mathcal{T}_y^j \cdot \rho(\mathbf{z}, t) = \rho(x_2, \cdot, x_j - y, \cdot, x_n)$. Now, we note that

$$\rho(\mathbf{z}, t) - \mathcal{T}_y^j \cdot \rho(\mathbf{z}, t) = y \int_0^1 \partial_{x_j} \rho(\mathbf{z}' + \tau y) \mathrm{d}\tau.$$

As a consequence,

$$\|\mathcal{N}_1(\rho)\|_{L^1(\mathbb{R}^n)} \leq \frac{\int_{\mathbb{R}} |\ddot{\varphi}|}{2} \|\nabla_{\mathbf{z}} \cdot \rho\|^2_{L^2(\mathbb{R}^{n-1})} \int_{\mathbb{R}} y^2 \mathcal{J}(y) \mathrm{d}y,$$

$$\leq C \|\nabla_{\mathbf{z}} \cdot \rho\|^2_{H^k(\mathbb{R}^{n-1})}.$$

Here we have used the fact that $\ddot{\varphi}$ is exponentially localized which is a direct consequence of the fact that $\mathcal{J} \in W^{1,1}_\eta(\mathbb{R})$ [7]. $\qquad\square$

5.3. **Proof of Theorem 1.3.** We can now conclude the proof of Theorem 1.3. By the variation of constants formula, the solution to (51) can be written as

$$v(t) = \mathcal{S}_{\mathcal{L}_{bc}}(t)v_0 + \int_0^t \mathcal{S}_{\mathcal{L}_{bc}}(t-s)\mathcal{Q}\left(\mathcal{H}(v(s)) + \widetilde{\mathcal{N}}(\rho(s), v(s)) + \mathcal{R}(\rho(s), v(s))\right)\mathrm{d}s,$$
$$(55\mathrm{a})$$

$$\rho(t) = \mathcal{S}_{\mathcal{A}_{n-1}}(t)\rho_0 + \int_0^t \mathcal{S}_{\mathcal{A}_{n-1}}(t-s)\mathcal{M}(\rho(s), v(s))\mathrm{d}s, \qquad (55\mathrm{b})$$

$$\omega(t) = \mathcal{S}_{\mathcal{A}_{n-1}}(t)\omega_0 + \int_0^t \nabla_{\mathbf{z}} \cdot \mathcal{S}_{\mathcal{A}_{n-1}}(t-s)\mathcal{M}(\rho(s), v(s))\mathrm{d}s. \qquad (55\mathrm{c})$$

In the last component of the above system, we used the fact that $\nabla_{\mathbf{z}} \cdot \mathcal{S}_{\mathcal{A}_{n-1}}(t)f = \mathcal{S}_{\mathcal{A}_{n-1}}(t)\nabla_{\mathbf{z}} \cdot f$. Once again, using standard semigroup theory, we obtain the local existence of solutions for the system (55) for initial condition $(v_0, \rho_0, \omega_0) \in \mathcal{X}$. Thus, let $T_* > 0$ be the maximal time of existence of a solution $(v, \rho, \omega) \in \mathcal{X}$ with initial condition $v_0 \in H^k(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$, and $\rho, \omega_0 \in H^k(\mathbb{R}^{n-1}) \cap L^1(\mathbb{R}^{n-1})$. For $t \in [0, T*)$ we define

$$\Phi_v(t) := \sup_{0 \le s \le t} (1+s)^{\frac{n+1}{2}} \|v(s)\|_{H^k(\mathbb{R}^n)},$$

$$\Phi_\rho(t) := \sup_{0 \le s \le t} (1+s)^{\frac{n-1}{4}} \|\rho(s)\|_{H^k(\mathbb{R}^{n-1})},$$

$$\Phi_\omega(t) := \sup_{0 \le s \le t} (1+s)^{\frac{n+1}{4}} \|\omega(s)\|_{H^k(\mathbb{R}^{n-1})}$$

and

$$\widetilde{\mathcal{E}}_0 := \|v_0\|_{L^1(\mathbb{R}^n)} + \|v_0\|_{H^k(\mathbb{R}^n)} + \|\rho_0\|_{W^{1,1}(\mathbb{R}^{n-1})} + \|\rho_0\|_{H^{k+1}(\mathbb{R}^{n-1})}.$$

Only the estimates for the v component will significantly changed and thus we only present the details of the computations in that case and let the ρ and ω estimates to the reader. Note that in that case, we obtain similar estimates as the one presented by Kapitula in the local setting [13]. By applying the semigroup estimates derived in the previous section, we obtain

$$\|v(t)\|_{H^k(\mathbb{R}^n)}$$
$$\lesssim e^{-\gamma_0 t}\mathcal{E}_0 + \int_0^t e^{-\gamma_0(t-s)} \left\|\mathcal{Q}\left(\mathcal{H}(v(s)) + \widetilde{\mathcal{N}}(\rho(s), v(s)) + \mathcal{R}(\rho(s), v(s))\right\|_{H^k(\mathbb{R}^n)}\right) \mathrm{d}s$$
$$\lesssim e^{-\gamma_0 t}\mathcal{E}_0$$
$$+ \int_0^t e^{-\gamma_0(t-s)} \left(\|v(s)\|_{H^k(\mathbb{R}^n)}^2 + \|\rho(s)\|_{H^k(\mathbb{R}^{n-1})} \|v(s)\|_{H^k(\mathbb{R}^n)} + \|\omega(s)\|_{H^k(\mathbb{R}^{n-1})}^2\right) \mathrm{d}s$$
$$\lesssim e^{-\gamma_0 t}\mathcal{E}_0 + \Phi_v^2(t)\int_0^t e^{-\gamma_0(t-s)}(1+s)^{-(n+1)}\mathrm{d}s$$
$$+ \Phi_v(t)\Phi_\rho(t)\int_0^t e^{-\gamma_0(t-s)}(1+s)^{-\frac{3n+1}{4}}\mathrm{d}s + \Phi_\omega^2(t)\int_0^t e^{-\gamma_0(t-s)}(1+s)^{-\frac{n+1}{2}}\mathrm{d}s$$
$$\lesssim e^{-\gamma_0 t}\mathcal{E}_0 + \Phi_v^2(t)(1+t)^{-(n+1)} + \Phi_v(t)\Phi_\rho(t)(1+t)^{-\frac{3n+1}{4}} + \Phi_\omega^2(t)(1+t)^{-\frac{n+1}{2}}.$$

As a consequence, reproducing similar computations for ρ and ω, one can find three constants $C_v > 0$, $C_\rho > 0$ and $C_\omega > 0$ such that for all $t \in [0, T_*)$ we have

$$\Phi_v(t) \le C_v\left(\widetilde{\mathcal{E}}_0 + \Phi_v^2(t) + \Phi_v(t)\Phi_\rho(t) + \Phi_\omega^2(t)\right),$$

$$\Phi_\rho(t) \le C_\rho\left(\widetilde{\mathcal{E}}_0 + \Phi_v^2(t) + \Phi_v(t)\Phi_\rho(t) + \Phi_\omega^2(t)\right),$$

$$\Phi_\omega(t) \le C_\omega\left(\widetilde{\mathcal{E}}_0 + \Phi_v^2(t) + \Phi_v(t)\Phi_\rho(t) + \Phi_\omega^2(t)\right).$$

Using similar arguments to that of Theorem 1.2, we conclude that the maximal time of existence $T_* = +\infty$ and that the solution (v, ρ, ω) of system (55) satisfies:

$$\sup_{t \geq 0} (1 + t)^{\frac{n+1}{2}} \|v(t)\|_{H^k(\mathbb{R}^n)} \leq C\widetilde{\mathcal{E}}_0,$$

$$\sup_{t \geq 0} (1 + t)^{\frac{n-1}{4}} \|\rho(t)\|_{H^k(\mathbb{R}^{n-1})} \leq C\widetilde{\mathcal{E}}_0,$$

$$\sup_{t \geq 0} (1 + t)^{\frac{n+1}{4}} \|\omega(t)\|_{H^k(\mathbb{R}^{n-1})} \leq C\widetilde{\mathcal{E}}_0.$$

6. Discussion.

Summary of main results. In this paper, we have proved the multidimensional stability of planar traveling waves for scalar nonlocal Allen-Cahn equation (1) using semigroup estimates. More precisely, we have shown that if the traveling wave is spectrally stable in one space dimension, then it is stable in n-space dimension, $n \geq 2$, with perturbations of the wave decaying like $t^{-(n-1)/4}$ as $t \to +\infty$ in $H^k(\mathbb{R}^n)$ for $k \geq \left[\frac{n+1}{2}\right]$. We have also obtained similar results by applying our method to a model proposed by Bates and Chen [1] generalizing to dimensions 2 and 3 their results.

Beyond smooth and small perturbations. One interesting avenue of future work is to investigate the multidimensional stability of planar traveling waves for equation (1) under weaker assumptions for the perturbations. For example, in the local case, Matano *et al.* have recently shown that [15] the multidimensional stability of planar traveling waves with possibly large initial perturbations that only decay at space infinity. It would be interesting to see if their techniques can adapted to our nonlocal setting.

Generalization to other Kernel. One of our key technical assumption for the kernel $\mathcal{K}$ is the Taylor expansion of its Fourier transform close to the origin. Namely, we have supposed

$$\widehat{\mathcal{K}}(\mathbf{k}) = 1 - d_0\|\mathbf{k}\|^2 + o(\|\mathbf{k}\|^2), \quad \text{as } \mathbf{k} \to 0.$$

A natural extension would be to study kernels with different Taylor expansion such as for example

$$\widehat{\mathcal{K}}(\mathbf{k}) = 1 - d_0\|\mathbf{k}\|^s + o(\|\mathbf{k}\|^s), \quad \text{as } \mathbf{k} \to 0,$$

with possibly $0 < s < 2$. Then one could conjecture that if the traveling wave is spectrally stable in one space dimension, then it is stable in n-space dimension, $n \geq 2$, with perturbations of the wave decaying like $t^{-(n-1)/(2s)}$ as $t \to +\infty$ in $H^k(\mathbb{R}^n)$ for $k \geq \left[\frac{n+1}{2}\right]$. We let this question as an open problem.

Generalization to other nonlocal problems. Recently, Miller and Zeng [16] have shown similar results in dimension $n \geq 4$ for an integrodifference equation of the form

$$u_{j+1} = \mathcal{K} *_{\mathbf{x}} g(u_j), \quad j \in \mathbb{N}, \tag{56}$$

with a Gaussian kernel $\mathcal{K}$ and a smooth nonlinearity. This type of equation belongs to the class of problems where the convolution term appears into the equation in a nonlinear fashion as it is often the case in physical or biological models. Within this class of problems, let for example mention the continuum neuronal model [9]

$$\partial_t u = -u + \mathcal{K} *_{\mathbf{x}} S(u) \tag{57}$$

where the smooth nonlinear function S is such that $-u + S(u)$ is of bistable type, or the continuum limit of an interacting particle system with Glauber dynamics and Kac potential [8]

$$\partial_t u = -u + \tanh\left(\beta \mathcal{K} *_{\mathbf{x}} u + h\right), \tag{58}$$

where $\beta > 1$ and $h > 0$. For both of these last two models (57) and (58), one can prove the existence and spectral stability of a traveling wave solution for the one dimensional problem [9, 8, 2]. As a consequence, under the same Hypothesis (H2) for the kernel, it should be straightforward to adapt our proof of Theorem 1.2 to show that if equations (56), (57) and (58) admit a traveling wave that is spectrally stable in one space dimension, then it is stable in n-space dimension, $n \geq 2$, with perturbations of the wave decaying like $t^{-(n-1)/4}$ as $t \to +\infty$ in $H^k(\mathbb{R}^n)$ for $k \geq \left[\frac{n+1}{2}\right]$.

Appendix A. **Some estimates.** The following lemma can be proved by direct computations, see [18].

Lemma A.1. *Suppose α, β, $\gamma > 0$, then*

1. $\int_0^{t/2}(1+t-s)^{-\beta}(1+s)^{-\gamma}\mathrm{d}s \lesssim (1+t)^{-\alpha}$, *if $\alpha \leq \beta$, $\alpha \leq \beta + \gamma - 1$, $\gamma \neq 1$ or if $\alpha < \beta$, $\alpha \leq \beta + \gamma - 1$, $\gamma = 1$;*
2. $\int_{t/2}^t(1+t-s)^{-\beta}(1+s)^{-\gamma}\mathrm{d}s \lesssim (1+t)^{-\alpha}$, *if $\alpha \leq \gamma$, $\alpha \leq \beta + \gamma - 1$, $\beta \neq 1$ or if $\alpha < \gamma$, $\alpha \leq \beta + \gamma - 1$, $\beta = 1$;*
3. $\int_0^t e^{-\beta(t-s)}(1+s)^{-\gamma}\mathrm{d}s \lesssim (1+t)^{-\gamma}$.

REFERENCES

[1] P. W. Bates and F. Chen, Spectral analysis and multidimensional stability of traveling waves for nonlocal Allen-Cahn equation, *J. Math. Anal. Appl.,* **273** (2002), 45–57.

[2] P. W. Bates and F. Chen, Spectral analysis of traveling waves for nonlocal evolution equations, *SIAM J. Math. Anal.,* **38** (2006), 116–126.

[3] P. W. Bates, P. C. Fife, X. Ren and X. Wang, Traveling waves in a convolution model for phase transitions, *Arch. Rational Mech. Anal.,* **138** (1997), 105–136.

[4] E. Chasseigne, M. Chaves and J. D. Rossi, Asymptotic behavior for nonlocal diffusion equations, *J. Math. Pures Appl.,* **86** (2006), 271–291.

[5] F. Chen, Uniform stability if multidimensional travelling waves for the nonlocal Allen-Cahn equation, *Electronic Journal of Differential Equations,* **10** (2003), 109–113.

[6] X. Chen, Existence, uniqueness, and asymptotic stability of traveling waves in nonlocal evolution equations, *Advences in Differential Equations,* **2** (1997), 125–160.

[7] J. Coville, *Equation de Réaction Diffusion Non-locale,* Thèse de doctorat del'université Pierre et Marie Curie, Paris 6, 2003.

[8] A. De Masi, T. Gobron and E. Presutti, Traveling fronts in non-local evolution equations, *Arch. Rat. Mech. Anal,* **132** (1995), 143–205.

[9] G. B. Ermentrout and J. B. McLeod, Existence and uniqueness of travelling waves for a neural network, *Proc. Roy. Soc. Edin.,* **123** (1993), 461–478.

[10] T. Gallay and A. Scheel, Diffusive stability of oscillations in reaction-diffusion systems, *Trans. Amer. Math. Soc.,* **363** (2011), 2571–2598.

[11] A. Hoffman, H. J. Hupkes and E. S. van Vleck, Multi-dimensional stability of waves travelling through rectangular lattices in rational directions, *Trans. Amer. Math. Soc.,* **367** (2015), 8757–8808.

[12] M. A. Johnson and K. Zumbrun, Nonlinear stability of periodic traveling wave solutions of systems of viscous conservation laws in the generic case, *J. Differential Equations,* **249** (2010), 1213–1240.

[13] T. Kapitula, Multidimensional stability of planar traveling waves, *Trans. Amer. Math. Soc.,* **349** (1997), 257–269.

[14] C. D. Levermore and J. X. Xin, Multidimensional stability of traveling waves in a bistable reaction-diffusion equation, II, *Comm. in Par. Diff. Equ.,* **17** (1992), 1901–1924.

[15] H. Matano, M. Nara and M. Taniguchi, Stability of planar waves in the Allen-Cahn equation, *Comm. in Par. Diff. Equ.,* **34** (2009), 976–1002.

[16] J. R. Miller and H. Zeng, Multidimensional stability of planar traveling waves for an integrodifference model, *Discrete and Continuous Dynamical Systems B,* **18** (2013), 741–751.

[17] M. Oh and K. Zumbrun, Stability and asymptotic behavior of traveling-wave solutions of viscous conservation laws in several dimensions, *Arch. Ration. Mech. Anal.,* **196** (2010), 1–20.

[18] J. X. Xin, Multidimensional stability of traveling waves in a bistable reaction-diffusion equation, I, *Comm. in Par. Diff. Equ.,* **17** (1992), 1889–1899.

E-mail address: gregory.faye@math.univ-toulouse.fr